Galaxies meet QCD

Feynman Integrals with Differential Equations

Andrea Pelloni

23.02.2024

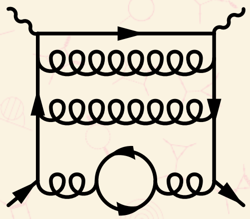
ETH zürich

PROFILE OF A COMPUTATION

Before anything let's start by defining what is the **problem** we are facing. We can split the whole computation in few key **steps** :

Generation → **Simplification** → **Computation** → **Assamble**

Generation



Define the problem:

- Model we want to study
- Level of precision (e.g. # of loops, # of final states)
- Use Feynman rules (UFO: link)

Generation of all diagrams:

- FeynArt (link)
- QGraf (link)
- MadGraph (link)
- many more...

Simplification

Cast into topologies:

- Contract external indices to obtain scalar integrals
- Tensor reduction with projectors
- Define container topologies of linearly idependent propagators

Reduce number of integrals:

- Collect by symmetry/algebra factors to resolve cancellation early on
- Use Integration by Part (**IBP**) identities
- Reduce to **master integrals** (e.g. FIRE, Kira, ...)

Computation

$$G(a_1,\dots,a_n;x) = \int_0^x \frac{\mathrm{d}t}{t-a_1} G(a_2,\dots,a_n;t)$$

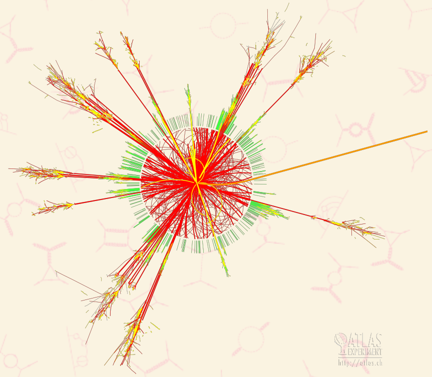
Look up for known masters:

- Loopedia

Compute the master integrals

- Create system of DEs
- Obtain expressions for boundary conditions
- Solve analytically/numerically

Assamble



Combine all the final expressions together in order to produce prediction for physical observables

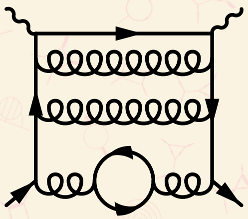
ETH zürich

D PHYS

Profile of a Computation

Before anything let's start by defining what is the **problem** we are facing. We can split the whole computation in few key **steps** :

Generation → **Simplification** → **Computation** → **Assamble**



Define the problem:

- Model we want to study
- Level of precision (e.g. # of loops, # of final states)
- Use Feynman rules (UFO: link)

Generation of all diagrams:

- FeynArt (link)
- QGraf (link)
- MadGraph (link)
- many more...

Cast into topologies:

- Contract external indices to obtain scalar integrals
- Tensor reduction with projectors
- Define container topologies of linearly idependent propagators

Reduce number of integrals:

- Collect by symmetry/algebra factors to resolve cancellation early on
- Use Integration by Part (**IBP**) identities
- Reduce to **master integrals** (e.g. FIRE, Kira, ...)

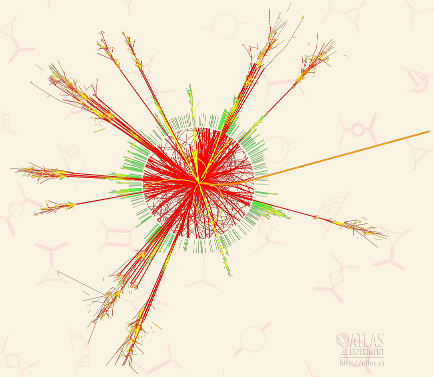
$$G(a_1,\dots,a_n;x) = \int_0^x \frac{dt}{t-a_1} G(a_2,\dots,a_n;t)$$

Look up for known masters:

- Loopedia

Compute the master integrals

- Create system of DEs
- Obtain expressions for boundary conditions
- Solve analytically/numerically



Combine all the final expressions together in order to produce prediction for physical observables

Defining a Topology

A general **Feynman integral** doesn't need to be scalar or have linearly independent propagators

$$\int \left[\prod_{i}^{\text{\# of loops}} \frac{\mathrm{d}^D k_i}{(2\pi)^D} \right] \frac{[\text{Color}]^{a,b} \cdot [\text{Spinor}]^{\mu\nu}}{\prod_{i}^{\text{\# of propagators}} \left(\left(\sum_{j}^{\text{\# of loops}} a_{ij} k_i + \sum_{j}^{\text{\# of externals}} b_{ij} p_i \right)^2 + m_j^2 \right)}$$

To set our machinery into motion we need **scalar integrals**

- Cross section with average/sum over external polarization will result in fully contracted indices
- We can isolate color factors outside of the integrand
- Open Lorentz indices can either be contracted with a **projection** or use **tensor reduction** to write the tensor structure using only loop-independent elements.

