Renormalization of the electromagnetic current

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g-2

Calculation of hardonic contribution to anomalous magnetic moment requires

$$\langle j_\mu(x) j_\nu(0) \rangle$$

with (continuum) electromagnetic current

$$j_\mu(x) = \sum_{f=1}^{N_f} q_f \overline{\psi}_f(x) \gamma_\mu \psi_f(x)$$

and single-flavor quark $\psi_f(x)$.
In order to obtain well-defined quantities when removing the lattice cutoff $a \to 0$, we need to renormalize our operators.

Concepts

Renormalization principles

1. Let A be a composite operator. The renormalized operator A_R can be written as

   $$A_\mathrm{R} = \sum_B Z_{AB} B,$$

   where the sum is over operators B with
   - the same symmetry properties as A and [Collins1984 (Section 9.1)]
   - lower or equal mass dimension than A. [Collins1984 (Section 6.4)]
2. A set of operators that is invariant under a symmetry transformation renormalizes with the same multiplicative renormalization constant.

Flavor symmetries

Consider N_f mass-degenerate quarks $\Psi = (u, d, \ldots)^T$ and the Wilson discretization of QCD. The transformation

$$\Psi'(x) = e^{i\alpha(x) t^a} \Psi(x)$$

is a symmetry of the theory for $t^a \in \mathfrak{u}(N_f)$ and $a = 0, \ldots, N_f^2 - 1$.

Example: $N_f = 2$, Pauli matrices

$$t^0 = \mathbb{1}, \quad t^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad t^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example: $N_f = 3$ Gell-Mann matrices

Conserved current

The corresponding Noether current is

$$V_\mu^{(\mathrm{con}),a}(x) = \frac{1}{2}\left(\overline{\Psi}(x+a\hat{\mu})(1+\gamma_\mu)U_\mu^\dagger(x)t^a\Psi(x) - \overline{\Psi}(x)(1-\gamma_\mu)U_\mu(x)t^a\Psi(x+a\hat{\mu})\right).$$

with flavor spinor $\Psi(x)$ flavor matrix $t^a \in \mathfrak{u}(N_f)$, $a = 0, \ldots, N_f^2 - 1$. Alternatively, the local current is

$$V_\mu^{(\mathrm{loc}),a}(x) = \overline{\Psi}(x)\gamma_\mu t^a \Psi(x),$$

- $\{V_\mu^a(x)\}_{a=1,\ldots,N_f^2-1}$ transforms in the adjoint representation.
- $\{V_\mu^0(x)\}$ transforms in the trivial representation.

Renormalization (chiral limit)

Consider N_f massless quarks. Renormalization is captured in a multiplicative constant for each operator. There are no operators that mix with the current.

$$
\begin{aligned}
V^a_{\mu,\mathrm{R}}(x) &= Z_V(g_0) V^a_\mu(x) \qquad a = 1,\ldots,N_f^2-1, \\
V^0_{\mu,\mathrm{R}}(x) &= Z_V(g_0) r_V(g_0) V^0_\mu(x).
\end{aligned}
$$

We neglect $O(a)$-improvement terms. See [Bhattacharya et al., arXiv:hep-lat/0511014] for details.

No-renormalization of currents

The conserved current arising from the flavor symmetry in QCD does not renormalize, i.e. $Z_V = 1$ and [Collins1984 (Eq. 6.6.32)]

$$V_{\mu,\mathrm{R}}^{(\mathrm{con}),a}(x) = V_{\mu}^{(\mathrm{con}),a}(x)$$

Remark: This does not hold for QED. [Collins et al., arXiv:hep-th/0512187]

Finite renormalization of currents

The local current arising from the flavor symmetry in QCD is finite

$$V_{\mu,\mathrm{R}}^{(\mathrm{loc}),a}(x) = Z_V(g_0)\, V_{\mu}^{(\mathrm{loc}),a}(x)$$

with renormalization constant

$$Z_V(g_0) = 1 + O(g_0^2).$$

[Vladikas, arXiv:1103.1323]

Renormalization (massive quarks)

Consider N_f massive quarks. There are operators of same mass dimension that mix with the current.

