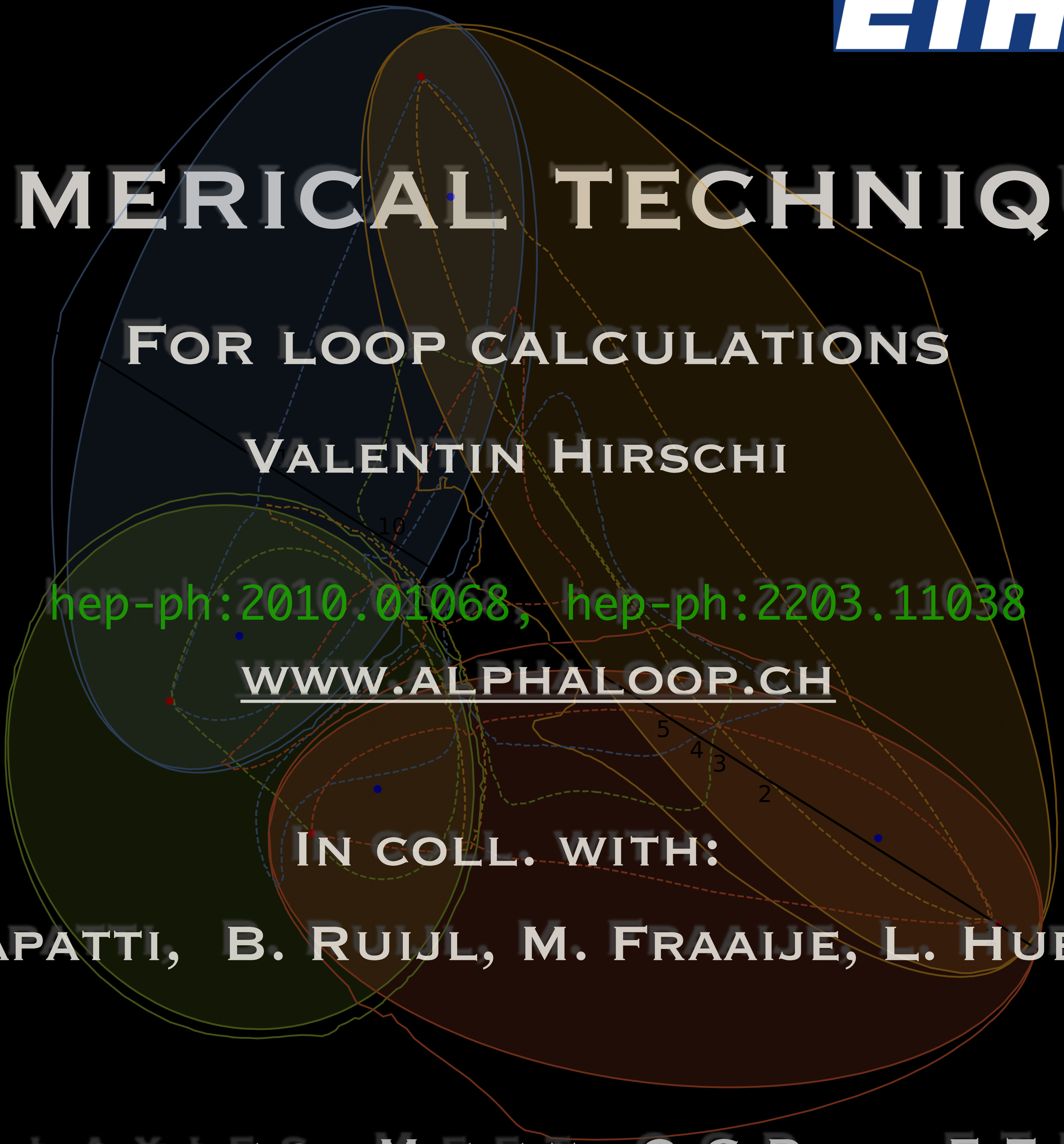


NUMERICAL TECHNIQUES

FOR LOOP CALCULATIONS

VALENTIN HIRSCHI

hep-ph:2010.01068, hep-ph:2203.11038

WWW.ALPHALOOP.CH

IN COLL. WITH:

Z. CAPATTI, B. RUIJL, M. FRAAIJE, L. HUBER

GALAXIES MEET QCD - ETHZ

FEBRUARY 23, 2024

{ ANARICAL }

$$\log(2)$$

EXPRESSION

$$\int_1^2 dx \frac{1}{x}$$

{ ANARICAL }

$$\log(2)$$

EXPRESSION

$$\int_1^2 dx \frac{1}{x}$$

ANALYTICAL ?

LAY MAN CLASSIFICATION

NUMERICAL ?

{ ANARICAL }

$$\log(2)$$

EXPRESSION

$$\int_1^2 dx \frac{1}{x}$$

ANALYTICAL ?

LAY MAN CLASSIFICATION

NUMERICAL ?

$$\sim \sum_{i=1}^{N} \frac{(1-x)^i}{-i}\Big|_{x=2}$$

IMPLEMENTATION

$$\sim \frac{1}{N}\sum^{N} \frac{1}{(1+\mathrm{rdm}())}$$

{ ANARICAL }

$\log(2)$	Expression	$\int_1^2 dx \frac{1}{x}$
Analytical ?	Lay man classification	**Numerical ?**
$\sim \sum_{i=1}^{N} \frac{(1-x)^i}{-i}\Big\vert_{x=2}$	Implementation	$\sim \frac{1}{N}\sum^{N}\frac{1}{(1+\mathrm{rdm}())}$
$t \propto D\log(D)^2$	Complexity for “D” accurate digits	$t \propto D^2$

WHERE DOES MY RESEARCH COME FROM ?

A PALE BLUE DOT.

ONE-DIMENSIONAL TOY EXAMPLE

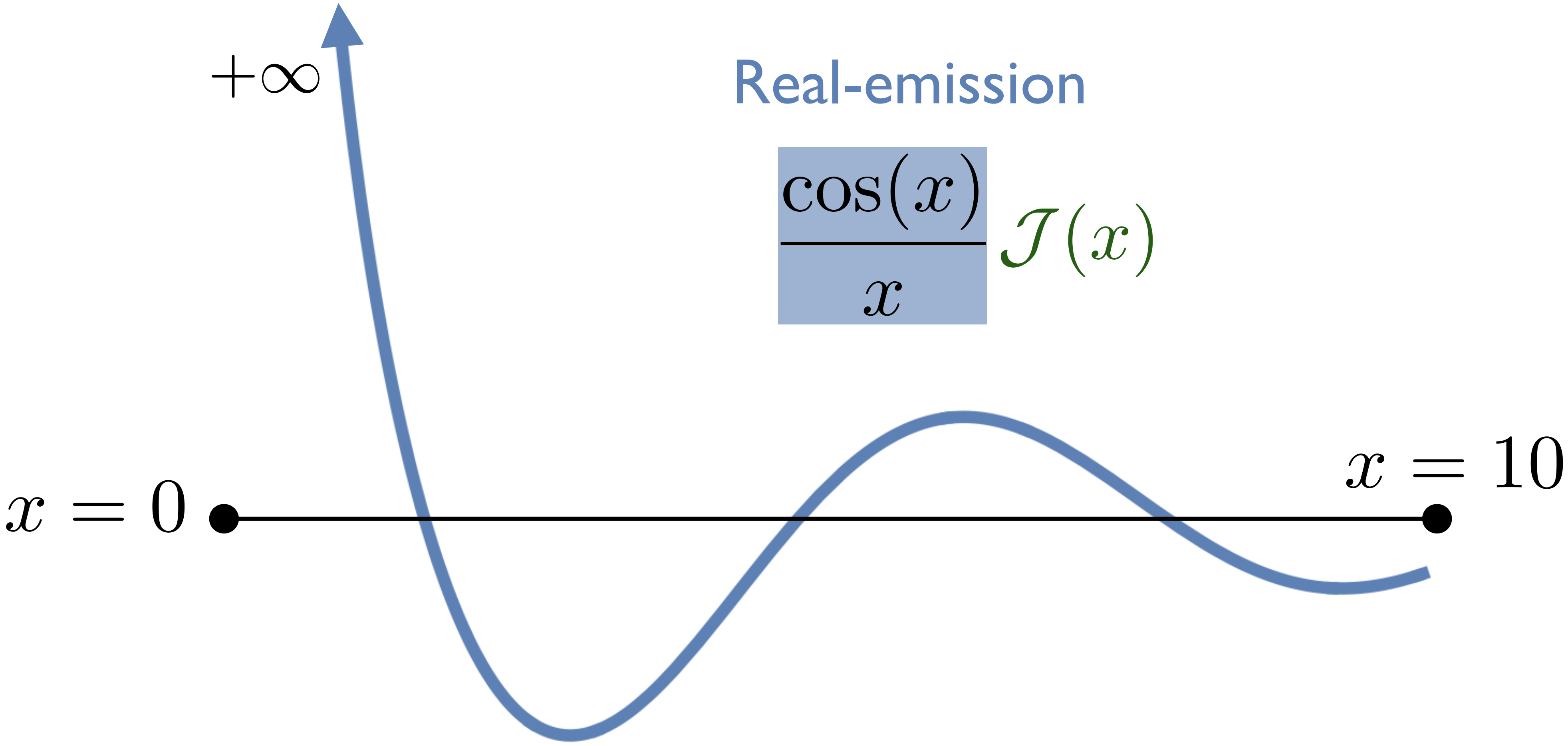
- Toy expression with $\mathcal{J}$ a measurement function, over $x \in [0, 10]$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \qquad\qquad\qquad \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$

ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with $\mathcal{J}$ a measurement function, over $x \in [0, 10]$

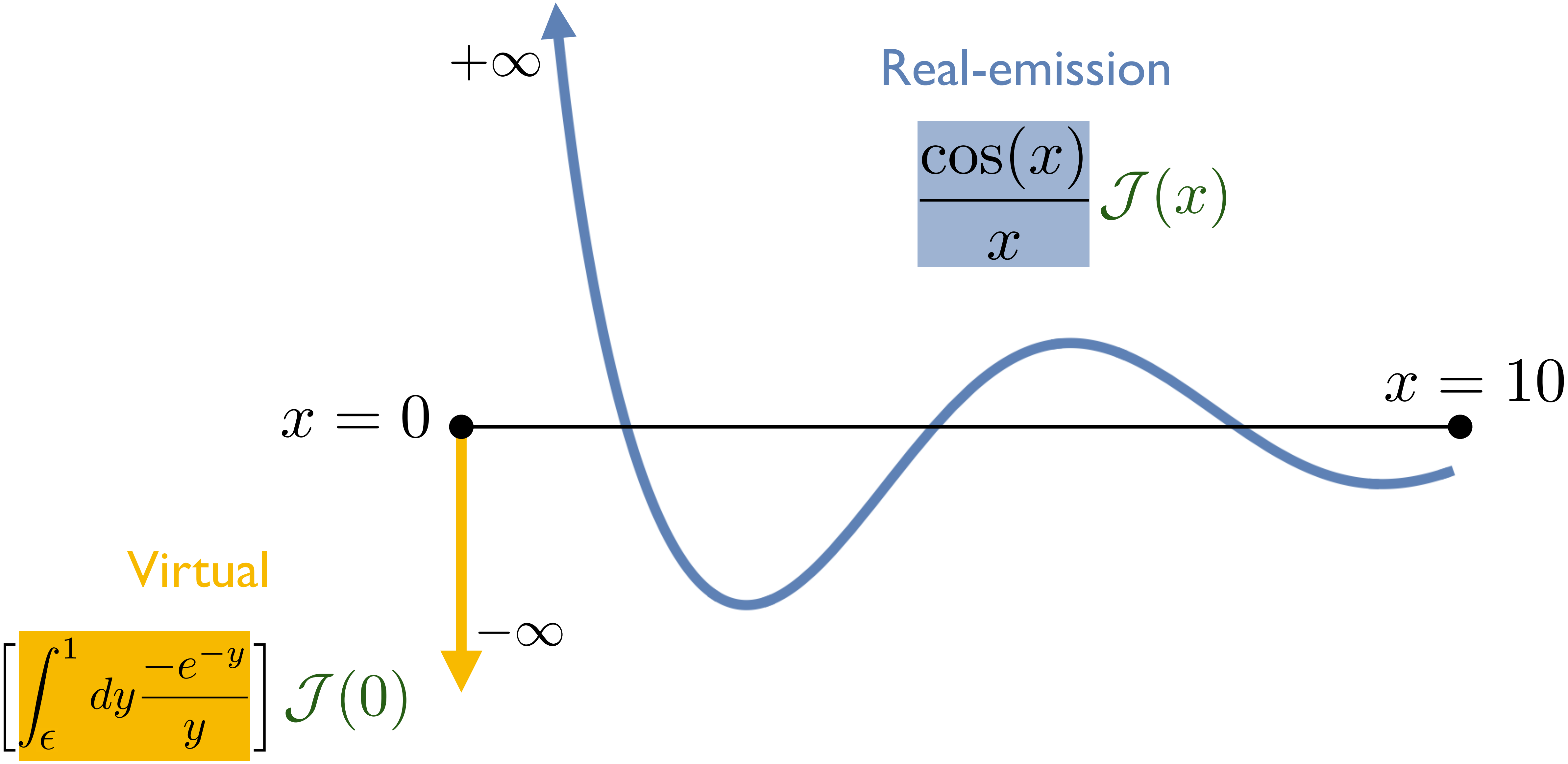
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \qquad\qquad \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with $\mathcal{J}$ a measurement function, over $x \in [0,10]$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \qquad\qquad \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



$+\infty$

Real-emission

$$\frac{\cos(x)}{x} \mathcal{J}(x)$$

$x = 0$

$x = 10$

Virtual

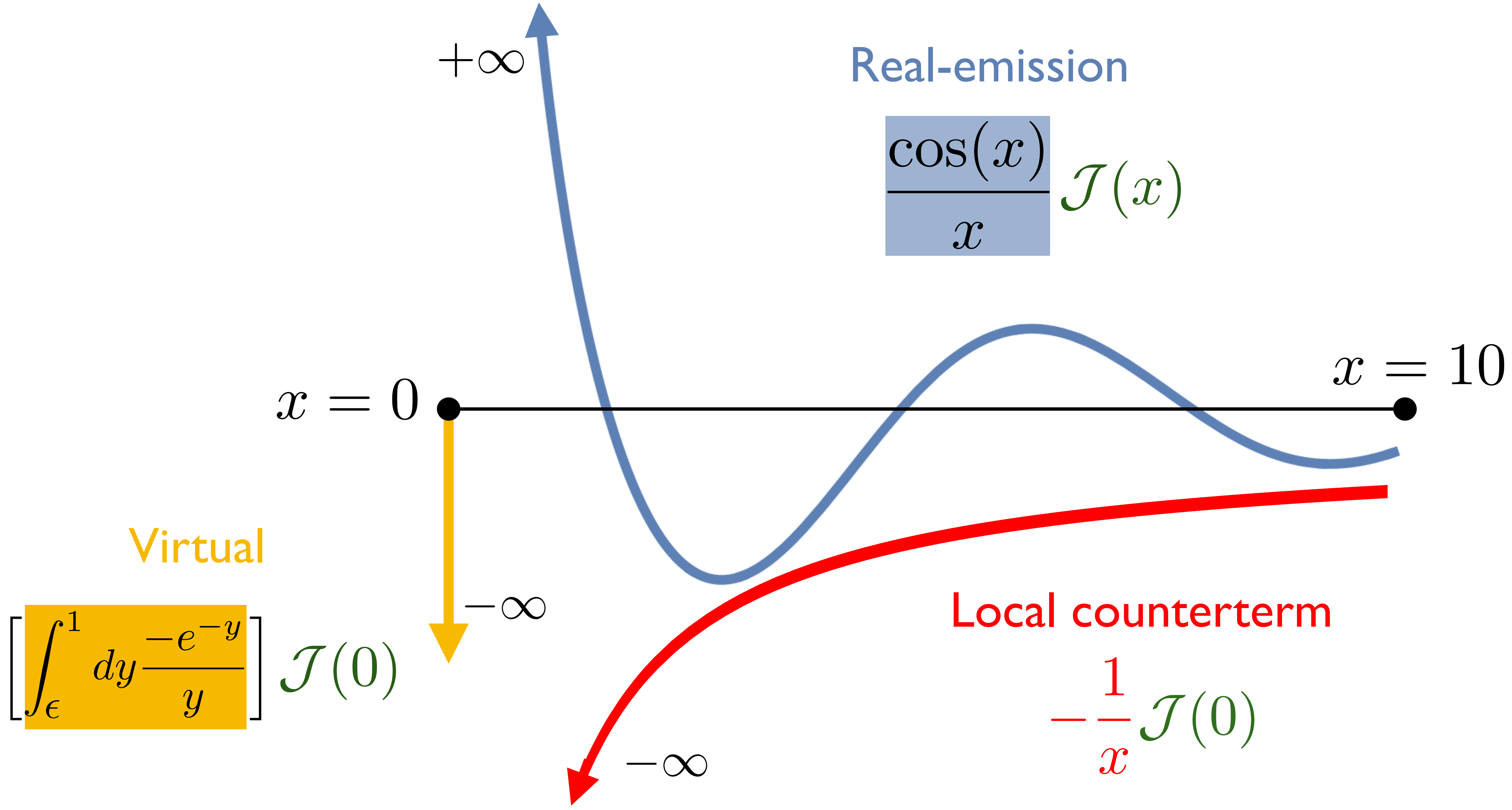
$$\left[\int_\epsilon^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$

$-\infty$

ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with $\mathcal{J}$ a measurement function, over $x \in [0, 10]$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left[\int_0^1 dy \frac{-e^{-y}}{y}\right]\mathcal{J}(0)$$



$+\infty$

Real-emission

$$\frac{\cos(x)}{x}\mathcal{J}(x)$$

$x = 0$

$x = 10$

Virtual

$$\left[\int_\epsilon^1 dy \frac{-e^{-y}}{y}\right]\mathcal{J}(0)$$

$-\infty$

Local counterterm

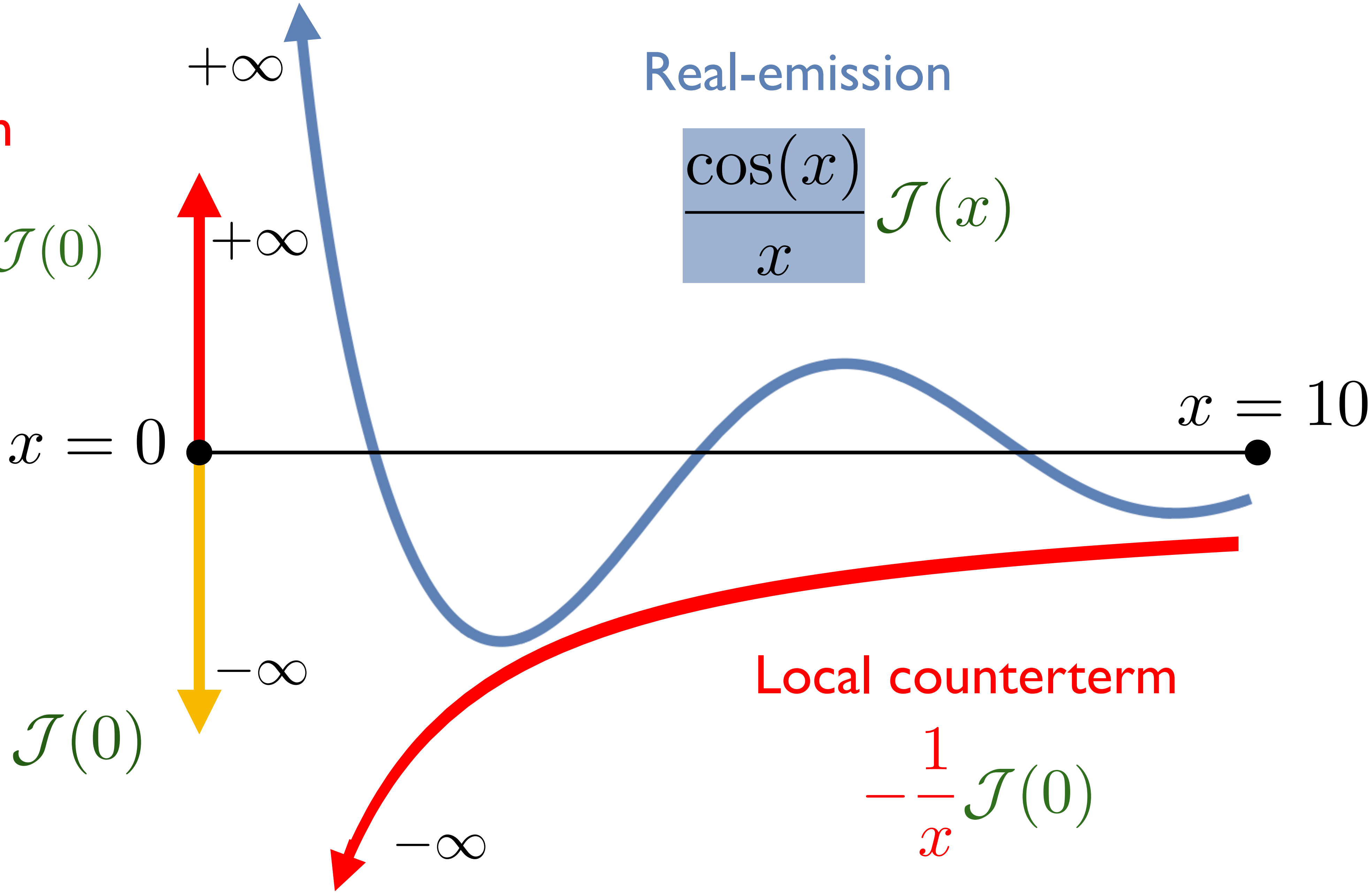
$$-\frac{1}{x}\mathcal{J}(0)$$

$-\infty$

ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with $\mathcal{J}$ a measurement function, over $x \in [0, 10]$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left[\int_0^1 dy \frac{-e^{-y}}{y}\right]\mathcal{J}(0) + \left[\int_0^{10} dx \frac{1}{x}\right]\mathcal{J}(0)$$