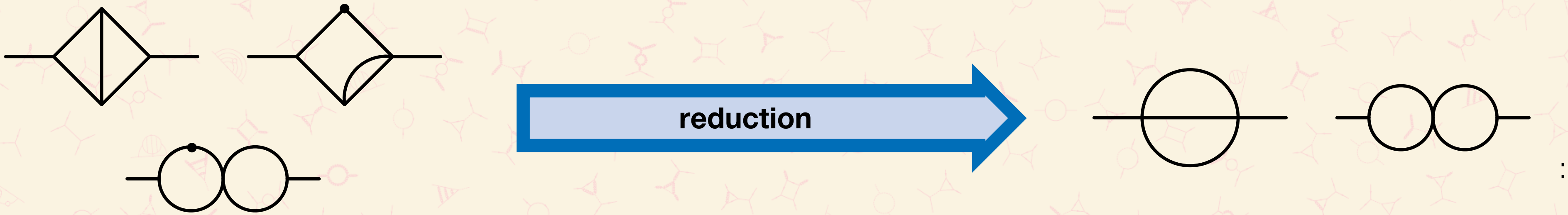
A **topology** is defined as

$$I_{\nu_1,\ldots,\nu_n} = \int \mathrm{d}^D k_i \frac{1}{D_1^{\nu_1} \ldots D_n^{\nu_n}}$$

- $\{D_i\}$ are a set of linearly independent propagators
- D_i can be any linear function of $k_i \cdot k_j$ and $k_i \cdot p_j$
- The topology is complete in the sense that any scalar product involving loop momenta can be written as

$$k_i \cdot v = \sum_i c_i D_i + (\text{ scalars })$$

REDUCTION TO MASTER INTEGRALS



- One of the standard tools to be used to resize the magnitude of the problem is to identify, by means of **Integration By Part** (IBP) identities, a set of Master integrals to span the space of the scalar integrals that appear in the computation:
- Within each **topology** we perform a reduction to **master integrals** :

$$\underbrace{I^{(n)}(s_{ij},\epsilon)}_{\text{Topology Integrals}} = \sum_i \underbrace{c_i(s_{ij},\epsilon)}_{\text{Coefficients}} \cdot \underbrace{M_i^{(n)}(s_{ij},\epsilon)}_{\text{Masters}}$$

  - c_i: Coefficient that depends on the external kinematics together with the dimensional regulator ϵ.

    Rational Functions

  - M_i: Master integrals (i.e. scalar integrals) that depend on the external kinematics together with the dimensional regulator.

    Special Functions (Multiple Polylogarithms, Elliptic Functions,...)

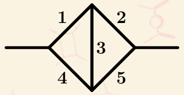
There are many publicly available programs that perform reductions to master integrals (AIR, FIRE, Reduze, Kira, LiteRed) each one with their strengths and weaknesses.

Reduction to Master Integrals: How?

The **reduction coefficients** are built through Integration By Part (IBP) relations to simplify the structure of the final integrals (lower powers)

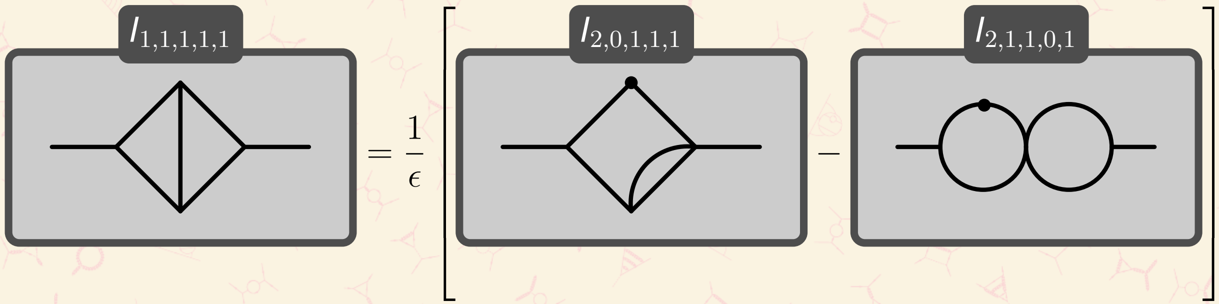
The set of **IBP identities** can be spawn from the condition that in dim-reg the integrand vanished at infinity:

$$0 = \int \left[\prod_i^L \frac{d^D k_i}{(2\pi)^D} \right] \frac{\partial}{\partial q^\mu} \left(v^\mu \left[\text{topology Integrand} \right] \right)$$