$$V^a_{\mu,\mathrm{R}} = Z_V(g_0)\left[(1 + a\bar{b}_V(g_0)\mathrm{tr}[M])V^a_\mu + \tfrac{1}{2}ab_V(g_0)\mathrm{tr}[\{t^a, M\}V_\mu] + af_V(g_0)\mathrm{tr}[t^a M]\,V^0_\mu\right],$$
$$V^0_{\mu,\mathrm{R}} = Z_V(g_0)r_V(g_0)\left[(1 + a\bar{d}_V(g_0)\mathrm{tr}[M])V^0_\mu + ad_V(g_0)\mathrm{tr}[MV_\mu]\right],$$

M is the (subtraced) bare quark mass matrix. [Bhattacharya2005 Eq. (15),(23)]

$N_f = 3 + 1$

We restrict to flavor-neutral currents $V_\mu^a(x)$ with $a = 0, 3, 8, 15$:

$$t^3 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, t^8 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -2 & \\ & & & 0 \end{pmatrix}, t^{15} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix}, t^0 = \mathbb{1}.$$

$$M = \mathrm{diag}(m_l, m_l, m_l, m_c).$$

$$\begin{pmatrix} V^3_{\mu,\mathrm{R}} \\ V^8_{\mu,\mathrm{R}} \\ V^{15}_{\mu,\mathrm{R}} \\ V^0_{\mu,\mathrm{R}} \end{pmatrix} = \begin{pmatrix} Z_{3,3} & Z_{3,8} & Z_{3,15} & Z_{3,0} \\ Z_{8,3} & Z_{8,8} & Z_{8,15} & Z_{8,0} \\ Z_{15,3} & Z_{15,8} & Z_{15,15} & Z_{15,0} \\ Z_{0,3} & Z_{0,8} & Z_{0,15} & Z_{0,0} \end{pmatrix} \begin{pmatrix} V^3_\mu \\ V^8_\mu \\ V^{15}_\mu \\ V^0_\mu \end{pmatrix}$$

Gray components vanish due to flavour degeneracy.

Renormalization (massive quarks)

Example: $N_f = 3 + 1$ and $M = \text{diag}(m_l, m_l, m_l, m_c)$:

$$Z_{3,3} = Z_{8,8} = Z_V(1 + a\overline{b}_V(3m_l + m_c) + ab_V m_l), \quad (1)$$

$$Z_{15,15} = Z_V(1 + a\overline{b}_V(3m_l + m_c) + ab_V \tfrac{m_l+3m_c}{4}), \quad (2)$$

$$Z_{0,0} = Z_V r_V(1 + a\overline{d}_V(3m_l + m_c) + ad_V \tfrac{3m_l+m_c}{4}), \quad (3)$$

$$Z_{15,0} = Z_V(ab_V \tfrac{3}{4}(m_l - m_c) + af_V 3(m_l - m_c)), \quad (4)$$

$$Z_{0,15} = Z_V r_V ad_V \tfrac{1}{4}(m_l - m_c). \quad (5)$$

Mass-dependent renormalization

- Once we have determined all constants, we know the renormalization at a fixed g_0 for arbitary quark masses (mass-independent renormalization)
- All additive contribution are $O(a)$ and can be considered improvement terms.
- However, for $N_f = 3 + 1$, we have $am_c \approx 0.3 \longrightarrow$ Consider mass-dependent renormalization scheme: [Andreas Risch, PhD thesis (2021)]

$$V^a_{\mu,\mathrm{R}}(x) = \sum_{b=0}^{N_f^2-1} Z_{a,b}(g_0, aM) V^b_\mu(x), \quad a = 0, \ldots, N_f^2 - 1$$

Implementation

Two-point function

Renormalization condition: Choose $k = 1, 2, 3$ and impose for $a = 3, 8, 15, 0$ [Martinelli (1994) doi:10.1016/0920-5632(94)90431-6]

$$\sum_{\vec{x}} \langle V_k^{(\mathrm{con}),a}(x) V_k^{(\mathrm{loc}),c}(0) \rangle \stackrel{!}{=} \sum_{b=3,8,15,0} Z_{a,b}(g_0, aM) \sum_{\vec{x}} \langle V_k^{(\mathrm{loc}),b}(x) V_k^{(\mathrm{loc}),c}(0) \rangle .$$