Integrated counterterm

$$+\left[\int_\epsilon^{10} dx \frac{1}{x}\right]\mathcal{J}(0)$$

Real-emission

$$\frac{\cos(x)}{x}\mathcal{J}(x)$$

Virtual

$$\left[\int_\epsilon^1 dy \frac{-e^{-y}}{y}\right]\mathcal{J}(0)$$

Local counterterm

$$-\frac{1}{x}\mathcal{J}(0)$$

ONE-DIMENSIONAL TOY EXAMPLE

- Local unitarity would align the measure between real and virtual

$$\left[\int_0^1 dy \frac{-e^{-y}}{y}\right]$$

$$\sigma^{(R+V)}\left(\mathcal{J}\right) = \int_0^{10} dx \left[\frac{\cos\left(x\right)}{x}\mathcal{J}\left(x\right) \qquad\qquad\qquad \right]$$

ONE-DIMENSIONAL TOY EXAMPLE

- Local unitarity would align the measure between real and virtual

$$\left[\int_0^1 dy \frac{-e^{-y}}{y}\right]$$

“ LU ”

$$y = x/10$$

$$\left[\int_0^{10} dx \frac{-e^{-x/10}}{x}\right]$$

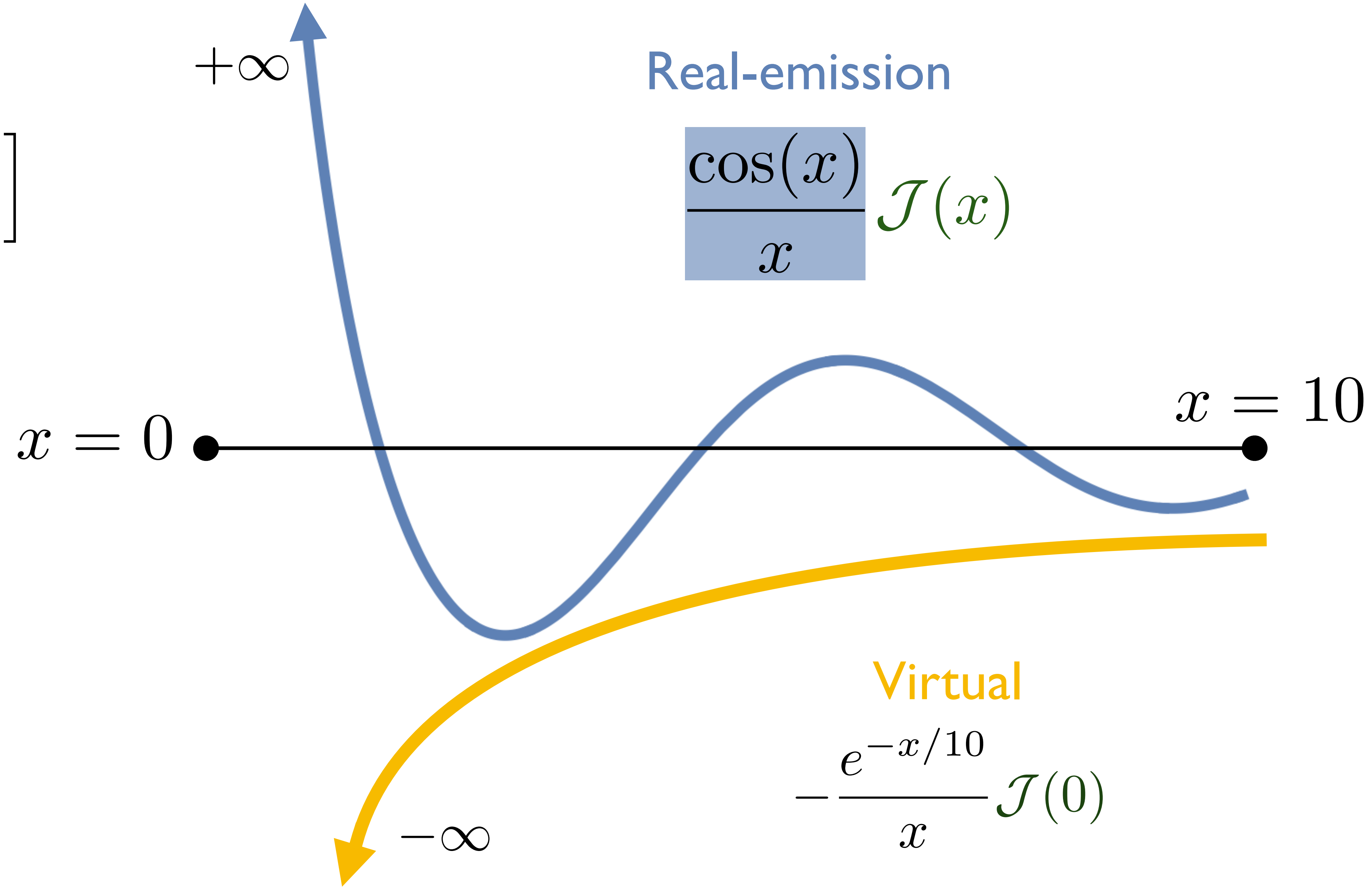
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{e^{-x/10}}{x}\mathcal{J}(0)\right]$$

ONE-DIMENSIONAL TOY EXAMPLE

- Local unitarity would align the measure between real and virtual

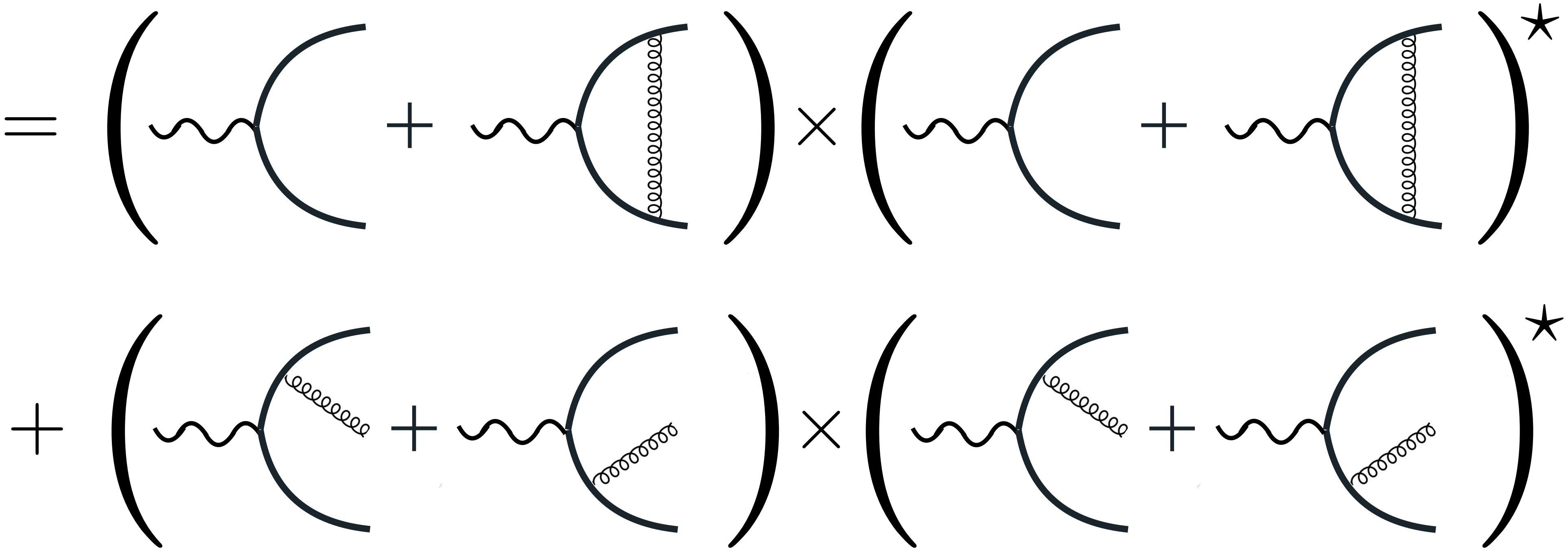
$$\begin{tikzcd} \left[\int_0^1 dy \frac{-e^{-y}}{y}\right] \arrow[d, "\text{`` LU ''}", "y = x/10"'] \\ \left[\int_0^{10} dx \frac{-e^{-x/10}}{x}\right] \end{tikzcd}$$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{e^{-x/10}}{x} \mathcal{J}(0) \right]$$



REMEDY: FORWARD-SCATTERING GRAPHS

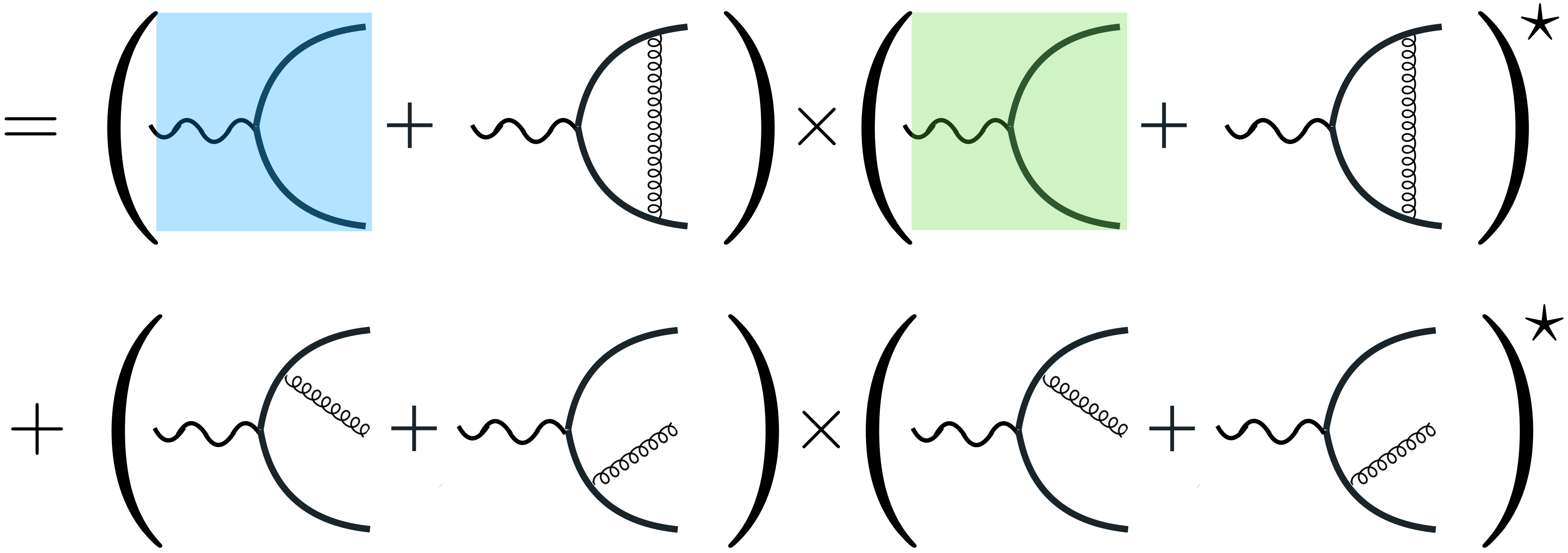
$$\sigma_{\gamma^\star \to d\bar{d}} =$$



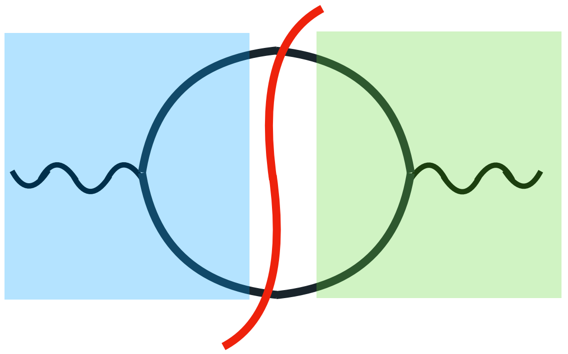
$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$$

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^\star \to d\bar{d}} =$$

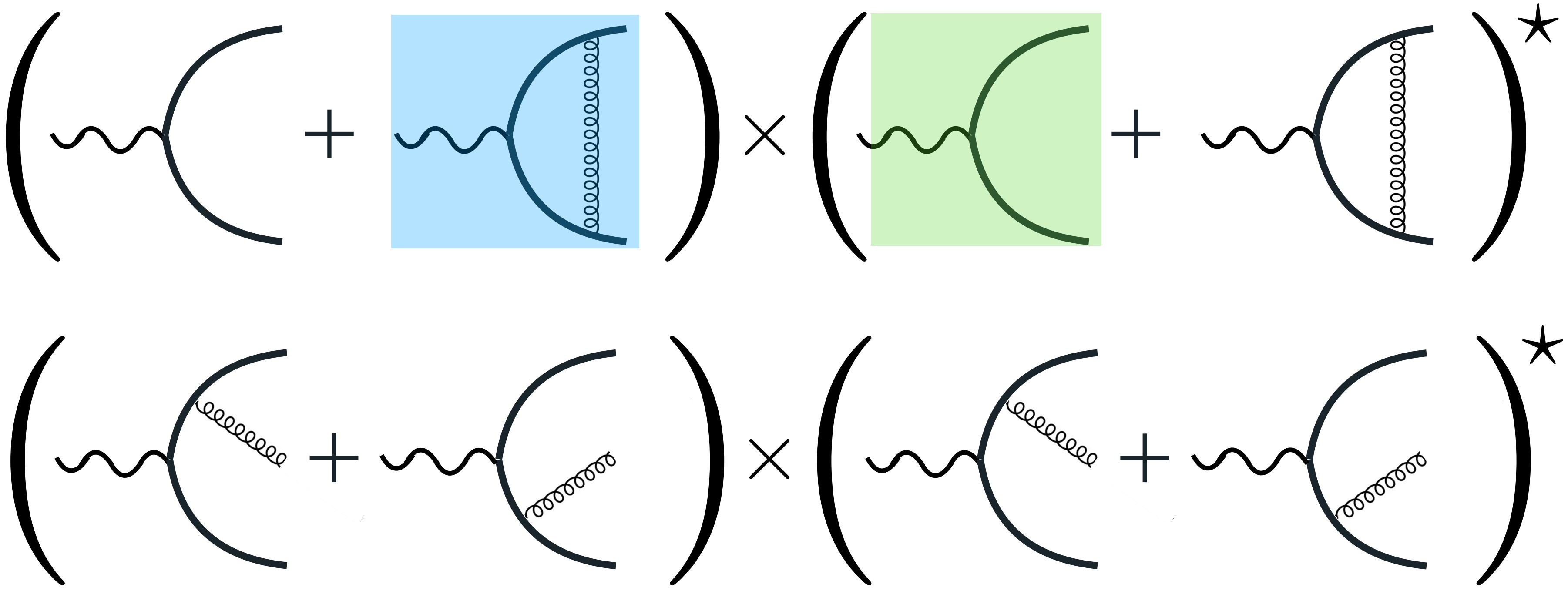


$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$$

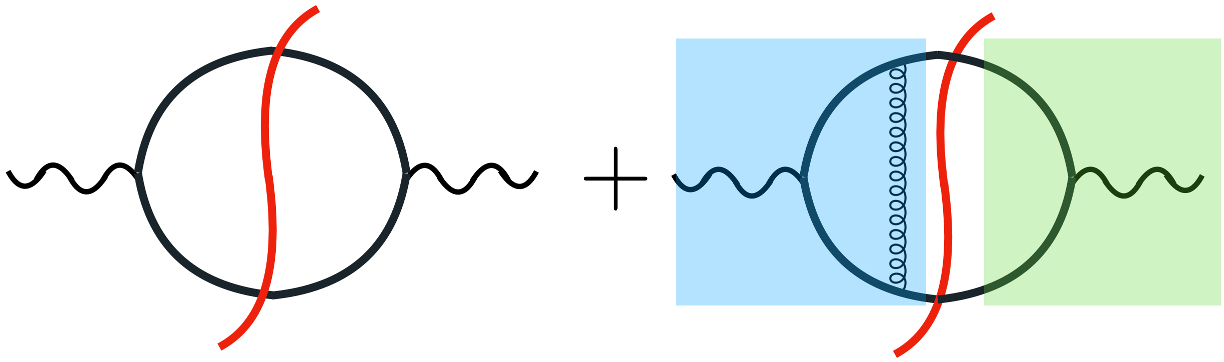


REMEDY: FORWARD-SCATTERING GRAPHS

$\sigma_{\gamma^\star \to d\bar{d}} =$



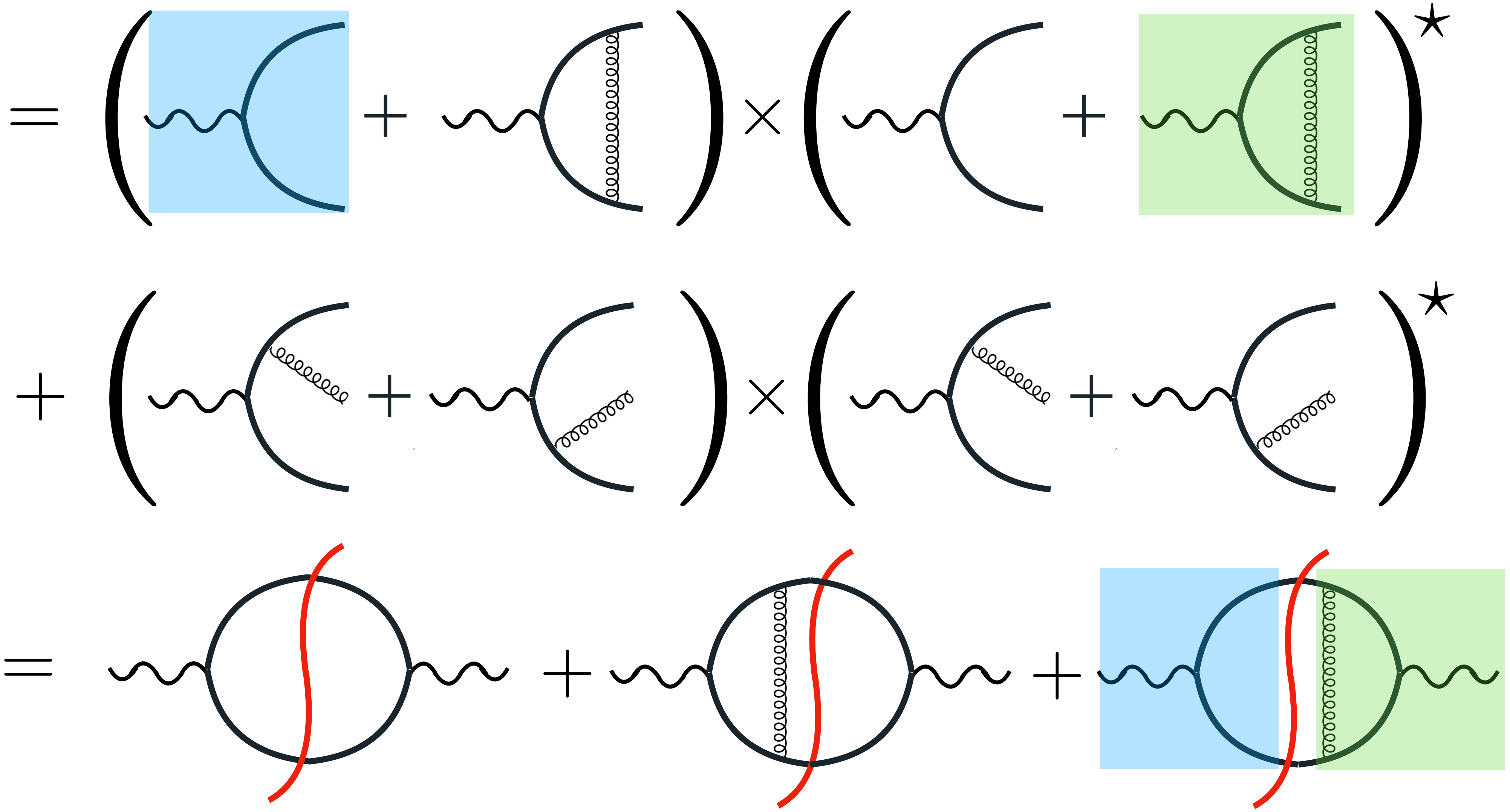
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$