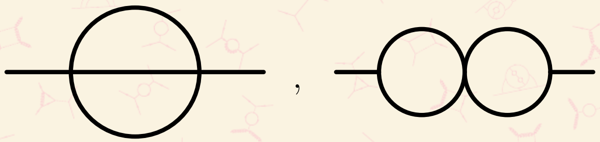
Example: Kite feynman integral 

$$D_1 = k_1^2, \qquad D_2 = k_2^2, \qquad D_3 = (k_1 + k_2)^2, \qquad D_4 = (k_1 - p)^2, \qquad , D_5 = (k_2 + p)^2$$

We can then write a relation with $q = k_1$ and $v = k_1 + k_2$



Within this topology, after combining all the IBP identities we find that we can write the integrals in terms of just two **master integrals** :



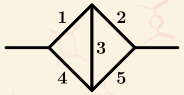
Note: The choice of a set of master integrals is not unique, the choice of one basis over another can greatly effect the computation

Reduction to Master Integrals: How?

The **reduction coefficients** are built through Integration By Part (IBP) relations to simplify the structure of the final integrals (lower powers)

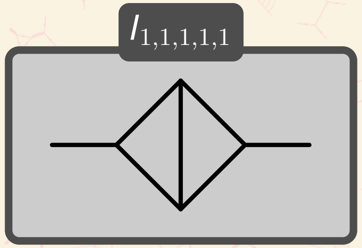
The set of **IBP identities** can be spawn from the condition that in dim-reg the integrand vanished at infinity:

$$0 = \int \left[\prod_i^L \frac{\mathrm{d}^D k_i}{(2\pi)^D}\right] \frac{\partial}{\partial q^\mu}\left(\frac{v^\mu}{\prod_j \mathrm{D}_j^{\alpha_j}}\right)$$

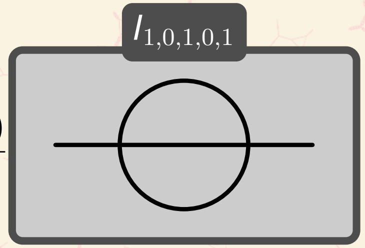
Example: Kite feynman integral 

$$D_1 = k_1^2, \qquad D_2 = k_2^2, \qquad D_3 = (k_1+k_2)^2, \qquad D_4 = (k_1-p)^2, \qquad , D_5 = (k_2+p)^2$$

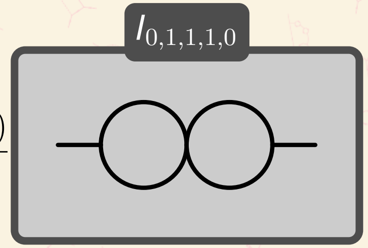
We can then write a relation with $q = k_1$ and $v = k_1 + k_2$



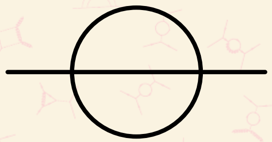
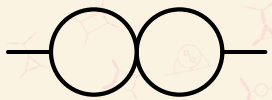
$$= \frac{2(3\epsilon-2)(3\epsilon-1)}{\epsilon^2 p^4}$$



$$- \frac{(2\epsilon-1)}{\epsilon p^2}$$



Within this topology, after combining all the IBP identities we find that we can write the integrals in terms of just two **master integrals** :

 , 

Note: The choice of a set of master integrals is not unique, the choice of one basis over another can greatly effect the computation

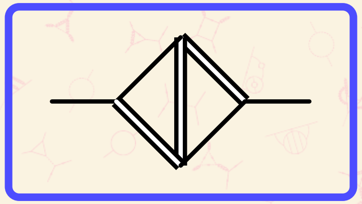
Reduction to Master Integrals: How?

A topology that will look more like the one in **EFTofLSS** is of the form:

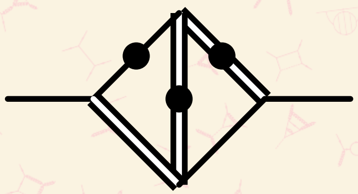
$$\int \mathrm{d}^D k_1 \mathrm{d}^D k_2 \frac{1}{D_1 D_2 D_3 D_4 D_5}$$