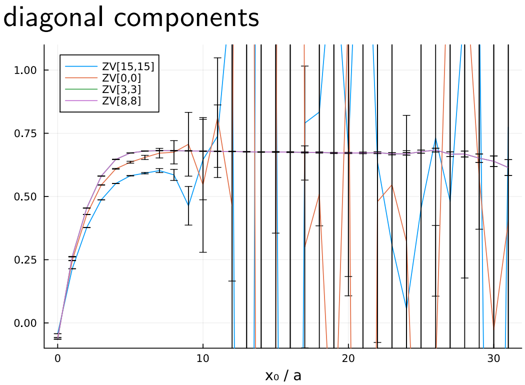
at some $x_0 = x_{\mathrm{ren}}$.

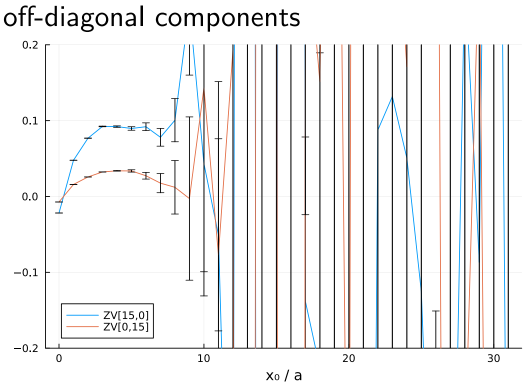
- \+ no extra calculation (since related to $g-2$)
- – signal-to-noise problem
- – improvement term contributes $\Rightarrow \mathrm{O}(a)$ ambiguity

Results

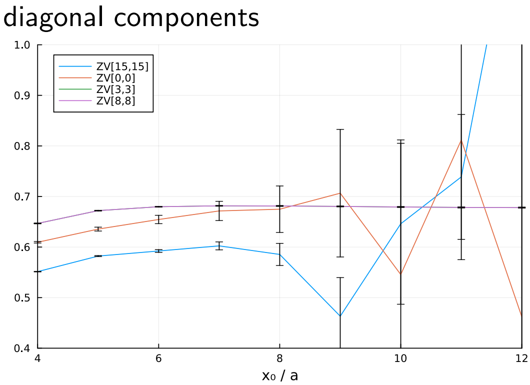
- A400a00b324: $m_\pi = 400\ \mathrm{MeV}, \alpha = 0.00, \beta = 3.24$.
- $C^\star$ boundary conditions
- $N_f = 3 + 1$

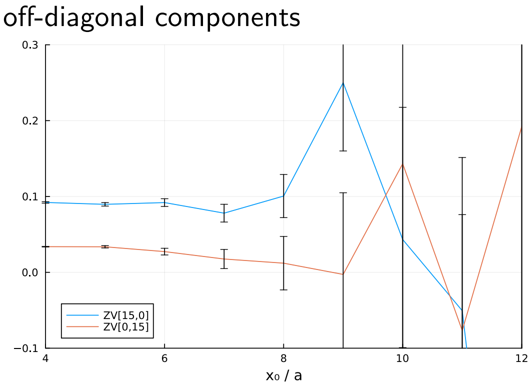
Plots


diagonal components
ZV[15,15]
ZV[0,0]
ZV[3,3]
ZV[8,8]
1.00
0.75
0.50
0.25
0.00
0
10
20
30
x₀ / a


off-diagonal components
ZV[15,0]
ZV[0,15]
0.2
0.1
0.0
−0.1
−0.2
0
10
20
30
x₀ / a

Plots


diagonal components
1.0
0.9
0.8
0.7
0.6
0.5
0.4
4
6
8
10
12
x0 / a
ZV[15,15]
ZV[0,0]
ZV[3,3]
ZV[8,8]


off-diagonal components
0.3
0.2
0.1
0.0
−0.1
4
6
8
10
12
x0 / a
ZV[15,0]
ZV[0,15]