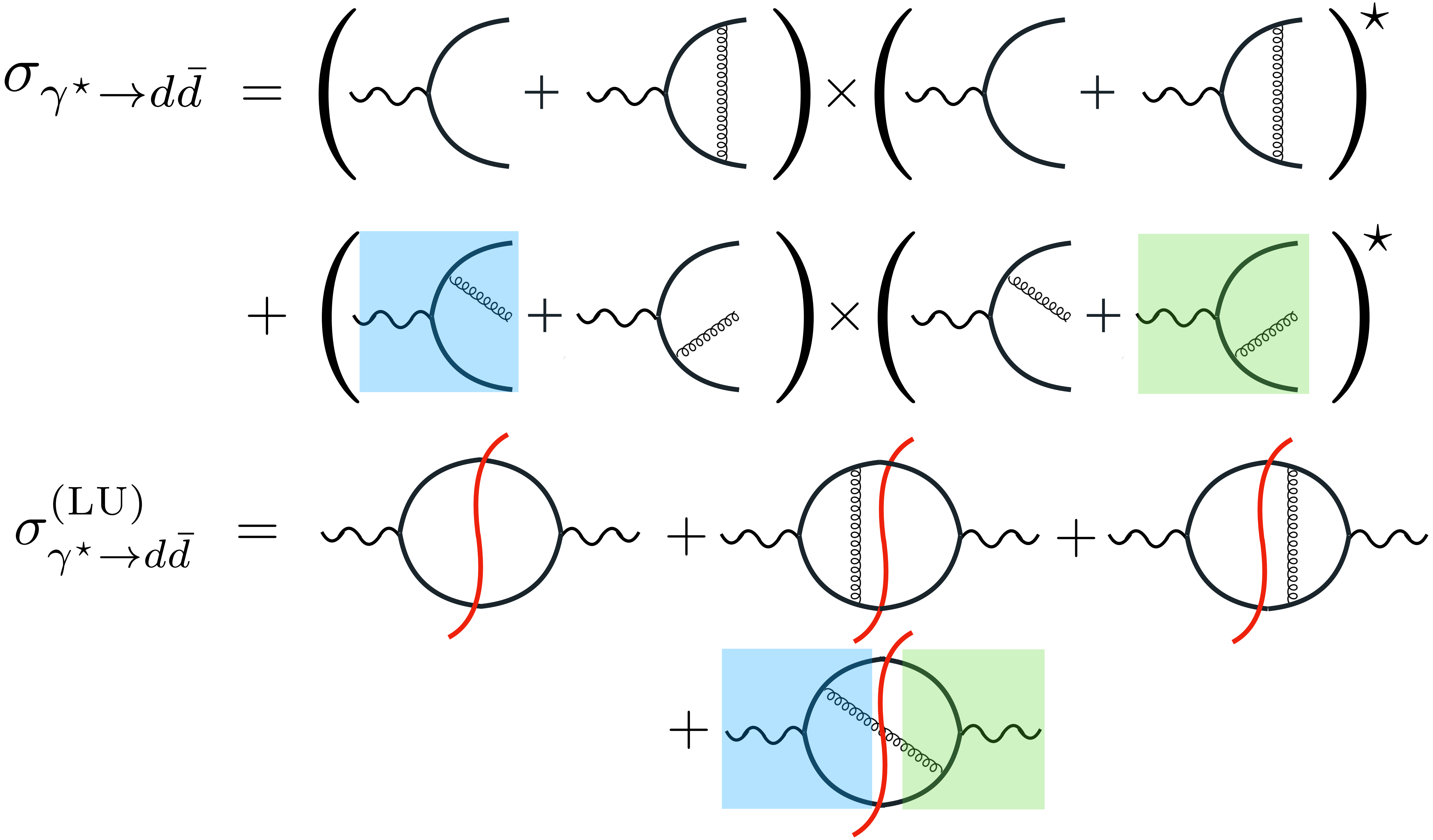
REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^\star \to d\bar{d}} =$$

$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$$



REMEDY: FORWARD-SCATTERING GRAPHS

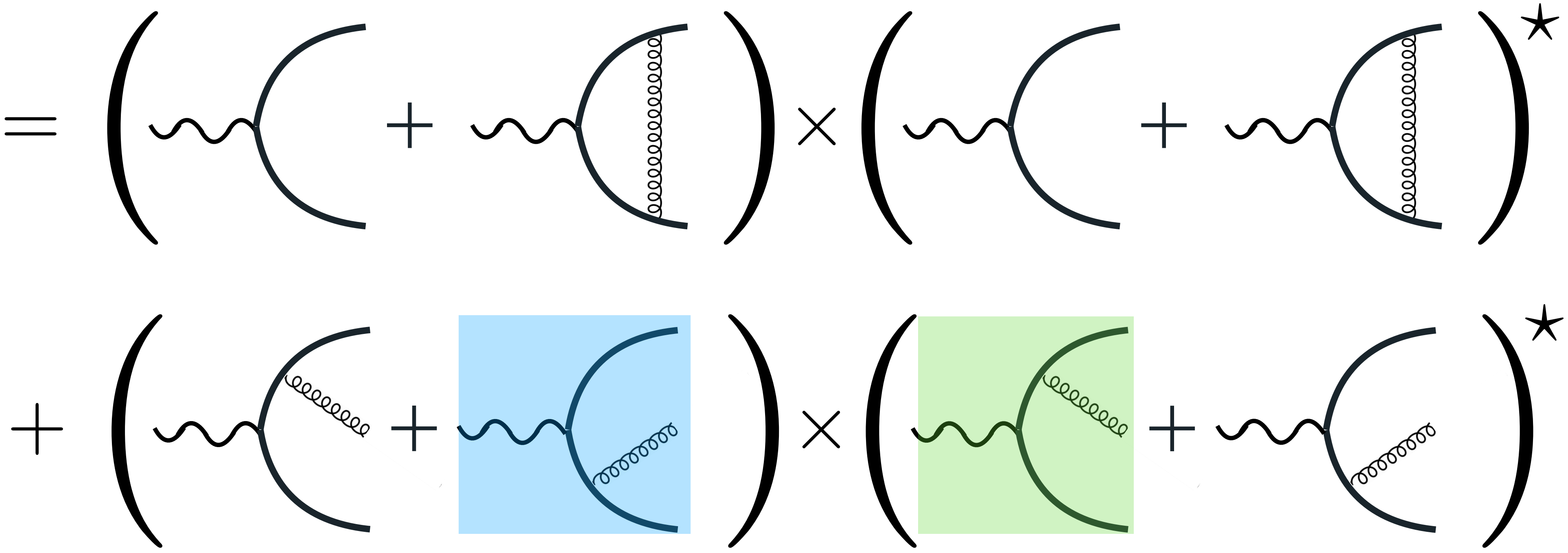


$$\sigma_{\gamma^\star \to d\bar{d}} =$$

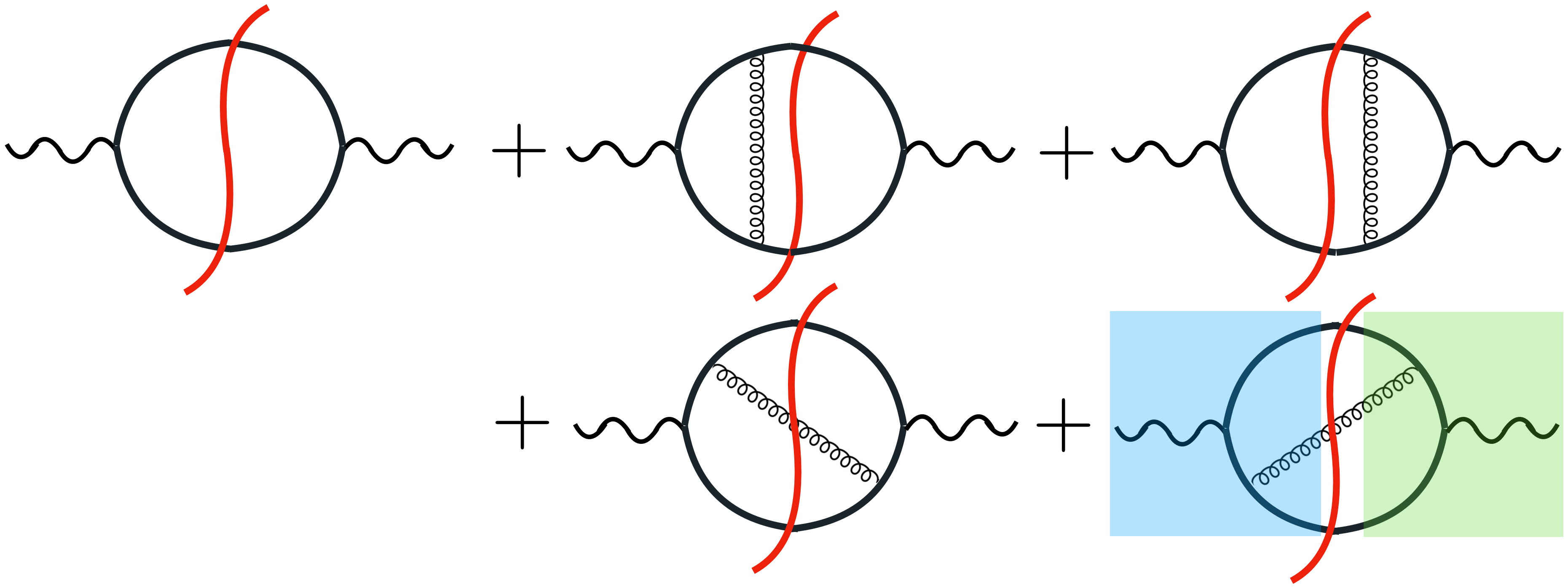
$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$$