with

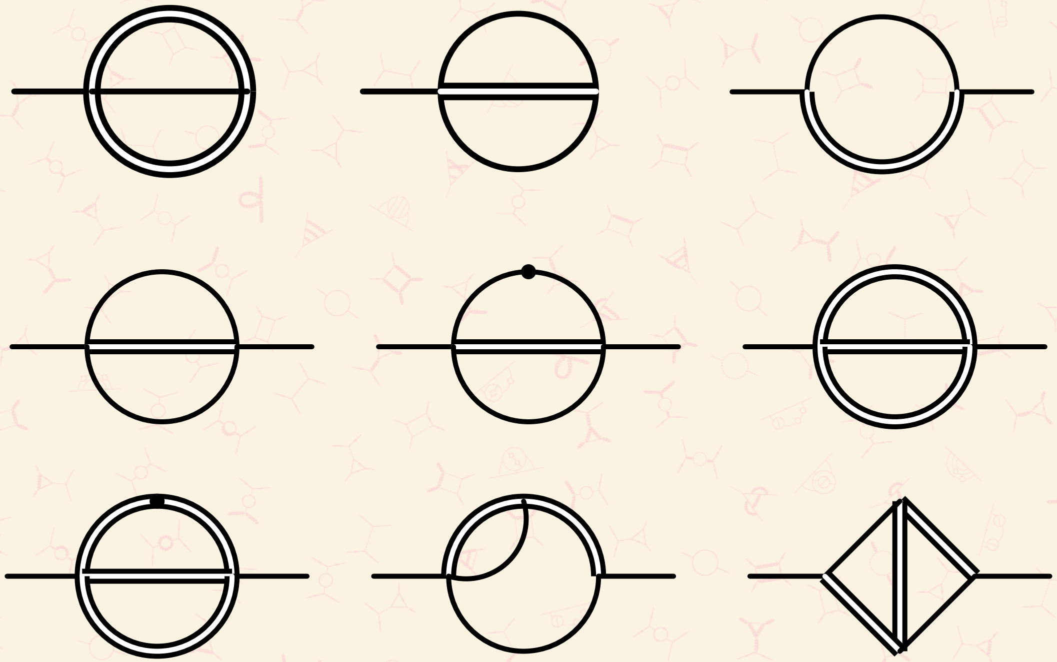
$$D_1 = k_1^2 + m_1^2, \qquad D_2 = k_2^2, \qquad D_3 = (k_1 + k_2)^2 + m_2^2, \qquad D_4 = (k_1 - p)^2, \qquad , D_5 = (k_2 + p)^2 + m_3^2$$



- LiteRed gives us a parametric reduction in 4h (single core)
- The master integrals look like (reduction using above rules in 10s)



$\sim$

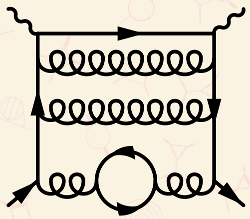


Profile of a Computation

Before anything let's start by defining what is the **problem** we are facing. We can split the whole computation in few key **steps** :

Generation → **Simplification** → **Computation** → **Assamble**

Generation



Define the problem:

- Model we want to study
- Level of precision (e.g. # of loops, # of final states)
- Use Feynman rules (UFO: link)

Generation of all diagrams:

- FeynArt (link)
- QGraf (link)
- MadGraph (link)
- many more...

Simplification

Cast into topologies:

- Contract external indices to obtain scalar integrals
- Tensor reduction with projectors
- Define container topologies of linearly idependent propagators

Reduce number of integrals:

- Collect by symmetry/algebra factors to resolve cancellation early on
- Use Integration by Part (**IBP**) identities
- Reduce to **master integrals** (e.g. FIRE, Kira, ...)

Computation

$$G(a_1, \dots, a_n; x) = \int_0^x \frac{\mathrm{d}t}{t - a_1} G(a_2, \dots, a_n; t)$$

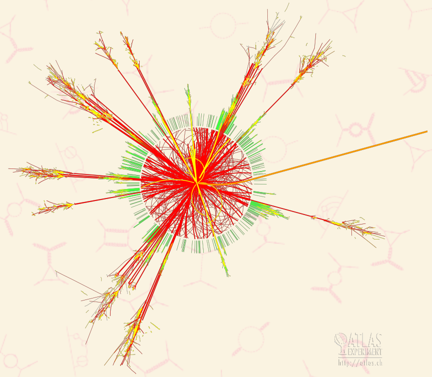
Look up for known masters:

- Loopedia

Compute the master integrals

- Create system of DEs
- Obtain expressions for boundary conditions
- Solve analytically/numerically

Assamble



Combine all the final expressions together in order to produce prediction for physical observables

Master Integrals: Computation

The **advantages** of the reduction to master integrals is first of all

- Order of magnitude fewer integrals to compute
- Allow to relate any integrals within the same topology and across computations
- The finite size of the basis allows to build a system of differential equations (see next slide)

Rich array of **techniques** related to them:

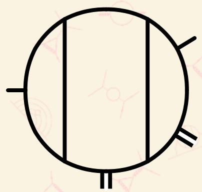
- Syzygys and Groebner basis for doped reduction
- Description of the solution in terms of a known basis of integrals (e.g. pentagon functions for 2-loop 1-mass expressions)
- **Differential equations**
- Deep learning to build a function that interpolates the solution (2312.02067)

The master integrals can also be computed **numerically**, for example by using

- Sector decomposition: (py)SecDec, FIESTA
- Two-point massless integrals (up to 4 loops): FORCER
- Direct numerical integration: LTD
- Tropical sampling: Feyntrop
- ...