Results

renormalization condition at $x_{\text{ren}} = 8a$

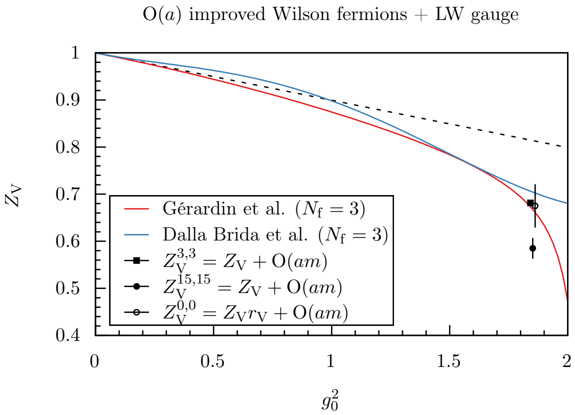
$$Z_V(g_0, aM) = \begin{pmatrix} 0.6813(7) & 0 & 0 & 0 \\ 0 & 0.6813(8) & 0 & 0 \\ 0 & 0 & 0.585(22) & 0.101(28) \\ 0 & 0 & 0.0012(35) & 0.675(46) \end{pmatrix}$$

renormalization condition at $x_{\text{ren}} = 10a$

$$Z_V(g_0, aM) = \begin{pmatrix} 0.6792(8) & 0 & 0 & 0 \\ 0 & 0.6792(8) & 0 & 0 \\ 0 & 0 & 0.65(16) & 0.043(17) \\ 0 & 0 & 0.14(24) & 0.55(27) \end{pmatrix}.$$

$a_\mu^{\text{HVP-LO}}$	u	d/s	c	disconnected	total
$x_{\text{ren}} = 8a$	121.7 ± 1.1	30.39 ± 0.36	5.01 ± 0.24	0.39 ± 0.69	187.9 ± 1.7
$x_{\text{ren}} = 10a$	121.3 ± 1.4	30.34 ± 0.54	4.4 ± 1.5	0.34 ± 0.62	186.7 ± 2.4

Perturbative QCD



[Dalla Brida et al. arXiV:2203.14754] [Gérardin, et al., arXiv:1811.08209]

Ward identity

$$\langle O \rangle := \int \mathcal{D}\Psi \mathcal{D}\overline{\Psi} \mathcal{D}U \, O e^{-S[\Psi, \overline{\Psi}, U]}$$

is invariant under flavor transformations $\Psi'(y) = e^{i\alpha(y)t^a}\Psi(y)$ with $t^a \in \mathfrak{u}(N_f)$.

$$\left\langle \frac{\partial O}{\partial \alpha(y)} \right\rangle = \left\langle \frac{\partial S}{\partial \alpha(y)} \, O \right\rangle$$

Ward identity

For the Wilson-Dirac action and $[M, t^a] = 0$

$$\sum_{\vec{y}} \left\langle \mathrm{tr}\left[(t^a)^T V_0^{(\mathrm{con})}(y) \right] O \right\rangle = \mathrm{const} \quad \text{for } y_0 \notin \mathrm{supp}(O).$$

Idea: Use Ward identities for conserved current $V_0^{(\mathrm{con})}$ and impose them to hold for $Z_V V_0^{(\mathrm{loc})}$. (for $N_f = 2 + 1$: [Gérardin, Harris, Meyer, arXiv:1811.08209])

Implementation

Consider for simplicity

- QCD Wilson action
- periodic boundary conditions
- $N_f = 3$ massless quarks
- non-singlet renormalization constant $Z_V(g_0)$

Three-point function

Use pseudoscalar operator $P(x) := \overline{u}(x)\gamma_5 d(x)$ in Ward identity:

$$\sum_{\vec{y}} \left\langle P^\dagger(x) V_0^{(\mathrm{con})}(y) P(z) \right\rangle = \left\langle P^\dagger(x) P(z) \right\rangle \quad \text{for } y_0 \in [z_0, x_0)$$

Renormalization condition:

$$Z_V(g_0) \sum_{\vec{y}} \left\langle \overline{P}(x) V_0^{(\mathrm{loc})}(y) P(z) \right\rangle \overset{!}{=} \left\langle P^\dagger(x) P(z) \right\rangle$$

+ related to conserved charge
+ improvement term does not contribute ⇒ $\mathrm{O}(a)$ improved
– not straightforward to generalize for $N_f = 3 + 1$
– elaborate calculations (baryon correlator required for singlet $r_V(g_0)$)