REMEDY: FORWARD-SCATTERING GRAPHS

$\sigma_{\gamma^\star \to d\bar{d}} =$

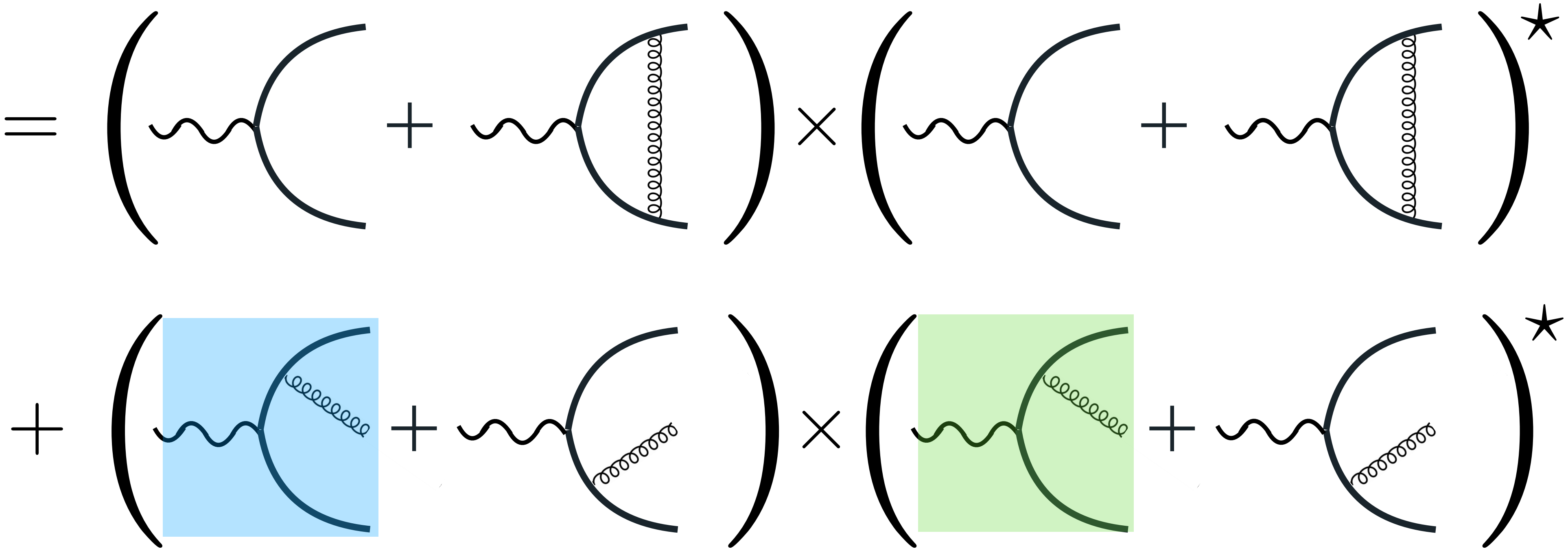


$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$

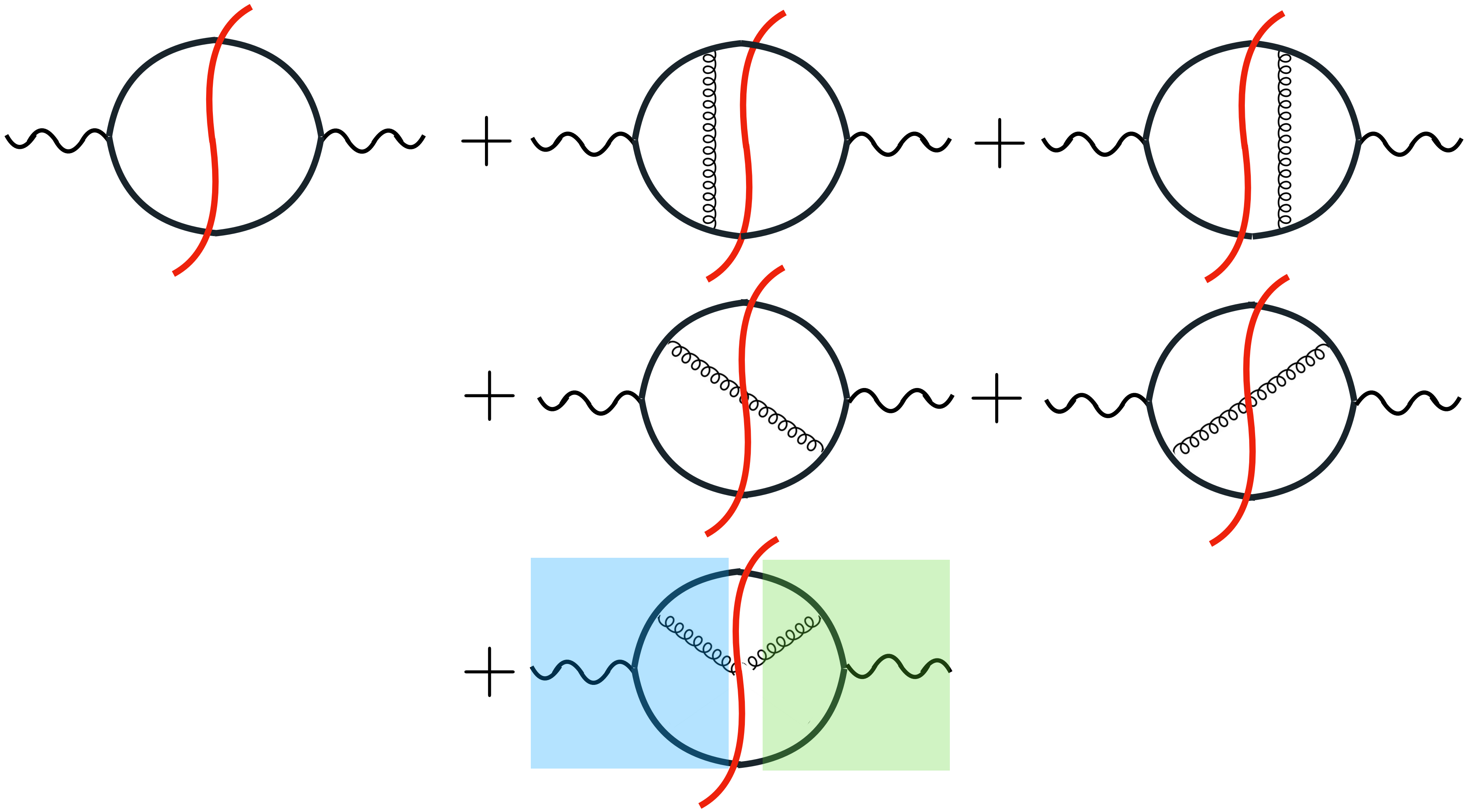


REMEDY: FORWARD-SCATTERING GRAPHS

$\sigma_{\gamma^\star \to d\bar{d}} =$

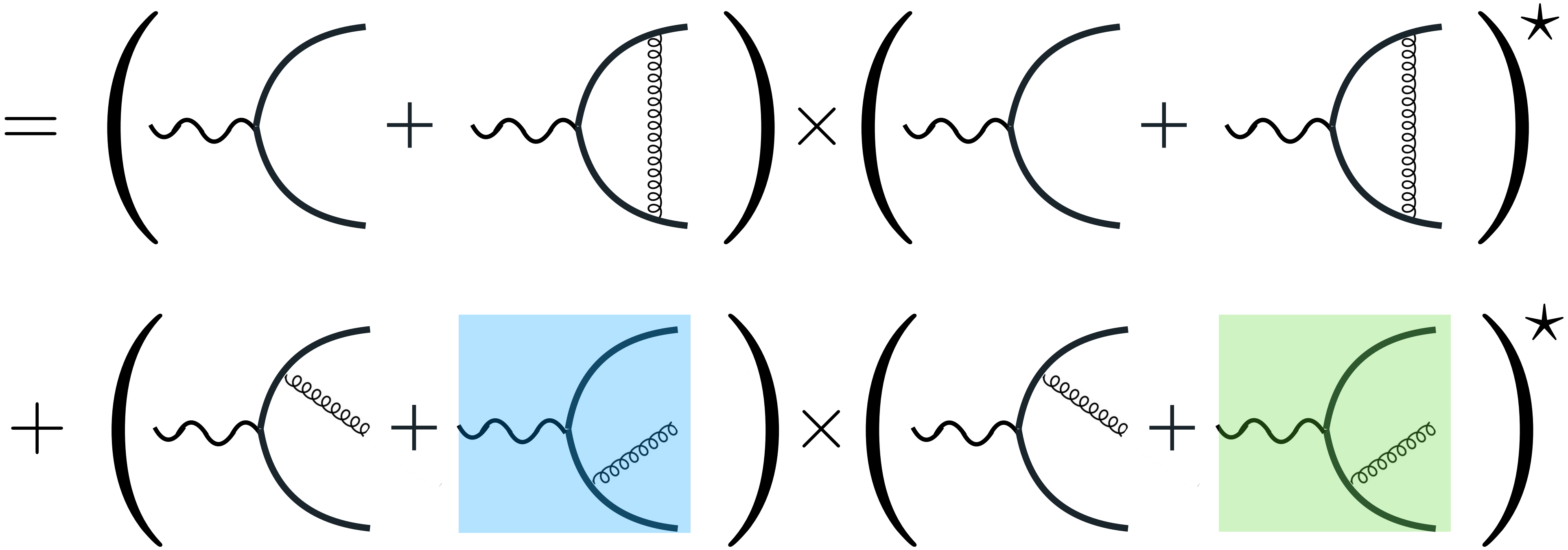


$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$

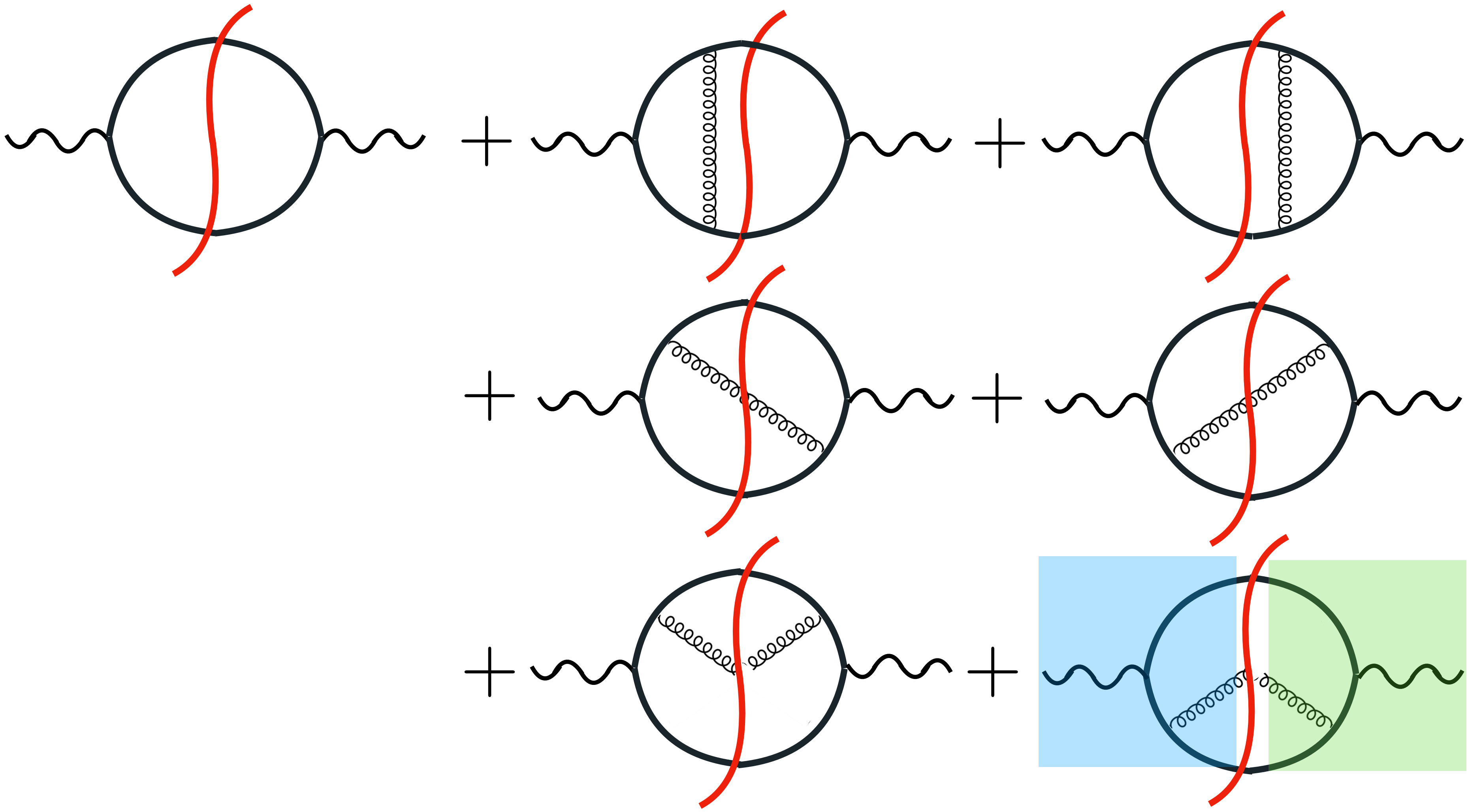


REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^\star\to d\bar{d}} =$$

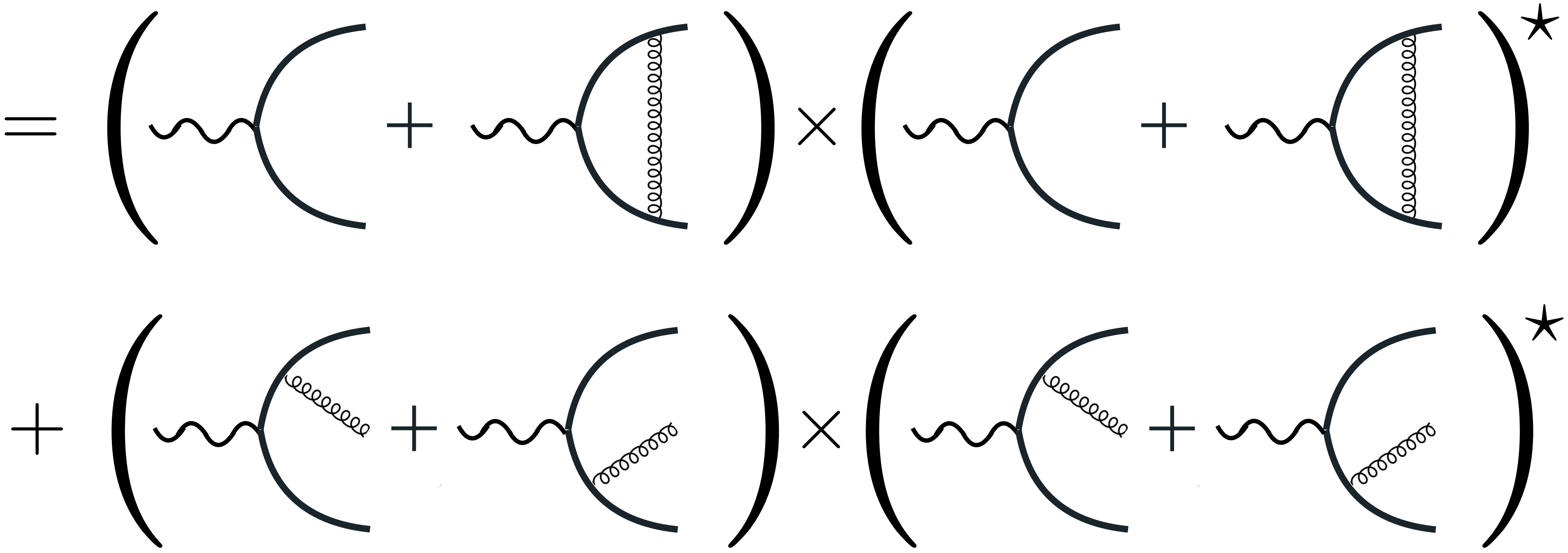


$$\sigma^{(\mathrm{LU})}_{\gamma^\star\to d\bar{d}} =$$

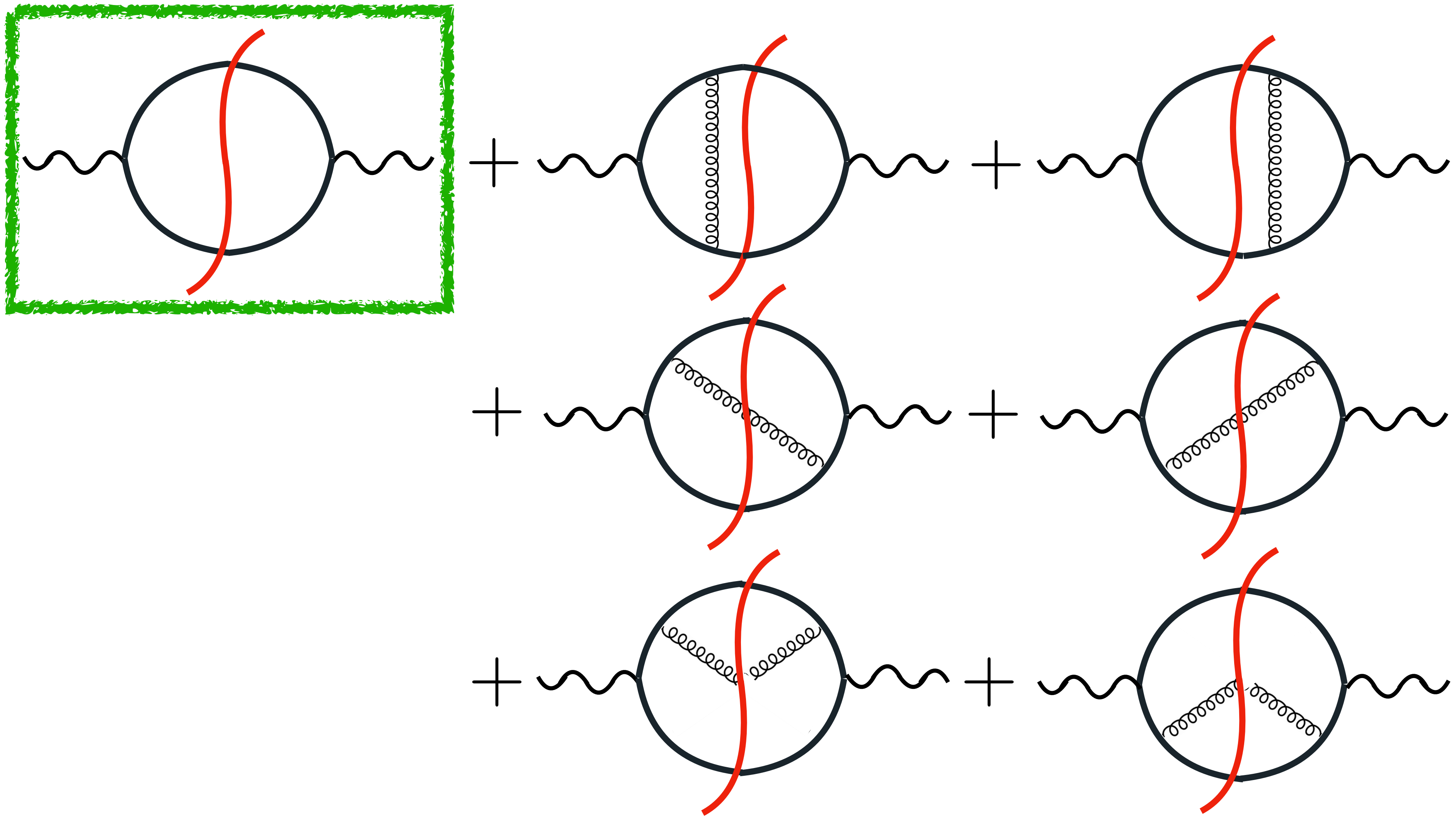


REMEDY: FORWARD-SCATTERING GRAPHS

$\sigma_{\gamma^\star \to d\bar{d}} =$



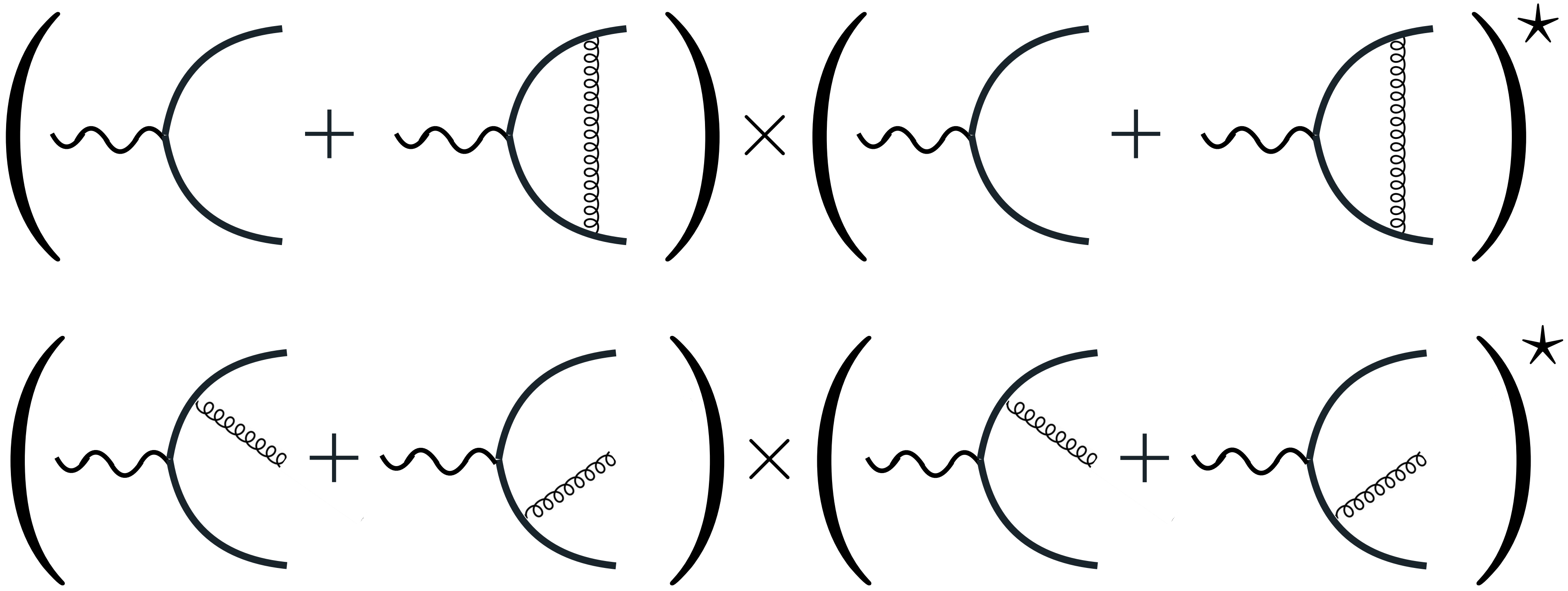
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$



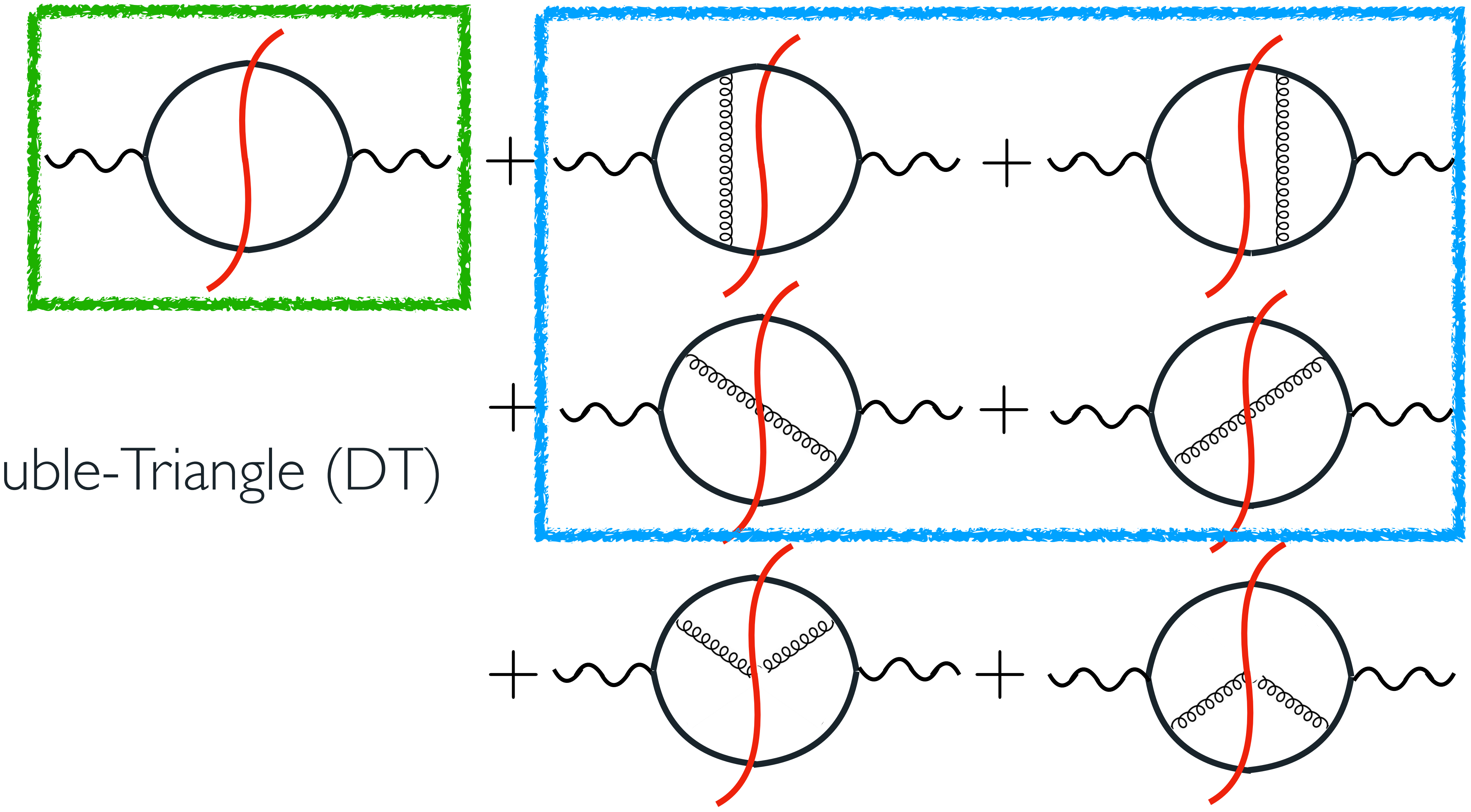
🟩 LO

REMEDY: FORWARD-SCATTERING GRAPHS

$\sigma_{\gamma^\star \to d\bar{d}} =$



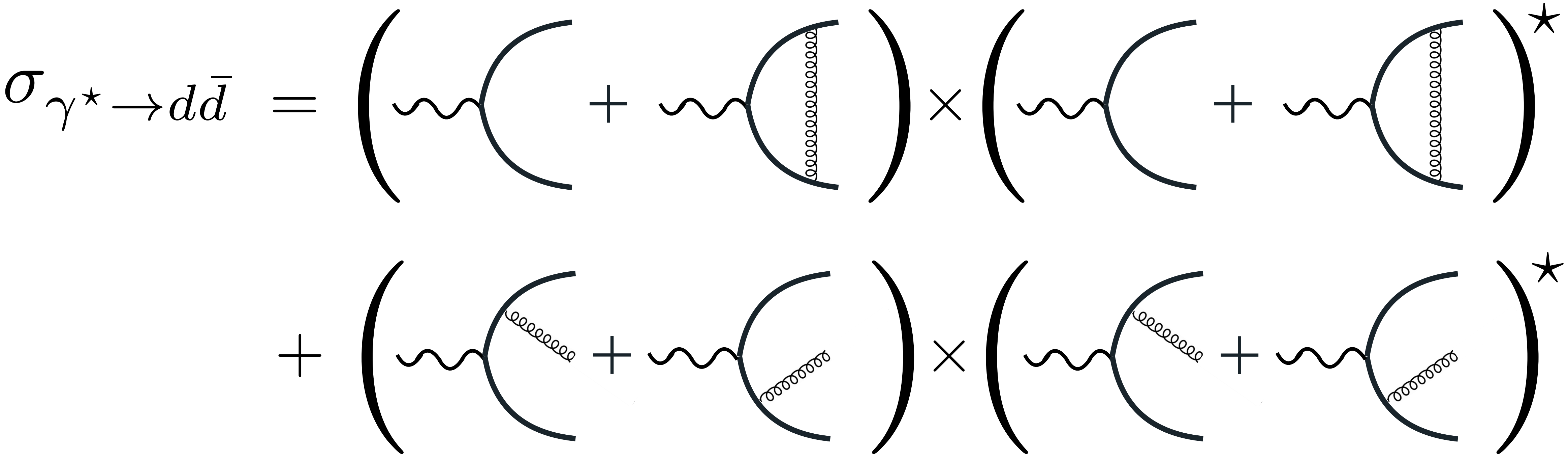
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$

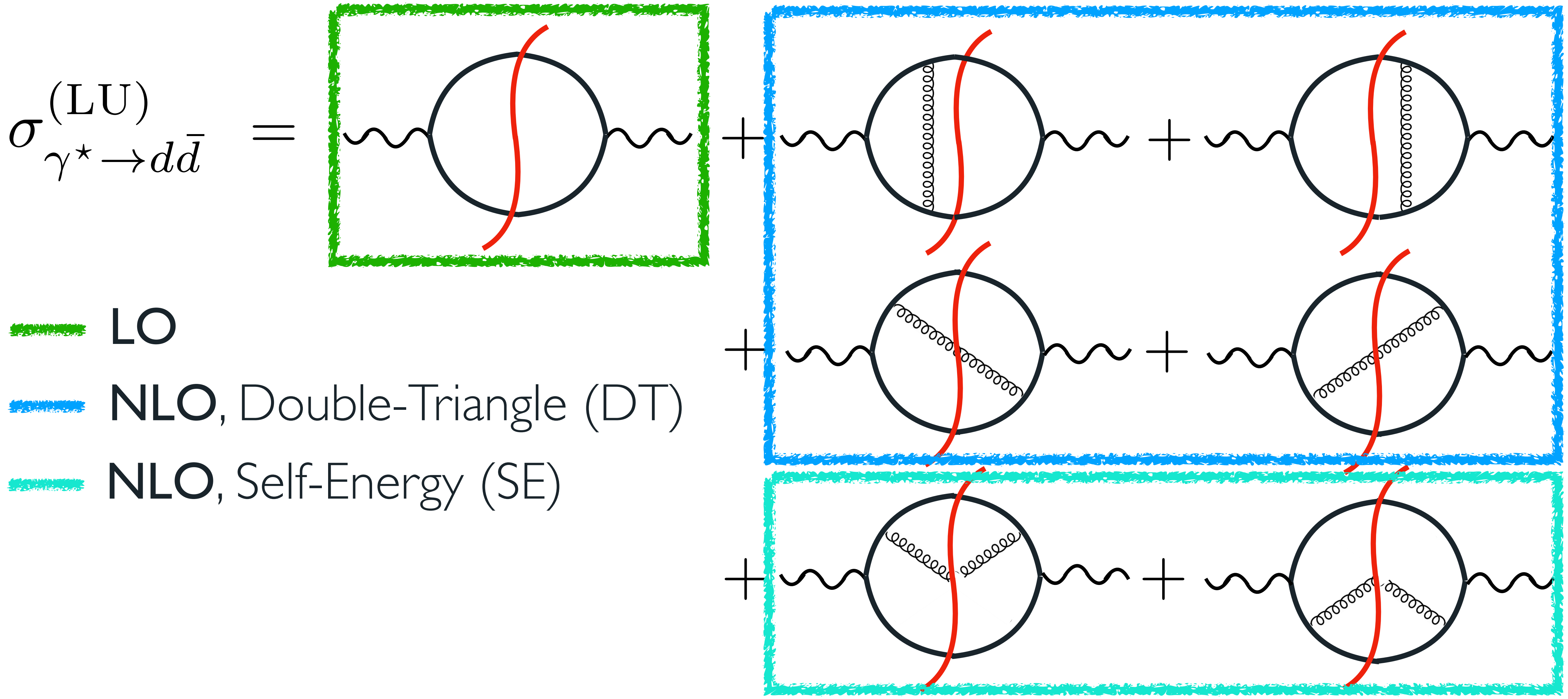


🟩 LO

🟦 **NLO**, Double-Triangle (DT)