More about numerical methods for loop integrals this afternoon (Valentin Hirshi)

System of Differential Equations



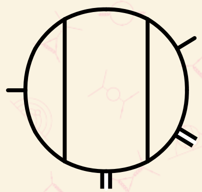
Differential Eqs $\Rightarrow$ $G(a_1, \ldots, a_n; x)$

In order to find the expressions that describe our **master integrals**, and by extension all of our integrals, we can set up a **system of differential equations** (DE).

- The master integrals allow to construct a **closed system** of differential equations:

$$\frac{\partial}{\partial t} \boxed{\vec{M}(t,\epsilon)} = F(t,\epsilon)$$

System of Differential Equations



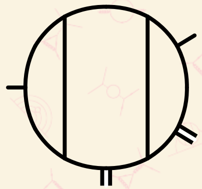
Differential Eqs $\Rightarrow$ $G(a_1, \ldots, a_n; x)$

In order to find the expressions that describe our **master integrals**, and by extension all of our integrals, we can set up a **system of differential equations** (DE).

- The master integrals allow to construct a **closed system** of differential equations:

$$\frac{\partial}{\partial t} \boxed{\vec{M}(t, \epsilon)} = \overset{\textbf{Reduce}}{\boxed{F(t, \epsilon)}}$$

System of Differential Equations



Differential Eqs $\longrightarrow$ $G(a_1, \ldots, a_n; x)$

In order to find the expressions that describe our **master integrals**, and by extension all of our integrals, we can set up a **system of differential equations** (DE).

- The master integrals allow to construct a **closed system** of differential equations:

$$\frac{\partial}{\partial t}\boxed{\vec{M}(t,\epsilon)} = \overset{\textbf{Differential Matrix}}{\boxed{A(t,\epsilon)}} \cdot \boxed{\vec{M}(t,\epsilon)}$$

  - The differentiation variable t can be any lorentz invariant of the system or a combination of them (e.g. $\gamma(t) = \{s_{11}(t), s_{12}(t), \ldots\}$)
  - We can have as many DEs as we have independent variables
  - The differential matrix $A(x,\epsilon)$ has rational function coefficients in the kinematics, the same as the reduction coefficients

It is possible to modify the form of the **Differential Matrix** by a change of **basis** with a liner transformation $\mathbf{T}(t,\epsilon)$:

$$\vec{M}(t,\epsilon) \longrightarrow \mathbf{T}(t,\epsilon)\cdot\vec{M}(t,\epsilon)$$

$$A(t,\epsilon) \longrightarrow \partial_x\mathbf{T}(t,\epsilon)\cdot\mathbf{T}^{-1}(t,\epsilon) + \mathbf{T}(x,\epsilon)\cdot A(t,\epsilon)\cdot\mathbf{T}^{-1}(t,\epsilon)$$

Canonical Basis

The differential equation we are studing can be written in the following form (see Moser (1959) and Barkatou (1995))

$$\boxed{\frac{\partial}{\partial t} M(t,\epsilon) = \left[\sum_k \frac{A_k(\epsilon)}{\alpha_k}\right] \cdot M(t,\epsilon)}, \qquad \{\alpha_k\} = \{t - a_k\} \text{ is the alphabet of the problem}$$

This expression still has rational function coefficients but it also has a **non-trivial** dependence in the dimensional regulator ϵ This is what is called a **Fuchsian form** , and there are public tools that implement algorithmic ways to obtain it:

- **Epsilon** [1701.00725], **Fuchsia** [1701.04269], **Libra** [2012.00279]

The **holy grail** is the canonical form, where a basis is found such that the ϵ dependence is completely factored out

Canonical Form

$$\boxed{\mathrm{d}M(t,\epsilon) = \epsilon\, \mathrm{d}\tilde{A} \cdot M(t,\epsilon)}, \qquad \tilde{A} = \sum_k A_k \log \alpha_k(t)$$

Then the solution can be written in terms of **Chen's iterated** integrals

$$M(t,\epsilon) = \mathbb{P}\exp\left[\epsilon \int_\gamma \mathrm{d}A\right] M_0$$

M_0 is the boundary term at $\gamma(t_0)$.