[Gérardin2018]

Two-point function

Susceptibility as renormalization condition:

$$\sum_{\vec{x}} \langle V_0^{(\text{con}),3}(x) V_0^{(\text{loc}),3}(0) \rangle \stackrel{!}{=} Z_V(g_0) \sum_{\vec{x}} \langle V_0^{(\text{loc}),3}(x) V_0^{(\text{loc}),3}(0) \rangle .$$

+ related to conserved charge
+ improvement term does not contribute $\Rightarrow \mathrm{O}(a)$ improved
+ can be generalized to $N_f = 3 + 1$
– susceptibility $\chi \sim T^2 = L_0^{-2}$ (in free theory) is small in the vacuum

One-point function

- finite temperature
- phase periodic boundary conditions (BC) : $\Psi(x + L_0\hat{0}) \sim -e^{i\theta_0}\Psi(x)$.
- $\langle V_0^{(\mathrm{con}),0}(x)\rangle$ relates to derivative of free energy density and is independent of x.

Renormalization condition:

$$\langle V_0^{(\mathrm{con}),0}\rangle \stackrel{!}{=} Z_V(g_0) r_V(g_0) \langle V_0^{(\mathrm{loc}),0}\rangle.$$

+ related to conserved charge
+ small error
– special boundary conditions/vanishes for periodic BC.

[Bresciani et al. (2022) arXiV:2203.14754]

Outlook

- many renormalization conditions
- mass-independent scheme not precise when charm is included.
- hard to find non-zero conserved charge with current lattices.

How to proceed?

1. brute-force increase statistics for disconnected contributions.
2. treat charm in quenched approximation [Mainz, arXiv:1904.03120]
3. generate dedicated lattices (finite temperature, phase-shifts)

Improvement terms

We can add operators with the same symmetry properties and *higher* mass dimension to obtain an improved operator $V^a_{\mu,\mathrm{I}}(x)$.

$$V^a_{\mu,\mathrm{I}}(x) = V^a_\mu(x) + ac_V\partial_\nu T^a_{\mu\nu}(x)$$

with $T_{\mu\nu}(x) = \frac{i}{2}\overline{\psi}(x)[\gamma_\mu, \gamma_\nu]t^a\psi(x)$. We then renormalize the improved operator

$$V^a_{\mu,\mathrm{R}}(x) = Z_V(g_0)V^a_{\mu,\mathrm{I}}(x).$$

[Bhattacharya2005] [Lüscher et al., arXiv:hep-lat/9605038]

Perturbative renormalization

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3$$

- Scalar ϕ^3-theory in $d = 6$.
- Dimensional regularization.
- Minimal subtraction: Put divergences in loop integral into renormalization constants.

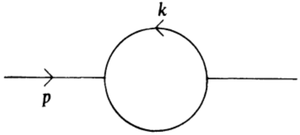


Figure: Loop correction to mass [Collins, Renormalization (1984)]

$$m_R = Z_m m = \left(1 + \frac{g^2}{64\pi^3}\frac{m^2}{d-6} + \mathcal{O}(g^4)\right) m.$$

Composite operators

In perturbative continuum field theory:

- After renormalizing fields, masses and couplings, there might still be divergent graphs for composite operators.
- Example: Scalar ϕ^3-theory in $d = 6$ with operator $\phi^2(z)$.

$$[\phi^2]_\mathrm{R} = Z_a[\phi^2] + Z_b m^2\phi + Z_c\partial^2\phi.$$

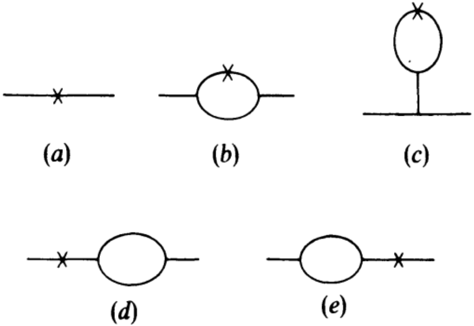


Figure: Lowest-order contributions to $\langle\phi(x)\phi(y)\phi^2(z)\rangle$. Cross denotes insertion of $[\phi^2]$.
[Collins1984 (Section 6.2)]