REMEDY: FORWARD-SCATTERING GRAPHS





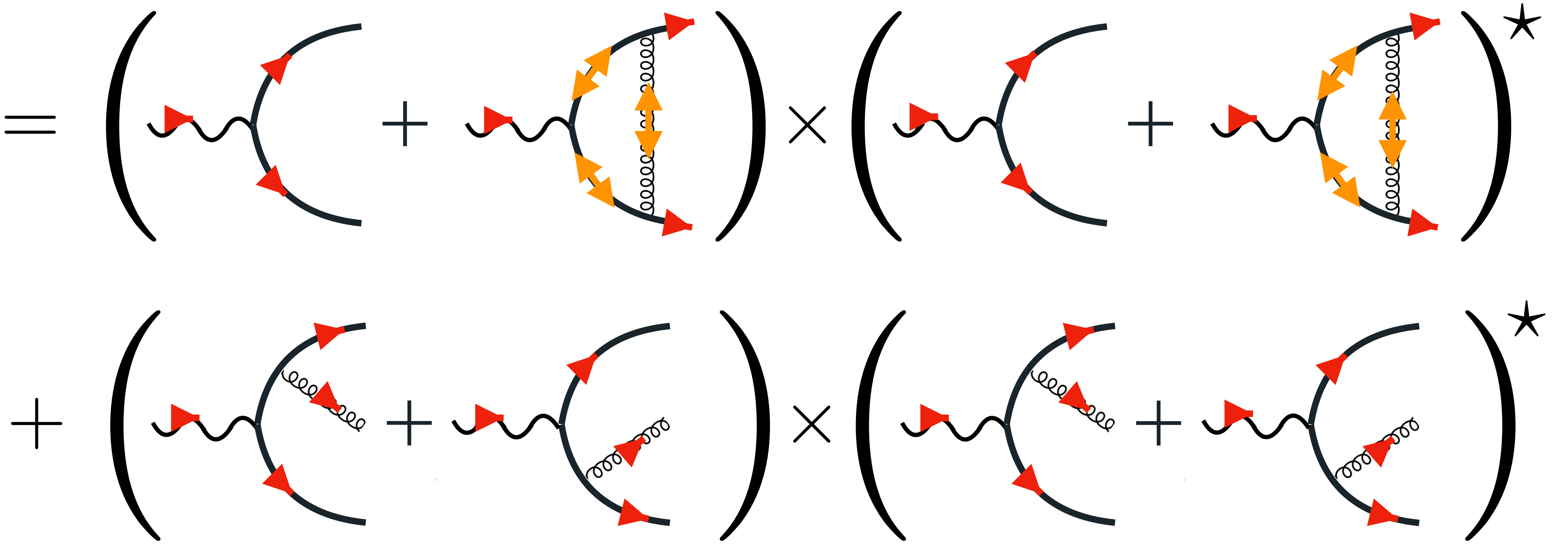
🟩 **LO**

🟦 **NLO**, Double-Triangle (DT)

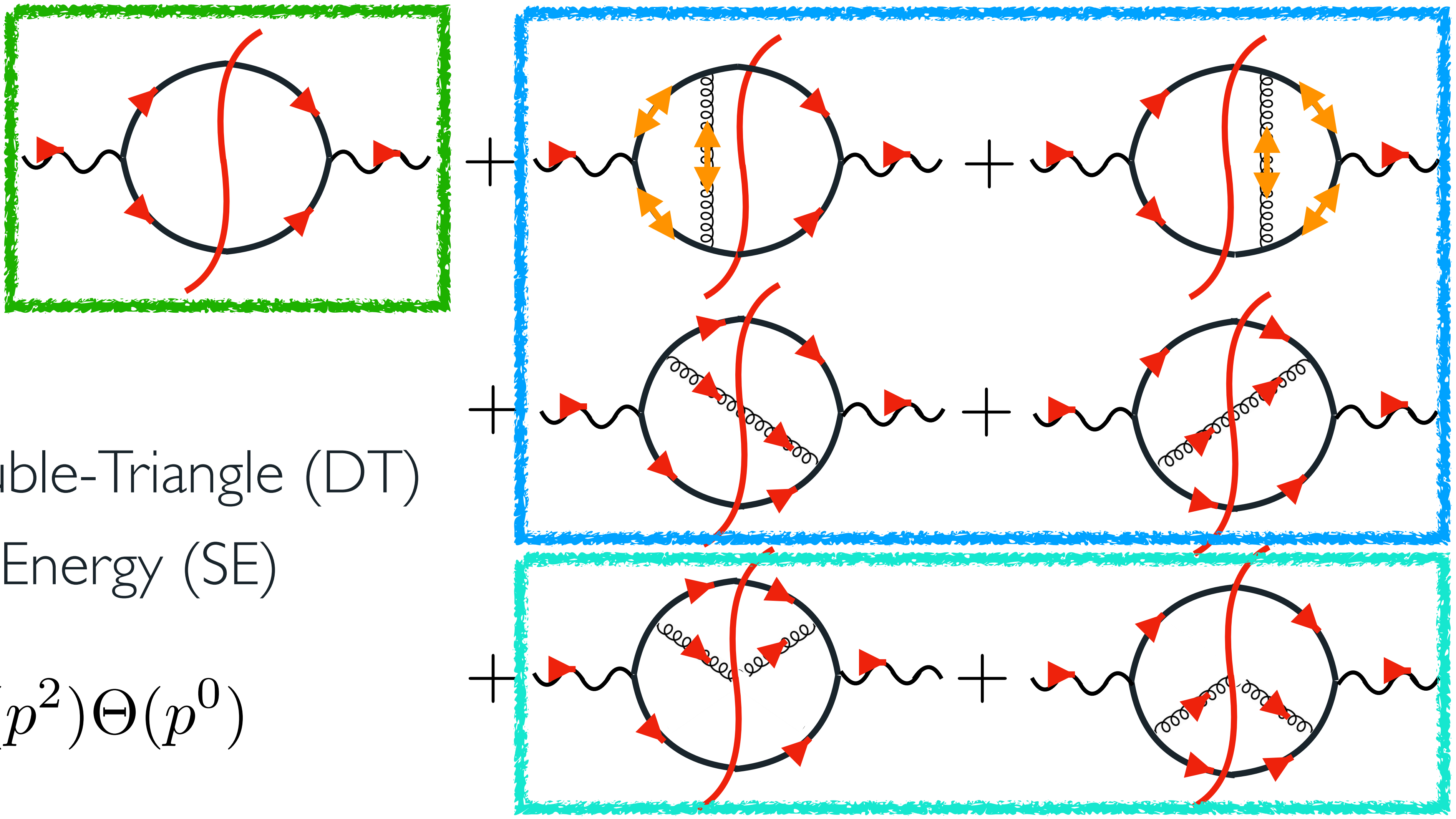
🟦 **NLO**, Self-Energy (SE)

REMEDY: FORWARD-SCATTERING GRAPHS

$\sigma_{\gamma^\star \to d\bar{d}} =$



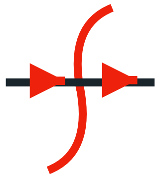
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$



🟩 LO

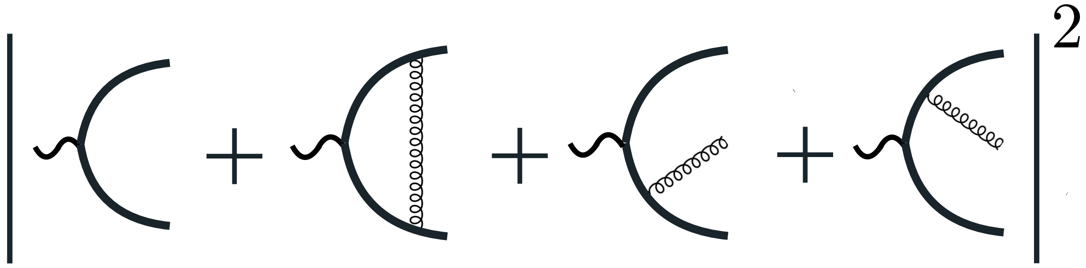
🟦 **NLO**, Double-Triangle (DT)

🟦 **NLO**, Self-Energy (SE)

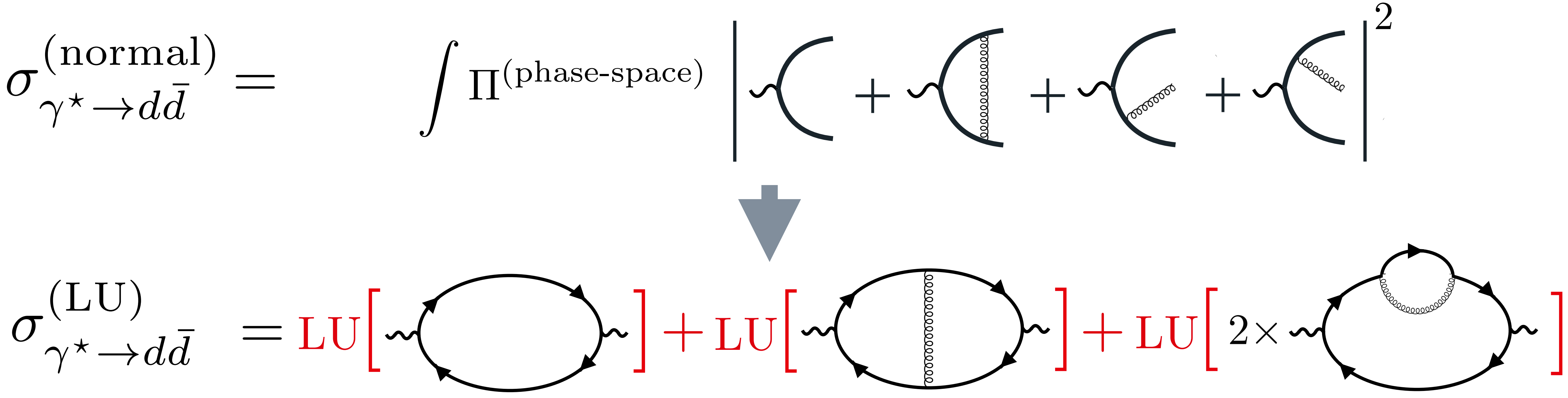
 $\equiv \frac{p^2}{2p^0}\delta(p^2)\Theta(p^0)$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma^{(\mathrm{normal})}_{\gamma^\star\to d\bar{d}} = \int \Pi^{(\text{phase-space})}$$

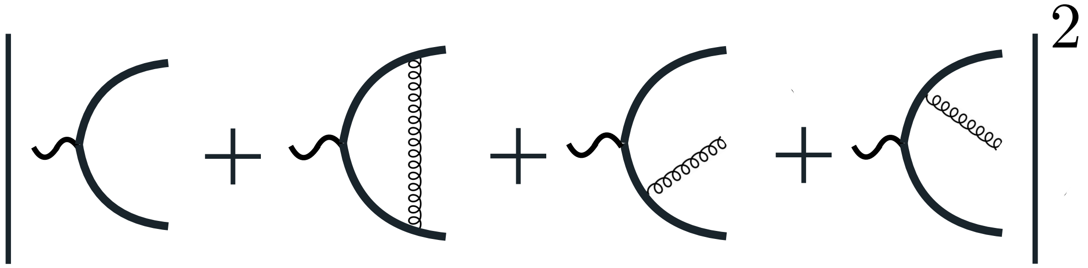


LOCAL UNITARITY: A CONCEPTUAL SHIFT

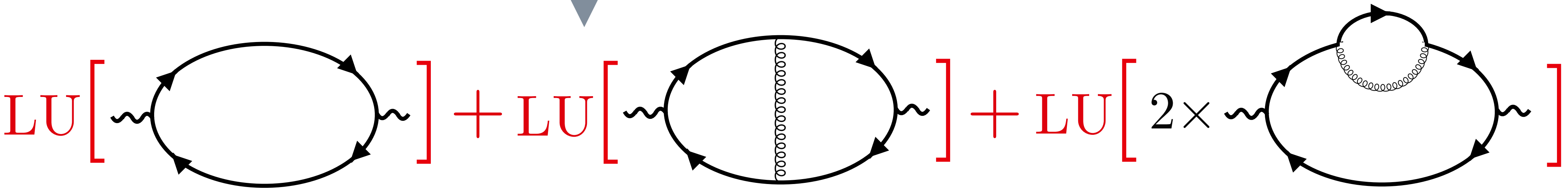


LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma^{(\text{normal})}_{\gamma^\star \to d\bar{d}} = \int \Pi^{(\text{phase-space})}$$



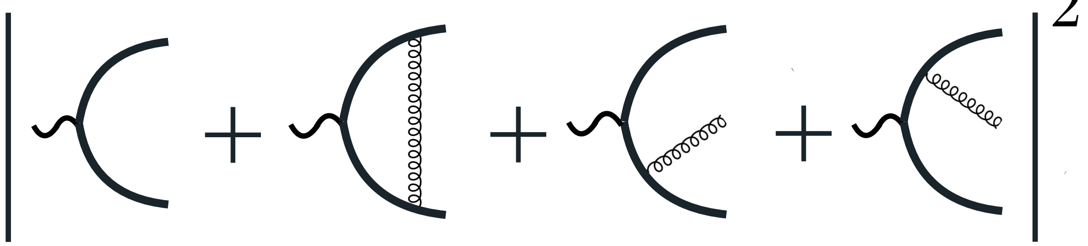
$$\sigma^{(\text{LU})}_{\gamma^\star \to d\bar{d}} =$$



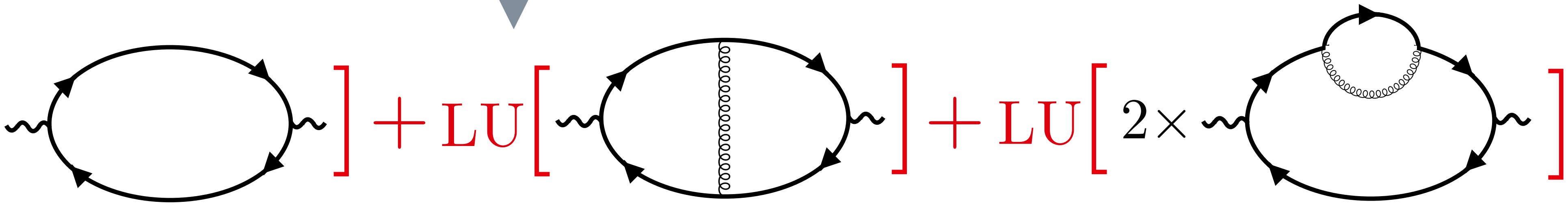
$$\sum_{c \in \{RRR, RRV, RVV, \ldots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|^2_{\text{truncated}}$$

IR-subtraction numerical $d = 4$ analytic $d = 4 - 2\epsilon$

LOCAL UNITARITY: A CONCEPTUAL SHIFT



$$\sigma^{(\text{normal})}_{\gamma^\star\to d\bar{d}} = \int \Pi^{(\text{phase-space})} \left| \quad + \quad + \quad + \quad \right|^2$$



$$\sigma^{(\text{LU})}_{\gamma^\star\to d\bar{d}} = \text{LU}\left[\quad\right] + \text{LU}\left[\quad\right] + \text{LU}\left[2\times\quad\right]$$

$$\sum_{c\in\{RRR,RRV,RVV,\dots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|^2_{\text{truncated}}$$

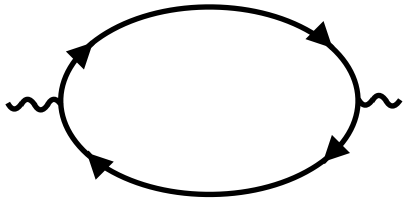
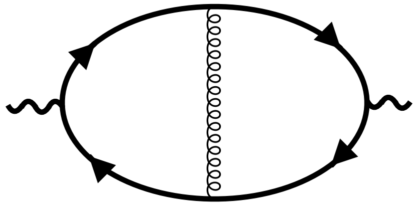
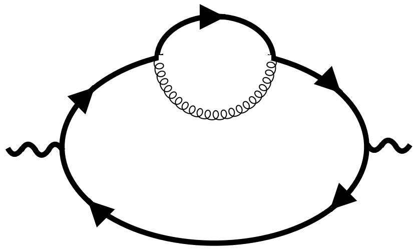
IR-subtraction numerical $d = 4$ analytic $d = 4 - 2\epsilon$

$$\sum_{j=1}^{n_{\text{supergraphs}}} \int \Pi \, g_j^{(\text{LU})}$$

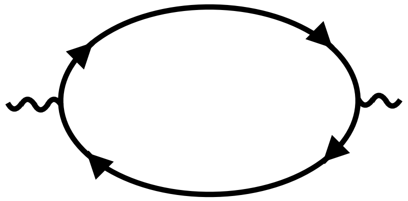
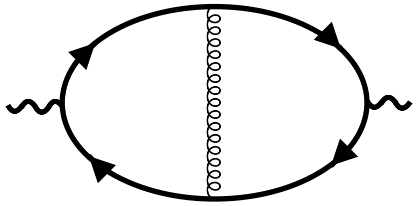
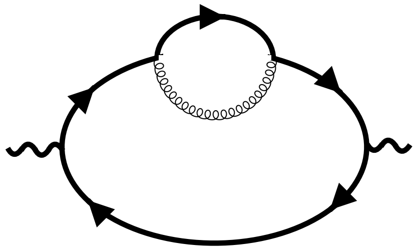
numerical $d = 4$

NO IR-subtraction

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

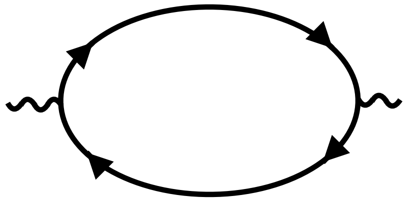
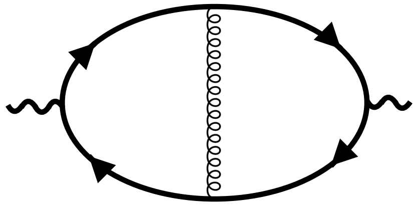
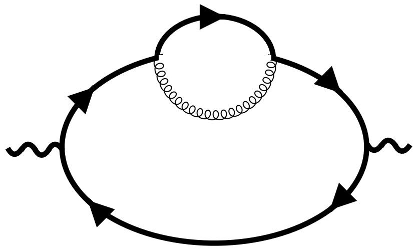
$\sigma^{(\mathrm{LU})}_{\gamma^\star\to d\bar{d}} = \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[2\times$  $\Big]$

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[\, 2\times$  $\Big]$

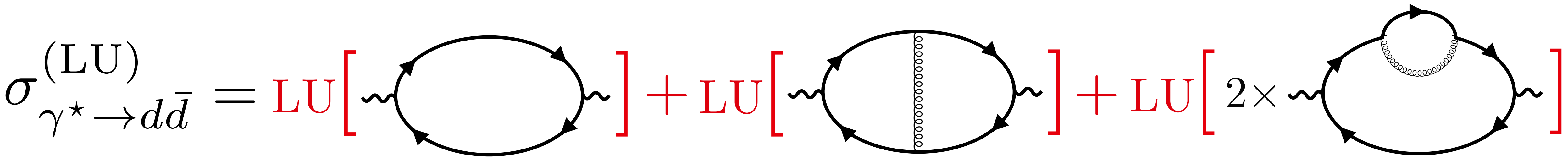
- At the **integral** level, the **Optical Theorem** simply gives: $\mathrm{LU}[\cdot] \propto \mathrm{Im}[\cdot]$

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[2\times$  $\Big]$

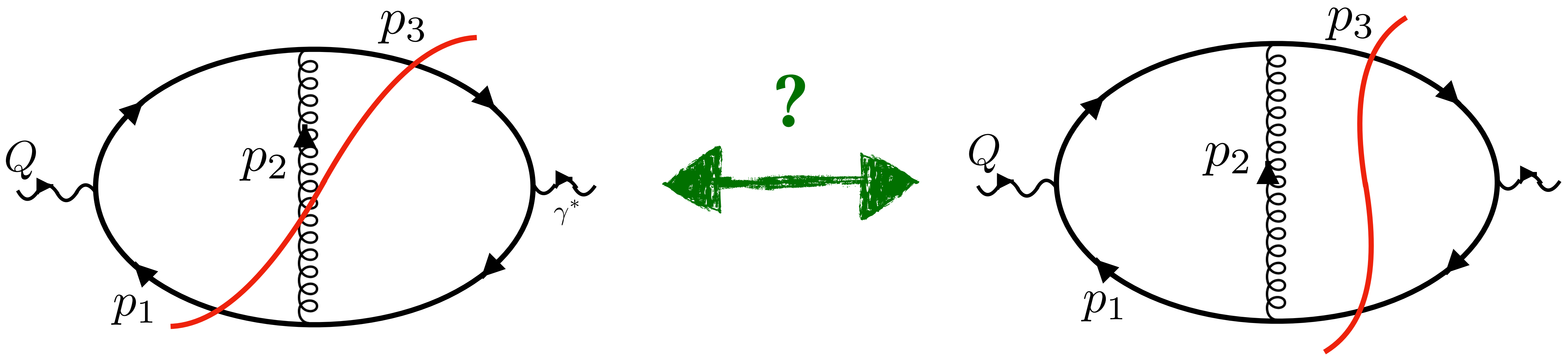
- At the **integral** level, the **Optical Theorem** simply gives: $\mathrm{LU}[\cdot] \propto \mathrm{Im}[\cdot]$
- The **differential** version of LU is our new paradigm: **Local Unitarity**

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?