When the coefficients of the differential matrix are rational functions we then we are looking at **multiple polylogarithms** (MPL)

$$G(a_1,\ldots,a_n;x) = \int_0^x \frac{\mathrm{d}t_1}{t_1 - a_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{t_2 - a_2} \cdots \int_0^{t_{n-1}} \frac{\mathrm{d}t_n}{t_n - a_n}$$

Well studied family of functions (see [1411.7538]), with efficient numerical evaluations (e.g. GiNaC)

Canonical Basis

Let's have a closer look at the general solution for

Canonical Form

$$\boxed{\mathrm{d}M(t,\epsilon) = \epsilon \, \mathrm{d}\tilde{A} \cdot M(t,\epsilon)}, \qquad \tilde{A} = \sum_k A_k \log \alpha_k(t)$$

We can see that the **depth** of the iteration (**trascendentality**) is connected to the power in ϵ

$$M(t,\epsilon) = const. + \epsilon \int_0^t \mathrm{d}t_1 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_1} + \epsilon^2 \int_0^t \mathrm{d}t_1 \frac{\mathrm{d}A}{\mathrm{d}t_1} \int_0^{t_1} \mathrm{d}t_2 \frac{\mathrm{d}A}{\mathrm{d}t_2} + \ldots$$

Some example of trasendental weights:

- trascendental weight **-1**: ϵ
- trascendental weight **1**: π, log
- trascendental weight **n**: $G(a_1,\ldots,a_n;x)$

$$G(a_1,\ldots,a_n;x) = \int_0^x \frac{\mathrm{d}t_1}{t_1-a_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{t_2-a_2} \cdots \int_0^{t_{n-1}} \frac{\mathrm{d}t_n}{t_n-a_n}$$

There are ways to find a suitable basis of integrals by looking for the property above:

- Unitarity cuts will fulfill the same canonical condition
- Use generalized cuts to find integrals of uniform trascendentality for our basis

Example: massive bubble

Topology Propagators: $D_1 = k^2 - m^2, \qquad D_2 = (k+p)^2 - m^2$

The topology is already **complete**, all scalar product with internal momenta can be expressed in terms of D_i's:

$$k^2 = D_1 + m^2, \qquad 2k\cdot p = D_2 - D_1 - p^2 + m^2$$

STEP 1 : $$\frac{\partial}{\partial x}\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{\epsilon}{4-x} & \frac{2+\epsilon x}{x(4-x)} \end{pmatrix} \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \qquad x = \frac{s}{m^2}$$

The two **1**master integrals are: $M_1 = I_{2,0}$ and $M_2 = I_{2,1}$.

- We can remove the ϵ^0 coefficient from the differential matrix with a change of basis

STEP 2 : $$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \to \begin{bmatrix} M_1 \\ \sqrt{1-\frac{4}{x}}M_2 \end{bmatrix} = \begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix}$$ the new DE $$\frac{\partial}{\partial x}\begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix} = \epsilon \begin{pmatrix} 0 & 0 \\ \frac{-1}{\sqrt{x(4-x)}} & \frac{1}{4-x} \end{pmatrix} \cdot \begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix}$$

- **Square roots** appearing in the matrix. This could mean the end of hoping to describe the master integrals in terms of MPL
- In some cases we can remove square roots by **rationalize** them

STEP 3 : $$x = \frac{(y+1)^2}{y}$$ the new DE w.r.t. the variable y reads $$\frac{\partial}{\partial y}\begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix} = \epsilon \begin{pmatrix} 0 & 0 \\ \frac{-1}{y} & \frac{1}{y} + \frac{2}{1-y} \end{pmatrix} \cdot \begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix}$$

The system can be solved as a series in ϵ in terms of MPLs

Also once we manage to find the canonical basis it could be that the system has some irreducible square roots:

- We end with special function such as Elliptic function (talk by Sebastian Pögel)

Expansion

The construction of a **canonical basis** is not always simple, and can become one very demanding task. We may simplify the problem with other **approximations** like expanding in one of the variables

- For example looking at the **threshold exapansion** for the Higgs cross-section (inclusive and differential)

The master formula for the expansion is given by

$$M(z,\ldots;\epsilon) = \sum_{a,n} z^{n-a\epsilon}\, m_{s,n}(z,\ldots;\epsilon)$$

- **n**: index of the Laurent series with bounded lower index
- **s**: sector of the expansion. Do not mix under DEs!

Only simple poles in z

$$A = \frac{A_{-1}}{\omega} + \sum_{k=0}^{\infty} A_k\, z^k \qquad \vec{M} = \sum_{k=0}^{\infty} \vec{m}_k\, z^k$$

$$\boxed{\underbrace{((k+1)\mathbb{1} - A_{-1})}_{:=B_k} \cdot \vec{m}_{k+1} = \sum_{j=0}^{k} A_j\, \vec{m}_{k-j}}$$

$\det(B_k) \neq 0$

Recursive Expression:

$$\vec{m}_{k+1} = B_k^{-1} \cdot \left(\sum_{j=0}^{k} A_j\, \vec{m}_{k-j}\right)$$

$\det(B_k) = 0$

Gaussian Elimination:

Required by a **finite number** of k

Expansion

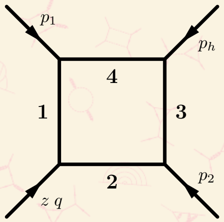
The master formula for the expansion is given by

$$M(z,\dots;\epsilon) = \sum_{a,n} z^{n-a\epsilon}\, m_{s,n}(z,\dots;\epsilon)$$

- **n**: index of the Laurent series with bounded lower index
- **s**: sector of the expansion. Do not mix under DEs!

The starting point of the series expansion and the sectors are fully fixed by the boundary conditions.

Example:



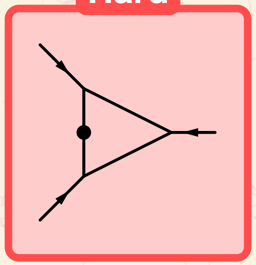
$$\begin{aligned} D_1 &= k^2 & p_1^2 &= 0 \\ D_2 &= (k+zq)^2 & p_2^2 &= 0 \\ D_3 &= (k+zq+p_2)^2 & q &= 0 \\ D_4 &= (k-p_1)^2 & p_h^2 &= m_h^2 \end{aligned}$$

Naive expansion only converges for large values of the loop momentum (Hard sector):

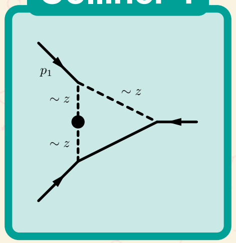
$$\frac{1}{D_2} = \frac{1}{(k+z\,2\,k\cdot q)^2} = \frac{1}{k^2}\sum_{n=0}^{\infty}\left(\frac{z\,2\,k\cdot q}{k^2}\right)^n$$

By Looking at the asymptotic behaviour around $z=0$ () we can see that there are three independent sectors parametrized by z

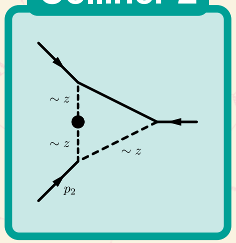
Hard



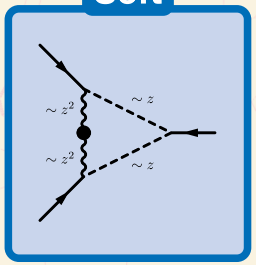
Colliner 1



Colliner 2



Soft



large loop momentum:
$k \gg z$

$$k \sim 1$$

loop momentum colliner to $p_{1,2}$:

$$k^2 \sim k\cdot p_{1,2} \sim z, \qquad k\cdot p_{2,1} \sim 1$$

phase space: $\mathrm{d}^D\Phi \sim z^{D/2}$

loop momentum soft:

$$k \sim z$$

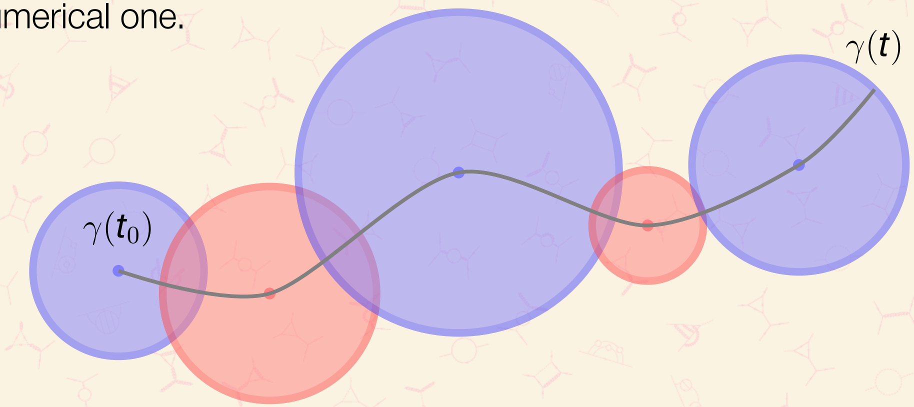
phase space: $\mathrm{d}^D\Phi \sim z^D$

SEMI-NUMERICAL

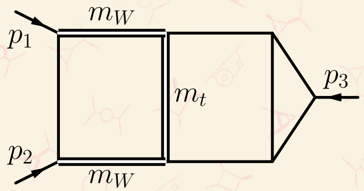
An alternative approach to the a fully analytic computation is a semi-numerical one.