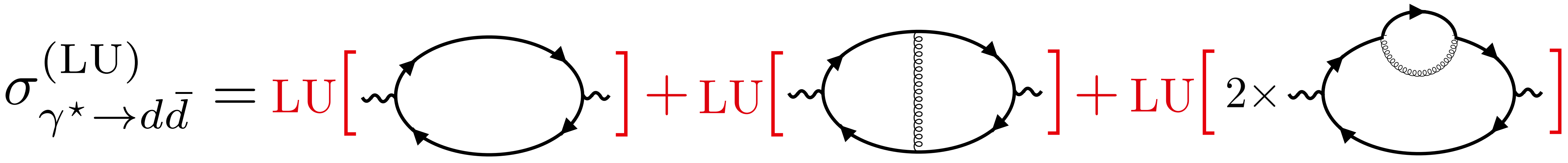
$$\sigma^{(\mathrm{LU})}_{\gamma^\star\to d\bar{d}} = \mathrm{LU}\Big[\quad\Big] + \mathrm{LU}\Big[\quad\Big] + \mathrm{LU}\Big[2\times\quad\Big]$$

- At the **integral** level, the **Optical Theorem** simply gives: $\mathrm{LU}[\cdot] \propto \mathrm{Im}[\cdot]$
- The **differential** version of LU is our new paradigm: **Local Unitarity**
- Each supergraph is **individually locally finite**, how you ask ?



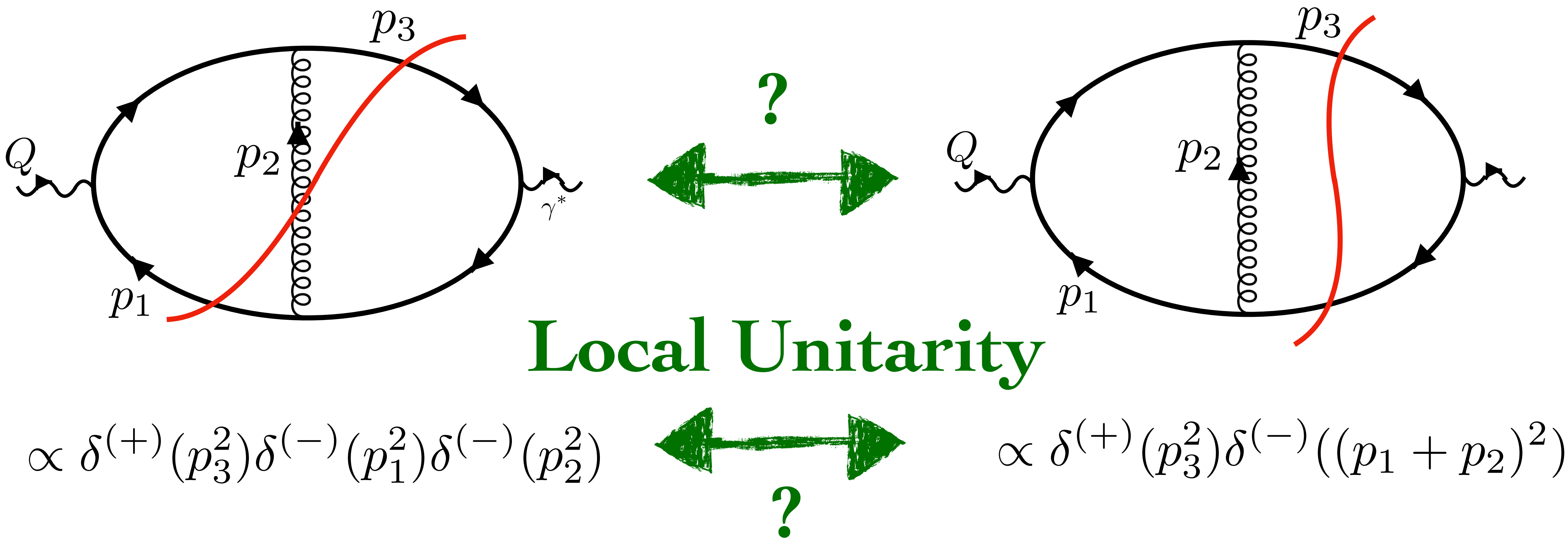
$$\propto \delta^{(+)}(p_3^2)\delta^{(-)}(p_1^2)\delta^{(-)}(p_2^2) \quad \underset{?}{\longleftrightarrow} \quad \propto \delta^{(+)}(p_3^2)\delta^{(-)}((p_1+p_2)^2)$$

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?



$$\sigma^{(\mathrm{LU})}_{\gamma^\star\to d\bar{d}} = \mathrm{LU}\Big[\ \Big] + \mathrm{LU}\Big[\ \Big] + \mathrm{LU}\Big[2\times\ \Big]$$

- At the **integral** level, the **Optical Theorem** simply gives: $\mathrm{LU}[\cdot] \propto \mathrm{Im}[\cdot]$
- The **differential** version of LU is our new paradigm: **Local Unitarity**
- Each supergraph is **individually locally finite**, how you ask ?



Local Unitarity

$\propto \delta^{(+)}(p_3^2)\delta^{(-)}(p_1^2)\delta^{(-)}(p_2^2)$

$\propto \delta^{(+)}(p_3^2)\delta^{(-)}((p_1+p_2)^2)$

LOOP TREE DUALITY

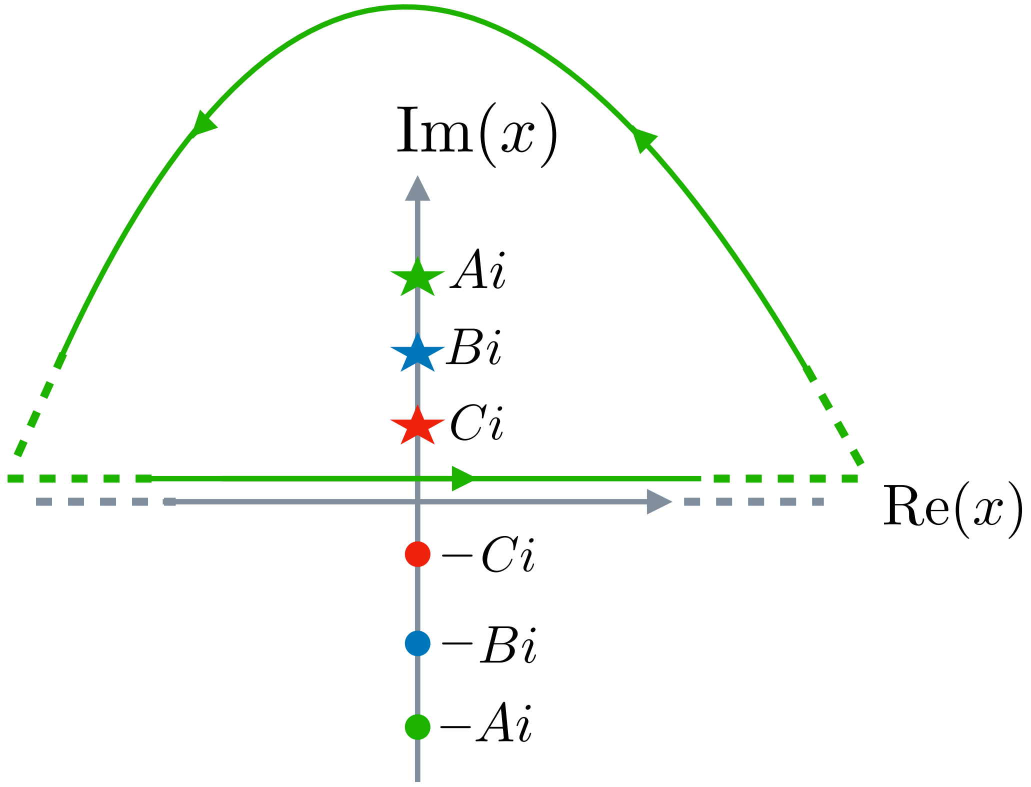
(CLOSELY RELATED TO FEYNMAN-TREE THEOREM
AND TIME-ORDERED PERTURBATION THEORY)

(ONE)-LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx F(x) \qquad F(x) = \frac{1}{x^2 + A^2} \frac{1}{x^2 + B^2} \frac{1}{x^2 + C^2}$$

$$F(x) = \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(x - Bi)(x + Bi)} \frac{1}{(x - Ci)(x + Ci)}$$

$$\left(\text{Assumptions} \rightarrow \{A > 0, B > 0, C > 0\}\right)$$



Cauchy: $\left(R(x^\star) \equiv \mathrm{Res}(F, x = x^\star) \right)$

$$I = (-2\pi i)[R(Ai) + R(Bi) + R(Ci)]$$

What does it correspond to for a one-loop integral?

(ONE-)LOOP TREE DUALITY

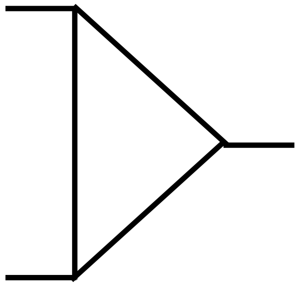
$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)\left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)\left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

Pole selected for each propagator

Then integrate the energy component using residue theorem



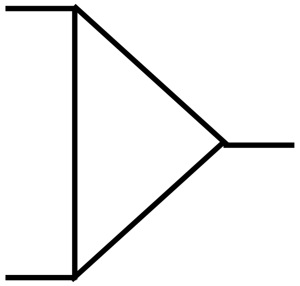
$$= \int \mathrm{d}^3\vec{k} \left[\mathrm{Res}_1\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_2\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_3\left[\frac{N}{D_1 D_2 D_3}\right]\right]$$

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)\left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

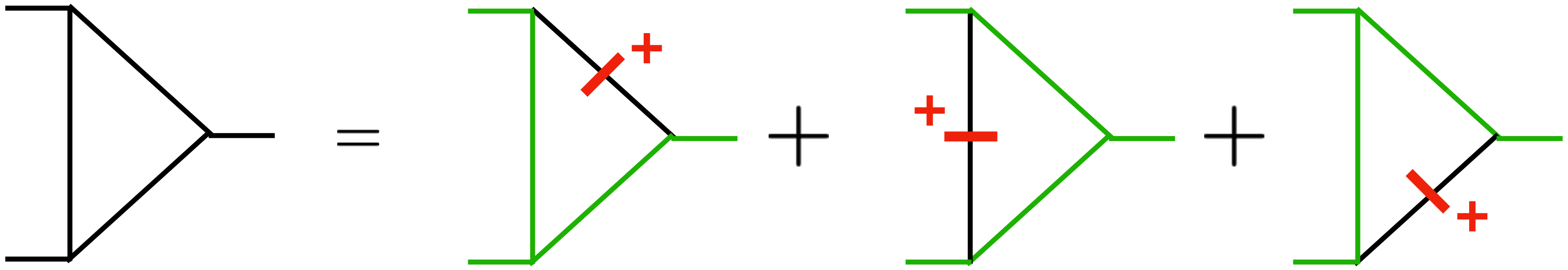
Pole selected for each propagator

Then integrate the energy component using residue theorem



$$= \int \mathrm{d}^3\vec{k} \left[\mathrm{Res}_1\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_2\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_3\left[\frac{N}{D_1 D_2 D_3}\right]\right]$$

Residues can be represented as cuts:

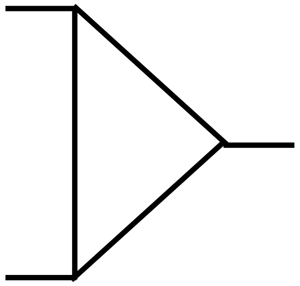


(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)\left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

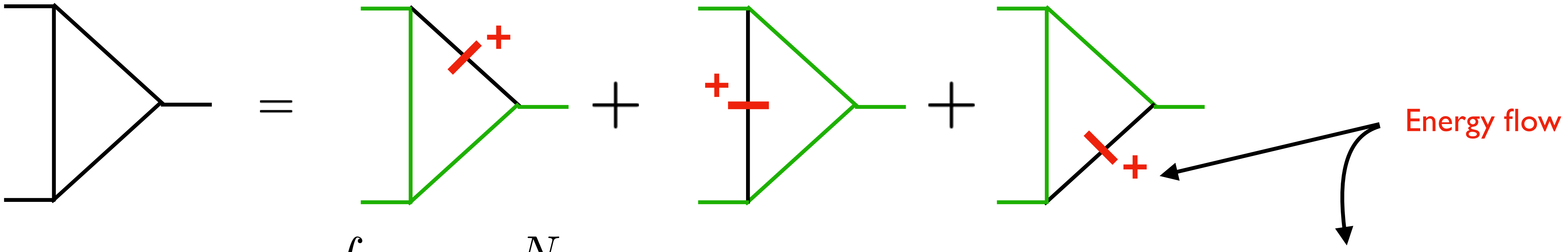
Pole selected for each propagator

Then integrate the energy component using residue theorem



$$= \int \mathrm{d}^3\vec{k} \left[\mathrm{Res}_1\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_2\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_3\left[\frac{N}{D_1 D_2 D_3}\right]\right]$$

Residues can be represented as cuts:



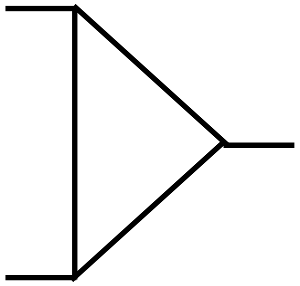
$$= \int \mathrm{d}^4 k \frac{N}{D_1 D_2 D_3}\left(D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3)\right)$$

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)\left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

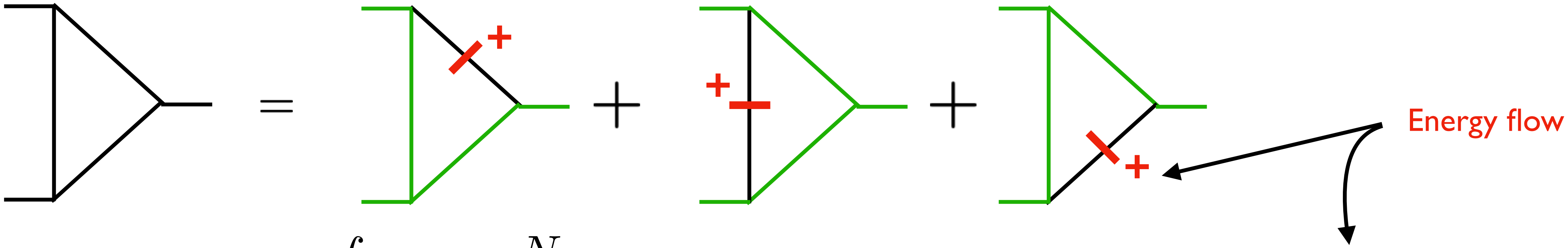
Pole selected for each propagator

Then integrate the energy component using residue theorem



$$= \int \mathrm{d}^3\vec{k} \left[\mathrm{Res}_1\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_2\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_3\left[\frac{N}{D_1 D_2 D_3}\right]\right]$$

Residues can be represented as cuts:



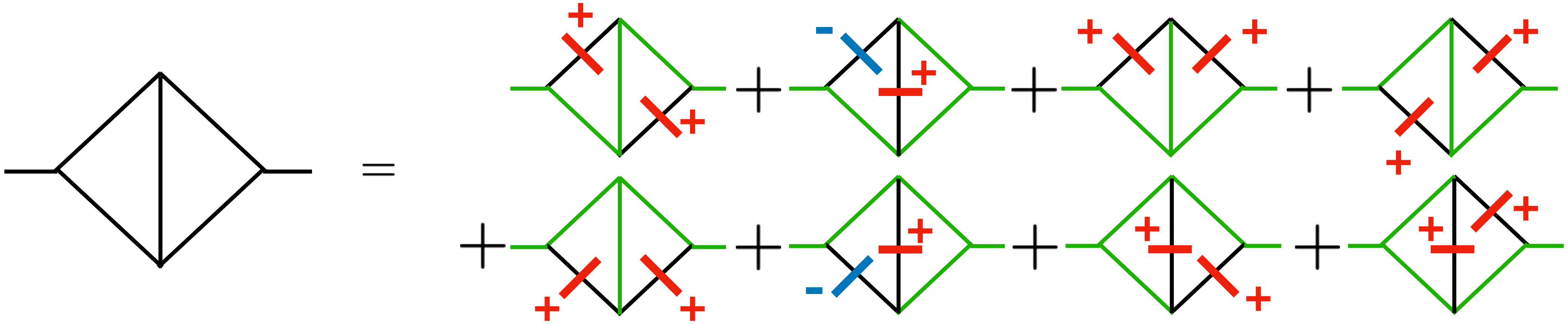
$$= \int \mathrm{d}^4 k \frac{N}{D_1 D_2 D_3}\left(D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3)\right)$$

(Note: exactly the same in Euclidean 3D integrals with:

$k_z \rightarrow \mathrm{i}k_z$ and $|\vec{k}|^2 \rightarrow \left(k_z - \sqrt{k_x^2 + k_y^2}\right)\left(k_z + \sqrt{k_x^2 + k_y^2}\right)$)

(MULTI-)LOOP TREE DUALITY

Applying LTD to a two-loop double-triangle: one residue per spanning tree



Interplay of momentum conservation and causal prescription is key to obtain the energy flow

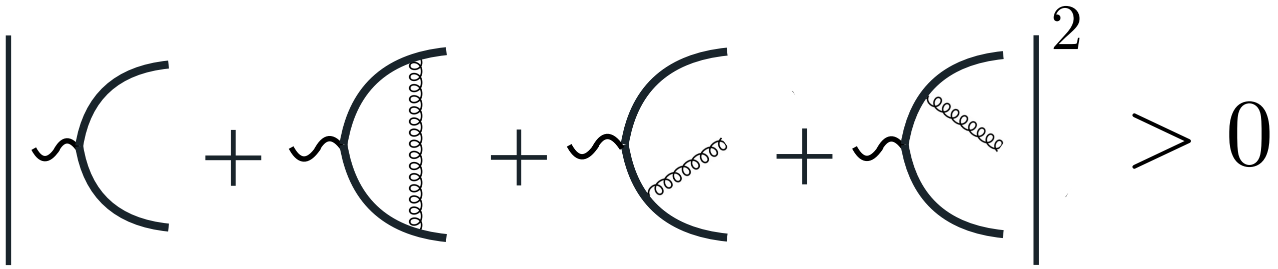
- **Distributional identities:** [Bierenbaum, Catani, Draggiotis, Rodrigo, arxiv: 1007.0194]
- **Averaging procedure:** [Runkel, Scór, Vesga, Weinzierl, arxiv: 1902.02135]
- **Iterative procedure:** [Capatti, VH, Kermanschah, Ruijl, arxiv: 1906.06138]
- **Manifestly causal:** [Capatti, VH, Kermanschah, Pelloni, Ruijl, arxiv: 2009.05509]
- **Cross-Free Family** [Capatti, arxiv: 2211.09653]
  (the best 3D repr.!)