There are several programs publicly available that provide provide a numerical result for the master integrals by using the DEs to evolve the solution from some boundary point $\gamma(t_0)$, to another region at $\gamma(t)$

- DiffExp [Hidding 2020],
- SeaSyde [Armadillo et al. 2022],
- AMFlow [Ma, Liu 2022]



Example:



compute in **euclidean region** at $(s, m_W, m_t) = (-2, 4, 16)$ and evolve to **physical** phase space point such as $(s, m_W, m_t) = (1, \frac{401925}{455938}, \frac{433000}{227969})$

The computation of the boundary condition in the euclidean region can be performed with

- (py)SecDec
- FIESTA
- Feyntrop

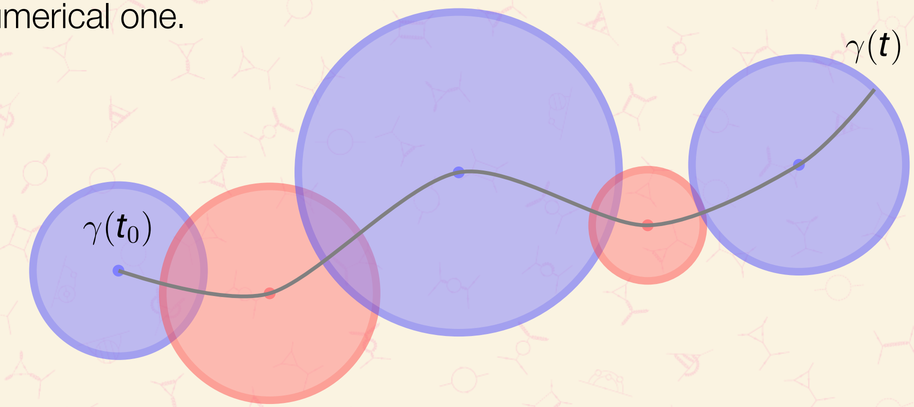
The evolution to the physical region is performed with high precision so the main source of uncertainty in many cases comes from the computed boundary terms.

SEMI-NUMERICAL

An alternative approach to the a fully analytic computation is a semi-numerical one.

There are several programs publicly available that provide provide a numerical result for the master integrals by using the DEs to evolve the solution from some boundary point $\gamma(t_0)$, to another region at $\gamma(t)$

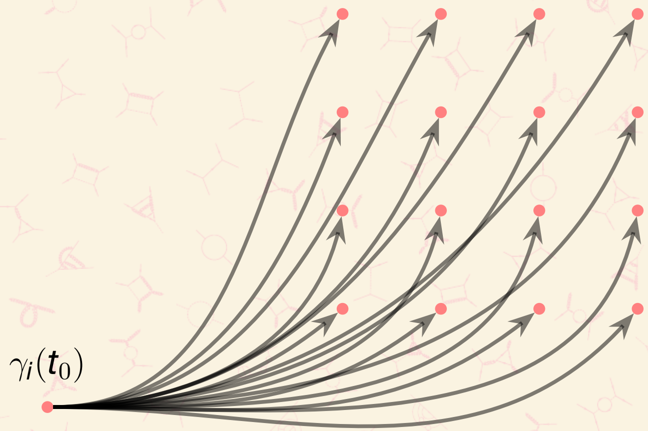
- DiffExp [Hidding 2020],
- SeaSyde [Armadillo et al. 2022],
- AMFlow [Ma, Liu 2022]



This hybrid approach allows to tackle problems that would be otherwise prohibitive, taking advantage of the knowledge of DEs

Key aspects :

- No need for casting the system in canonical form
- Agnostic on the type and complexity of the expressions in the solution.
- Elliptic functions can be computed by looking at a pre-canonical form with rational coefficients in the DE
- Expansion in the regulator ϵ to allow for possible cancellations once recombined in the original expression
- Possible to create interpolation grids if necessary for the computation



Conclusions

Differential equations are a powerful tool for the compuation of master integrals and a well established method in QCD

- Use IBP identities for a **reduction to master integrals** and build a system of differential equations
- Seek for a **canonical basis** with integrals ofhomogeneous transcendental weight
- **MPL** may not be sufficient for the problem at hand and require special functions
- Rich array of dedicated tools

The integrals appearing in **EFTofLSS** have euclidean metric and $D = 3$

- The DEs and the reductions are written in terms of scalars and independent of the type of matrix used
- Usually working in **dim-reg** having D as a variable