Codes : [https://github.com/apelloni/cLTD]

[https://bitbucket.org/wjtorresb/lotty]

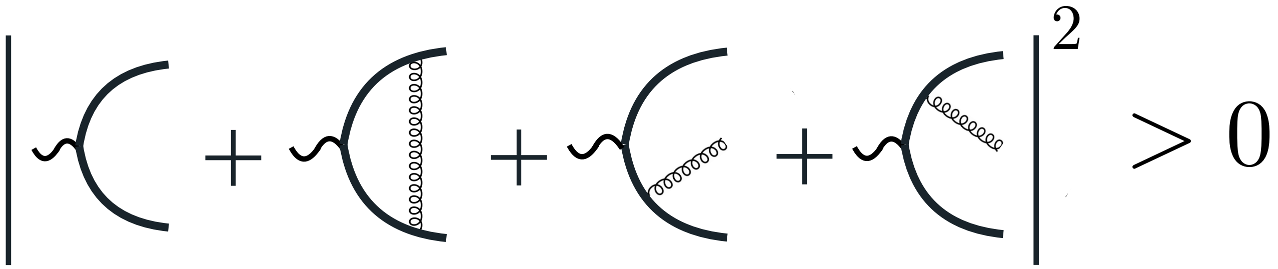
{ PICK YOUR CANDY: CANNOT ALL BE MANIFEST }

POSITIVITY



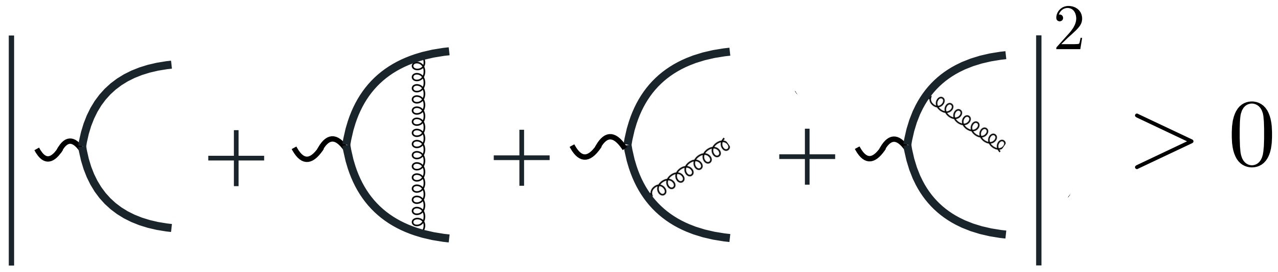
{ PICK YOUR CANDY: CANNOT ALL BE MANIFEST }

~~POSITIVITY~~



{ PICK YOUR CANDY: CANNOT ALL BE MANIFEST }

~~POSITIVITY~~

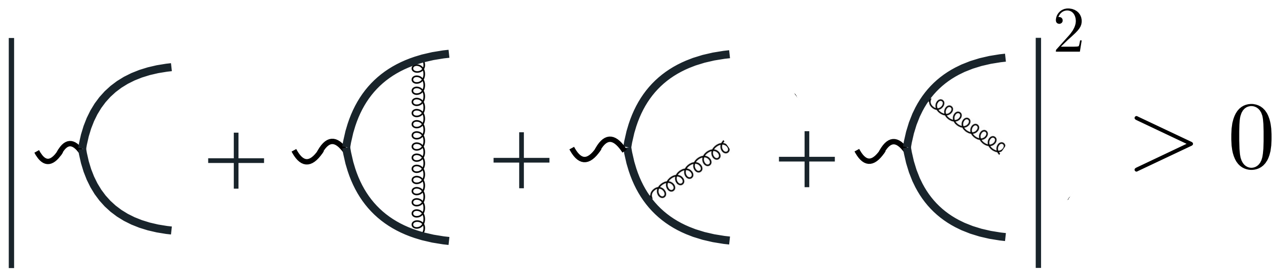


LORENTZ INVARIANCE

$$\mathcal{M}^{\mu}\left(\{p_i^{\mu}\}\right) = \Lambda^{\mu}_{\nu}\mathcal{M}^{\nu}\left(\{\Lambda^{\mu}_{\nu}p_i^{\nu}\}\right)$$

{ PICK YOUR CANDY: CANNOT ALL BE MANIFEST }

~~POSITIVITY~~

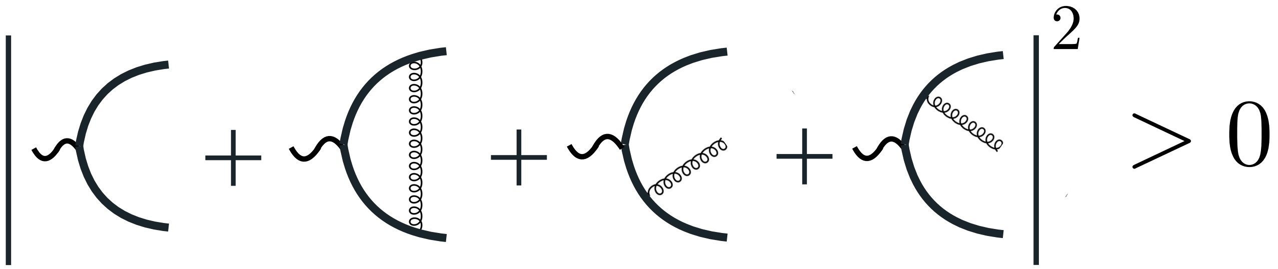


~~LORENTZ INVARIANCE~~

$$\mathcal{M}^{\mu}\left(\{p_i^{\mu}\}\right) = \Lambda^{\mu}_{\nu}\mathcal{M}^{\nu}\left(\{\Lambda^{\mu}_{\nu}p_i^{\nu}\}\right)$$

{ PICK YOUR CANDY: CANNOT ALL BE MANIFEST }

~~POSITIVITY~~



~~LORENTZ INVARIANCE~~

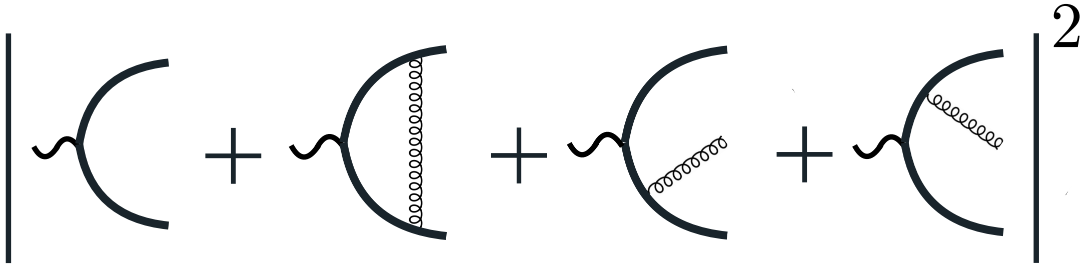
$$\mathcal{M}^{\mu}\left(\{p_i^{\mu}\}\right) = \Lambda^{\mu}_{\nu}\mathcal{M}^{\nu}\left(\{\Lambda^{\mu}_{\nu}p_i^{\nu}\}\right)$$

GAUGE INVARIANCE

$$k^{\mu}\mathcal{M}_{\mu} = 0$$

{ Pick your candy: cannot all be manifest }

~~Positivity~~



> 0

~~Lorentz Invariance~~

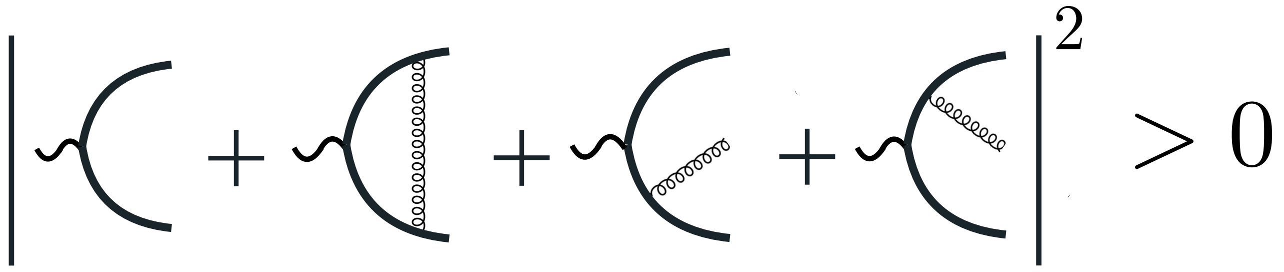
$$\mathcal{M}^{\mu}\left(\{p_i^{\mu}\}\right) = \Lambda^{\mu}_{\nu}\mathcal{M}^{\nu}\left(\{\Lambda^{\mu}_{\nu}p_i^{\nu}\}\right)$$

~~Gauge Invariance~~

$$k^{\mu}\mathcal{M}_{\mu} = 0$$

{ Pick your candy: cannot all be manifest }

~~Positivity~~



~~Lorentz Invariance~~

$$\mathcal{M}^{\mu}\left(\{p_i^{\mu}\}\right) = \Lambda^{\mu}_{\nu}\mathcal{M}^{\nu}\left(\{\Lambda^{\mu}_{\nu}p_i^{\nu}\}\right)$$

~~Gauge Invariance~~

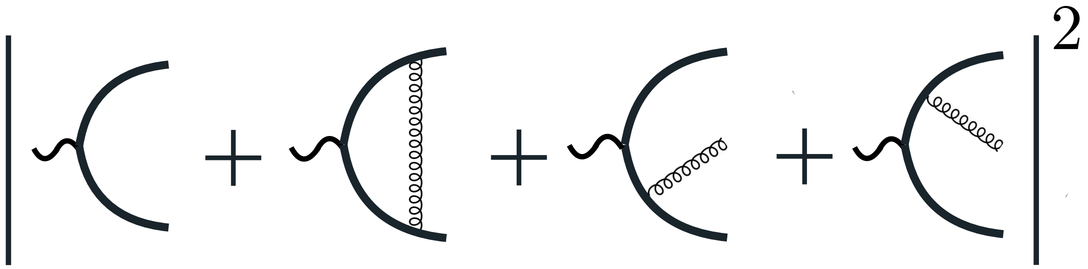
$$k^{\mu}\mathcal{M}_{\mu} = 0$$

Unitarity

$$\mathrm{i}(T^{\dagger} - T) = T^{\dagger}T$$

{ PICK YOUR CANDY: CANNOT ALL BE MANIFEST }

~~POSITIVITY~~



> 0

~~LORENTZ INVARIANCE~~

$$\mathcal{M}^{\mu}\left(\{p_i^{\mu}\}\right) = \Lambda^{\mu}_{\nu}\mathcal{M}^{\nu}\left(\{\Lambda^{\mu}_{\nu}p_i^{\nu}\}\right)$$

~~GAUGE INVARIANCE~~

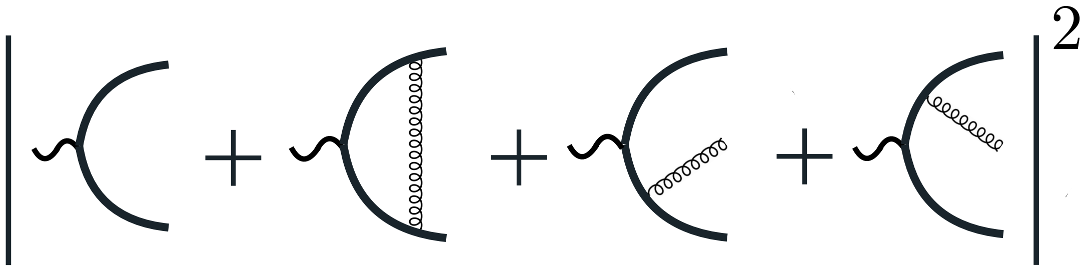
$$k^{\mu}\mathcal{M}_{\mu} = 0$$

UNITARITY

$$\mathrm{i}(T^{\dagger} - T) = T^{\dagger}T$$

{ PICK YOUR CANDY: CANNOT ALL BE MANIFEST }

~~POSITIVITY~~



> 0

~~LORENTZ INVARIANCE~~

$$\mathcal{M}^{\mu}\left(\{p_i^{\mu}\}\right) = \Lambda^{\mu}_{\nu}\mathcal{M}^{\nu}\left(\{\Lambda^{\mu}_{\nu}p_i^{\nu}\}\right)$$

~~GAUGE INVARIANCE~~

$$k^{\mu}\mathcal{M}_{\mu} = 0$$

UNITARITY

$$\mathrm{i}(T^{\dagger} - T) = T^{\dagger}T$$

LOCALLY FINITE

$$\lim_{k\,\to\,\text{soft,colli,UV}} I(k) = \mathcal{O}(1)$$

FOUR TYPES OF SINGULARITIES :

INFRARED SINGULARITIES | LOCAL UNITARITY

FOUR TYPES OF SINGULARITIES :

INFRARED SINGULARITIES | LOCAL UNITARITY

THRESHOLDS | CONTOUR-DEFORMATION

FOUR TYPES OF SINGULARITIES :

INFRARED SINGULARITIES	LOCAL UNITARITY
THRESHOLDS	CONTOUR-DEFORMATION
ULTRAVIOLET SINGULARITIES	LOCAL BPHZ

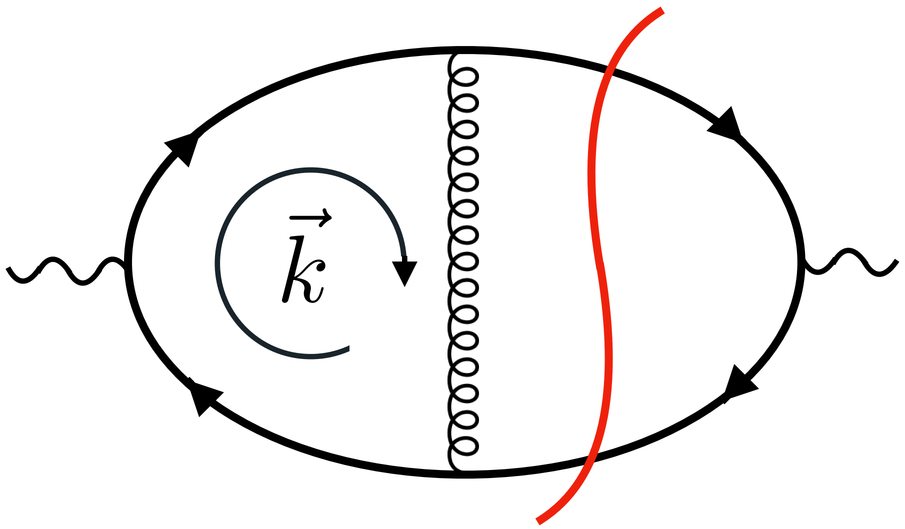
FOUR TYPES OF SINGULARITIES :

INFRARED SINGULARITIES	LOCAL UNITARITY
THRESHOLDS	CONTOUR-DEFORMATION
ULTRAVIOLET SINGULARITIES	LOCAL BPHZ
INTEGRABLE SINGULARITIES	SAMPLING

FOUR TYPES OF SINGULARITIES :

INFRARED SINGULARITIES	LOCAL UNITARITY
THRESHOLDS	CONTOUR-DEFORMATION
ULTRAVIOLET SINGULARITIES	LOCAL BPHZ
INTEGRABLE SINGULARITIES	SAMPLING

LOCALISED RENORMALISATION: BPHZ



$$\lim_{|\vec{k}|\to\infty} I^{(\text{Local Unitarity})} \to \infty$$

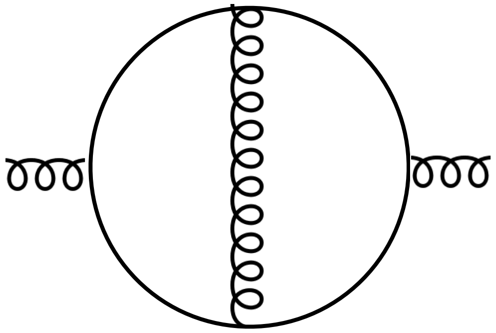
LOCALISED RENORMALISATION

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

$$R(\Gamma) = \sum_{S\in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S\in W(\gamma)\setminus\gamma} \gamma \setminus S * \prod_{\gamma'\in S} Z(\gamma')\right)$$

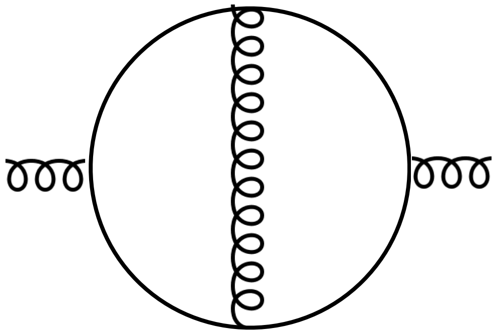
LOCALISED RENORMALISATION

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

$R\Big(\Gamma =$  $\Big)$ = $\sum_{S\in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S\in W(\gamma)\setminus\gamma} \gamma \setminus S * \prod_{\gamma'\in S} Z(\gamma')\right)$

LOCALISED RENORMALISATION

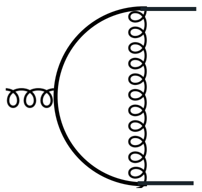
[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

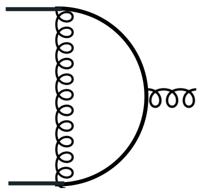
$R\Big(\Gamma =$  $\Big) = \sum_{S\in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S\in W(\gamma)\setminus\gamma} \gamma \setminus S * \prod_{\gamma'\in S} Z(\gamma')\right)$

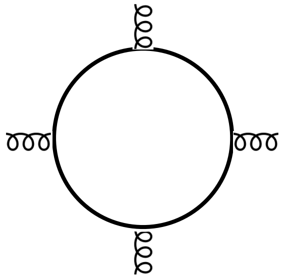
UV subgraphs :

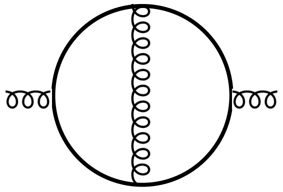
$\mathrm{dod}\left(\gamma_{\{1,2,3\}}\right) = 0$

$\mathrm{dod}\left(\gamma_4\right) = 2$

$\gamma_1 =$ 

$\gamma_2 =$ 

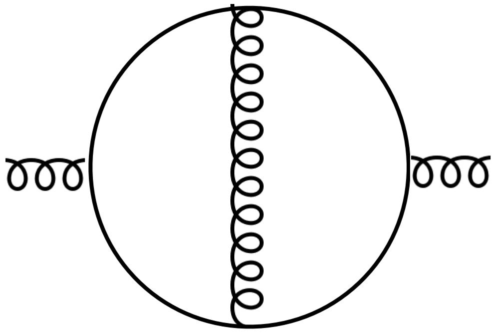
$\gamma_3 =$ 

$\gamma_4 =$ 

LOCALISED RENORMALISATION

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

R (Γ =



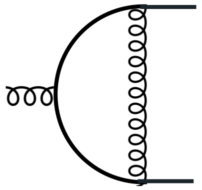
) $$= \sum_{S\in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S\in W(\gamma)\setminus\gamma} \gamma\setminus S * \prod_{\gamma'\in S} Z(\gamma')\right)$$

UV subgraphs :

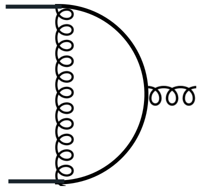
$\mathrm{dod}\left(\gamma_{\{1,2,3\}}\right) = 0$

$\mathrm{dod}\left(\gamma_4\right) = 2$

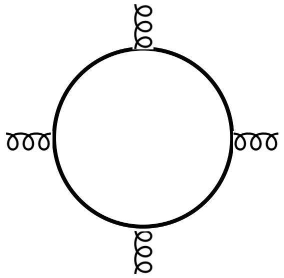
$\gamma_1 =$



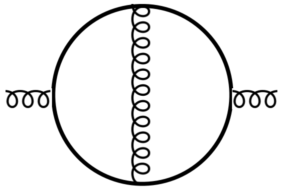
$\gamma_2 =$



$\gamma_3 =$



$\gamma_4 =$

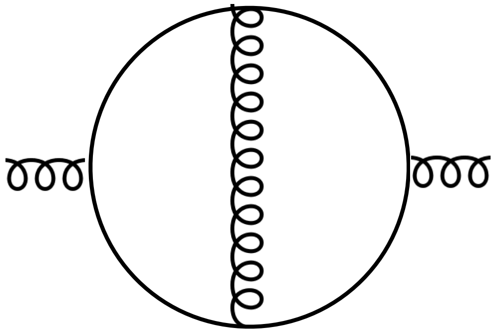


$$R(\Gamma) = \Gamma - K(\gamma_1)*\Gamma\setminus\gamma_1 - K(\gamma_2)*\Gamma\setminus\gamma_2 - K(\gamma_3)*\Gamma\setminus\gamma_3 - K(\gamma_4)*\Gamma\setminus\gamma_4$$

$$+ K\big(K(\gamma_1)*\Gamma\setminus\gamma_1\big) + K\big(K(\gamma_2)*\Gamma\setminus\gamma_2\big) + K\big(K(\gamma_3)*\Gamma\setminus\gamma_3\big)$$

LOCALISED RENORMALISATION

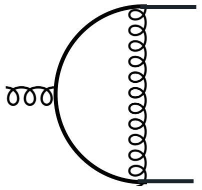
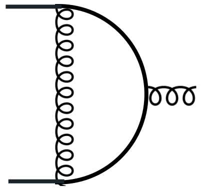
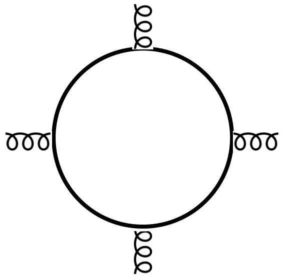
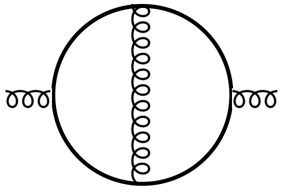
[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

R $\Bigg(\Gamma =$  $\Bigg) = \sum_{S\in W(\Gamma)} \Gamma\setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\Bigg(\sum_{S\in W(\gamma)\setminus\gamma} \gamma\setminus S * \prod_{\gamma'\in S} Z(\gamma')\Bigg)$

UV subgraphs :

$\mathrm{dod}\left(\gamma_{\{1,2,3\}}\right) = 0$

$\mathrm{dod}\left(\gamma_4\right) = 2$

$\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

$$\mathrm{R}(\Gamma) = \Gamma - \mathrm{K}(\gamma_1) * \Gamma\setminus\gamma_1 - \mathrm{K}(\gamma_2) * \Gamma\setminus\gamma_2 - \mathrm{K}(\gamma_3) * \Gamma\setminus\gamma_3 - \mathrm{K}(\gamma_4) * \Gamma\setminus\gamma_4$$

$$+ \mathrm{K}\big(\mathrm{K}(\gamma_1) * \Gamma\setminus\gamma_1\big) + \mathrm{K}\big(\mathrm{K}(\gamma_2) * \Gamma\setminus\gamma_2\big) + \mathrm{K}\big(\mathrm{K}(\gamma_3) * \Gamma\setminus\gamma_3\big)$$

What is the operator K(γ) ? Anything we want ! so long as it:

- Locally cancels UV divergences of γ , even in the presence of nestings
- Yields results immediately renormalised in the chosen scheme ($\overline{\mathrm{MS}} + \mathrm{OS}$)
- Minimal analytics: at most single-scale all-massive vacuum integrals

LOCAL RENORMALISATION OPERATOR K

Our solution: $K(\gamma) := T(\gamma)$

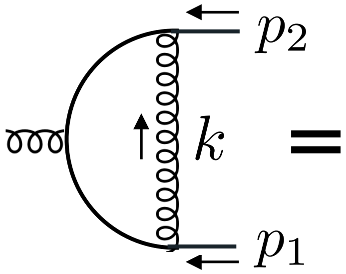
$T(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

LOCAL RENORMALISATION OPERATOR K

Our solution: K(γ) := T(γ)

T(γ) := **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

γ_1 =



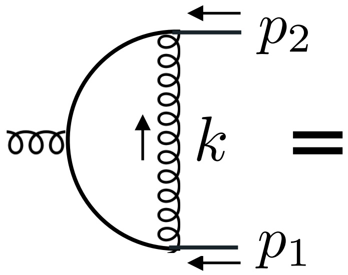
$$= \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$$

Local renormalisation operator K

Our solution: $\mathrm{K}(\gamma) := \mathrm{T}(\gamma)$

$\mathrm{T}(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

$\gamma_1 =$



$$= \frac{\mathcal{N}_{\gamma_1}(k,p_1,p_2,m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$$

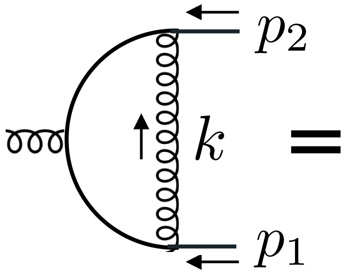
$$\gamma_1^{\lambda} := \frac{\mathcal{N}_{\gamma_1}(k,\lambda p_1,\lambda p_2,\lambda m)}{(k-\lambda p_1)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2))(k^2 - m_{\mathrm{UV}}^2)((k+\lambda p_2)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2)\ )}$$

LOCAL RENORMALISATION OPERATOR K

Our solution: $\mathrm{K}(\gamma) := \mathrm{T}(\gamma)$

$\mathrm{T}(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

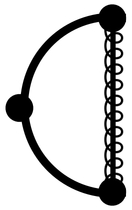
$\gamma_1 =$



$$= \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$$

$$\gamma_1^{\lambda} := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k-\lambda p_1)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2))(k^2 - m_{\mathrm{UV}}^2)((k+\lambda p_2)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2)\ )}$$

$$\mathrm{T}(\gamma) = \mathrm{T}_{\mathrm{dod}(\gamma)}(\gamma^{\lambda}) = \sum_{j=0}^{\mathrm{dod}(\gamma)} \frac{1}{j!} \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} \gamma^{\lambda} \bigg|_{\lambda=0}, \qquad \mathrm{T}_0(\gamma_1) = \frac{\mathcal{N}(k,0,0,0)}{(k^2 - m_{\mathrm{UV}}^2)^3} \sim$$

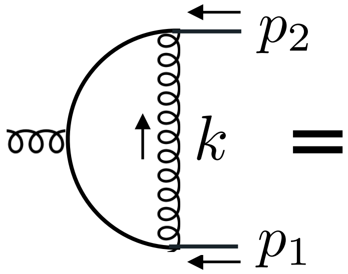


LOCAL RENORMALISATION OPERATOR K

Our solution: $K(\gamma)$:= $T(\gamma)$ - $[T](\gamma)$

$T(\gamma)$:= **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

γ_1 =

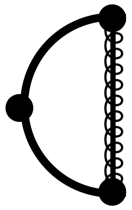


$$= \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$$

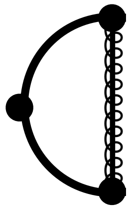
$$\gamma_1^{\lambda} := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k-\lambda p_1)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2))(k^2 - m_{\mathrm{UV}}^2)((k+\lambda p_2)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2)\ )}$$

$$T(\gamma) = T_{\mathrm{dod}(\gamma)}(\gamma^{\lambda}) = \sum_{j=0}^{\mathrm{dod}(\gamma)} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\gamma^{\lambda}\Big|_{\lambda=0},$$

$$T_0(\gamma_1) = \frac{\mathcal{N}(k,0,0,0)}{(k^2 - m_{\mathrm{UV}}^2)^3} \sim$$



$[T](\gamma)$:= **Integrated CT** , $[T](\gamma_1) = \left(\frac{\mu_r^2}{4\pi e^{-\gamma_{\mathrm{E}}}}\right)^{\epsilon} \int d^{4-2\epsilon}k$

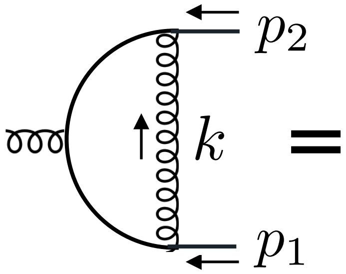


$$= \sum_{k=-\infty}^{+\infty} \alpha_k \epsilon^k$$

LOCAL RENORMALISATION OPERATOR K

Our solution: $K(\gamma) := T(\gamma) - [T](\gamma) + \delta^{\overline{\text{MS}}+\text{OS}}(\gamma)$

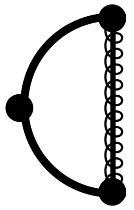
$T(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\text{dod}(\gamma)$



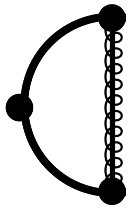
$$\gamma_1 = \text{[diagram]} = \frac{\mathcal{N}_{\gamma_1}(k,p_1,p_2,m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$$

$$\gamma_1^{\lambda} := \frac{\mathcal{N}_{\gamma_1}(k,\lambda p_1,\lambda p_2,\lambda m)}{(k-\lambda p_1)^2 - m_{\text{UV}}^2 - \lambda^2(m^2 - m_{\text{UV}}^2))(k^2 - m_{\text{UV}}^2)((k+\lambda p_2)^2 - m_{\text{UV}}^2 - \lambda^2(m^2 - m_{\text{UV}}^2)\ )}$$

$$T(\gamma) = T_{\text{dod}(\gamma)}(\gamma^{\lambda}) = \sum_{j=0}^{\text{dod}(\gamma)} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\gamma^{\lambda}\Big|_{\lambda=0}, \qquad T_0(\gamma_1) = \frac{\mathcal{N}(k,0,0,0)}{(k^2 - m_{\text{UV}}^2)^3} \sim$$



$[T](\gamma) :=$ **Integrated CT** , $[T](\gamma_1) = \left(\frac{\mu_r^2}{4\pi e^{-\gamma_{\text{E}}}}\right)^{\epsilon} \int d^{4-2\epsilon}k$

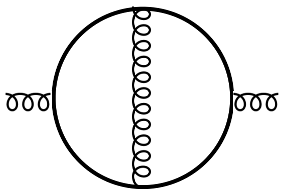


$$= \sum_{k=-\infty}^{+\infty} \alpha_k \epsilon^k$$

$\delta^{X}(\gamma) :=$ **Renormalisation CT** in scheme X , $(-[T] + \delta^{\overline{\text{MS}}}) := \bar{K}$, $\bar{K}(\gamma_1) = \sum_{k=0}^{+\infty} \alpha_k \epsilon^k$

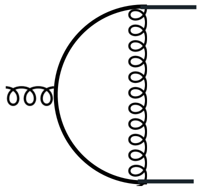
R-OPERATOR UNFOLDING

Γ =

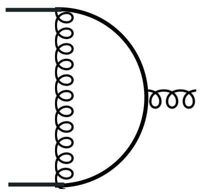


with UV subgraphs

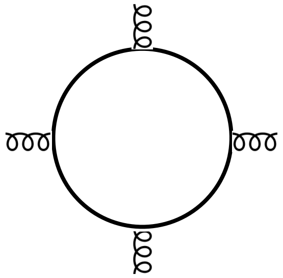
γ_1 =



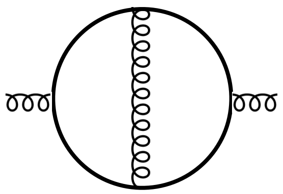
γ_2 =



γ_3 =



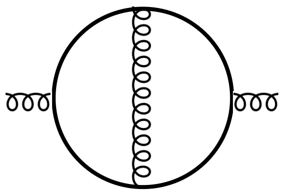
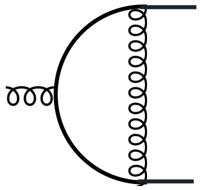
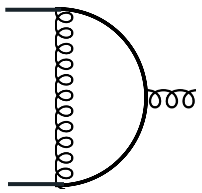
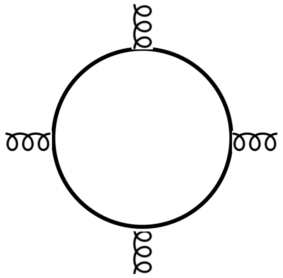
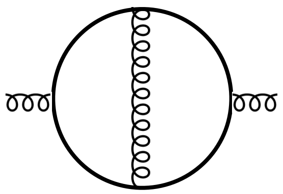
γ_4 =



$$\mathrm{R}(\Gamma) = \Gamma - \mathrm{T}_0(\gamma_1) * \Gamma\backslash\gamma_1 - \mathrm{T}_0(\gamma_2) * \Gamma\backslash\gamma_2 - \mathrm{T}_0(\gamma_3) * \Gamma\backslash\gamma_3 - \mathrm{T}_2(\gamma_4) * \Gamma\backslash\gamma_4$$

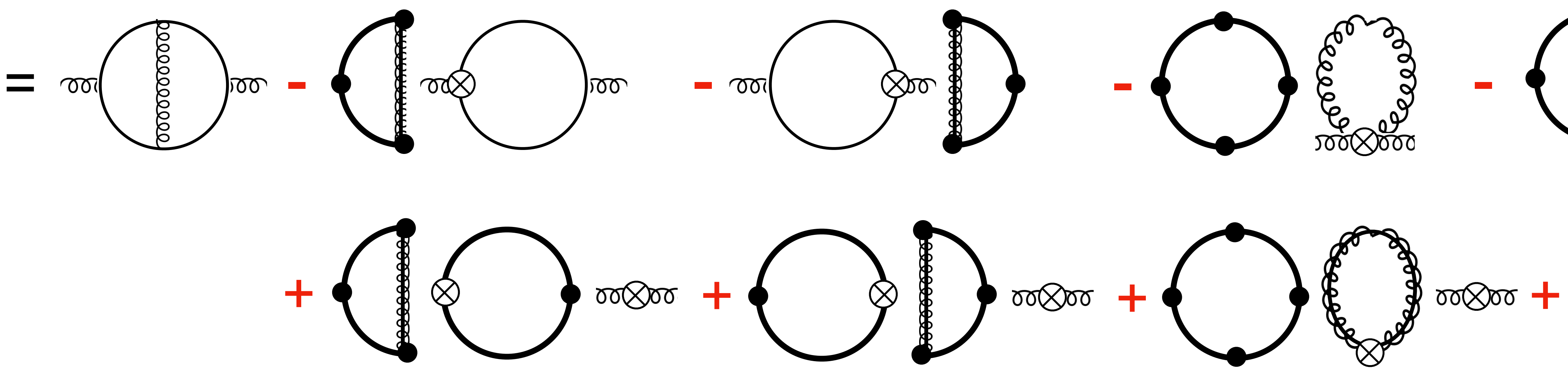
$$+ \mathrm{T}_2(\mathrm{T}_0(\gamma_1) * \Gamma\backslash\gamma_1) + \mathrm{T}_2(\mathrm{T}_0(\gamma_2) * \Gamma\backslash\gamma_2) + \mathrm{T}_2(\mathrm{T}_0(\gamma_3) * \Gamma\backslash\gamma_3) + \bar{K} \text{ terms}$$

R-OPERATOR UNFOLDING

Γ =  with UV subgraphs γ_1 =  γ_2 =  γ_3 =  γ_4 = 

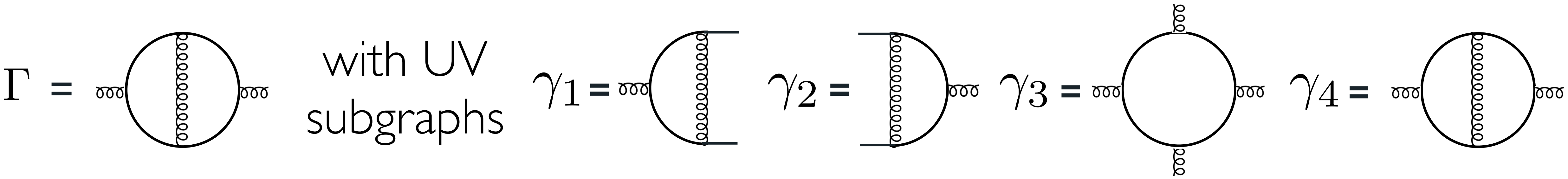
$$R(\Gamma) = \Gamma - T_0(\gamma_1) * \Gamma\backslash\gamma_1 - T_0(\gamma_2) * \Gamma\backslash\gamma_2 - T_0(\gamma_3) * \Gamma\backslash\gamma_3 - T_2(\gamma_4) * \Gamma\backslash\gamma_4$$

$$+ T_2(T_0(\gamma_1) * \Gamma\backslash\gamma_1) + T_2(T_0(\gamma_2) * \Gamma\backslash\gamma_2) + T_2(T_0(\gamma_3) * \Gamma\backslash\gamma_3) + \bar{K} \text{ terms}$$



+ $\bar{K}$ terms

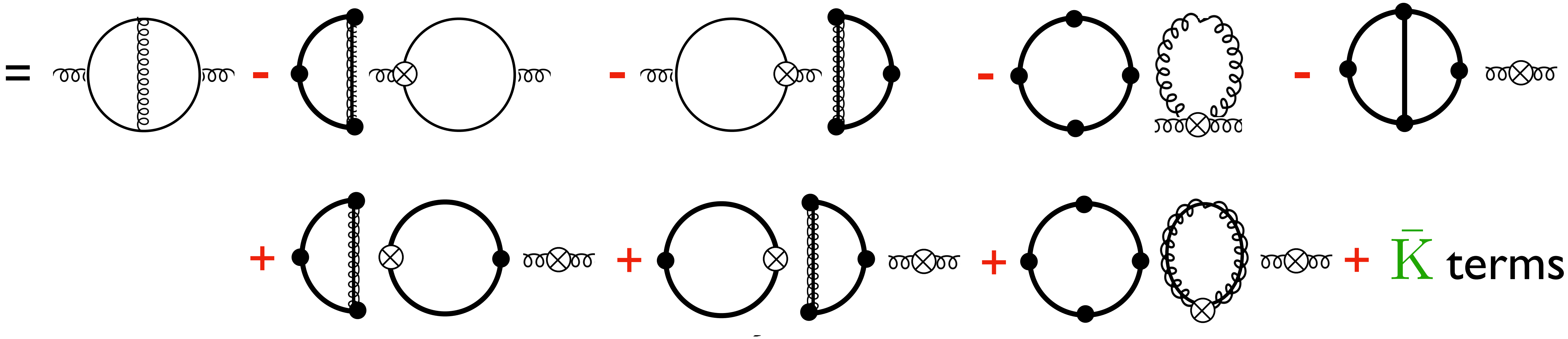
R-OPERATOR UNFOLDING



Γ = with UV subgraphs γ_1 = γ_2 = γ_3 = γ_4 =

$$R(\Gamma) = \Gamma - T_0(\gamma_1)*\Gamma\backslash\gamma_1 - T_0(\gamma_2)*\Gamma\backslash\gamma_2 - T_0(\gamma_3)*\Gamma\backslash\gamma_3 - T_2(\gamma_4)*\Gamma\backslash\gamma_4$$

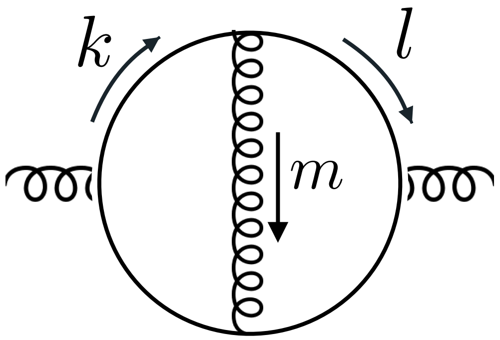
$$+T_2(T_0(\gamma_1)*\Gamma\backslash\gamma_1) + T_2(T_0(\gamma_2)*\Gamma\backslash\gamma_2) + T_2(T_0(\gamma_3)*\Gamma\backslash\gamma_3) + \bar{K}\text{ terms}$$



= − − − −

+ + + + $\bar{K}$ terms

The four different types of UV limits are now **finite** !



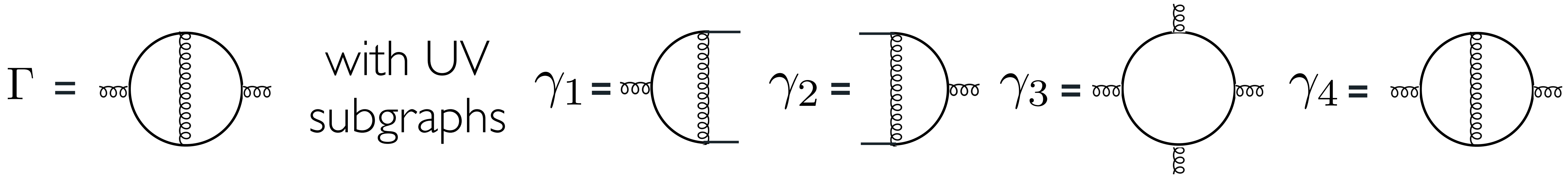
$k, m \to \infty$, l finite

$l, m \to \infty$, k finite

$k, l \to \infty$, m finite

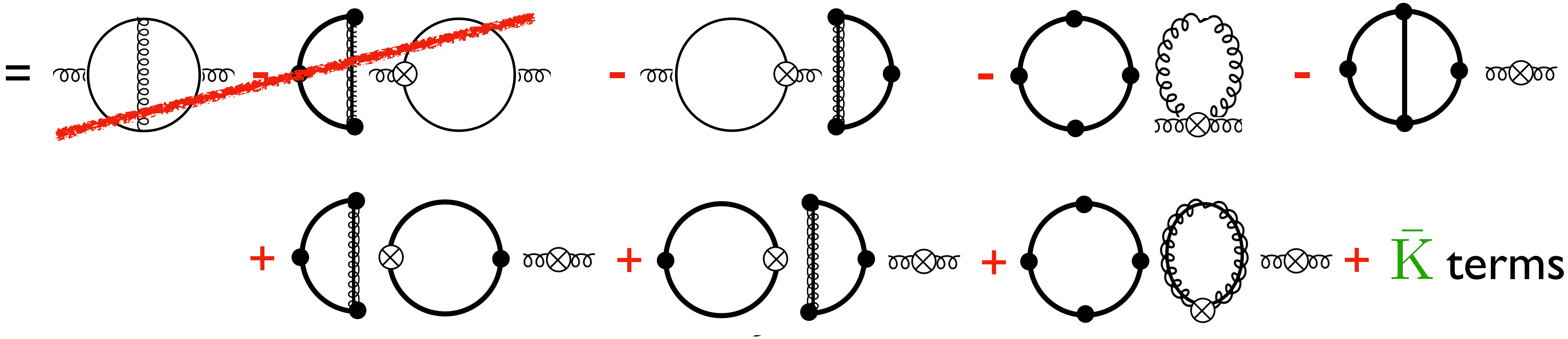
$k, l, m \to \infty$

R-OPERATOR UNFOLDING

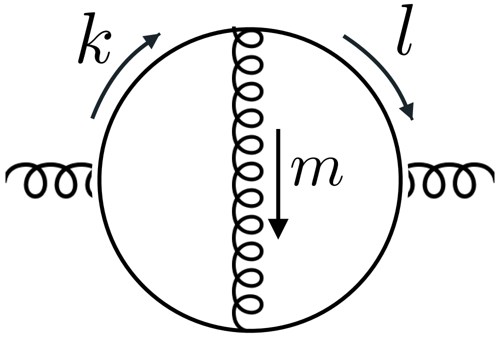


$$R(\Gamma) = \Gamma - T_0(\gamma_1)*\Gamma\backslash\gamma_1 - T_0(\gamma_2)*\Gamma\backslash\gamma_2 - T_0(\gamma_3)*\Gamma\backslash\gamma_3 - T_2(\gamma_4)*\Gamma\backslash\gamma_4$$

$$+T_2(T_0(\gamma_1)*\Gamma\backslash\gamma_1) + T_2(T_0(\gamma_2)*\Gamma\backslash\gamma_2) + T_2(T_0(\gamma_3)*\Gamma\backslash\gamma_3) + \bar{K} \text{ terms}$$



The four different types of UV limits are now **finite** !



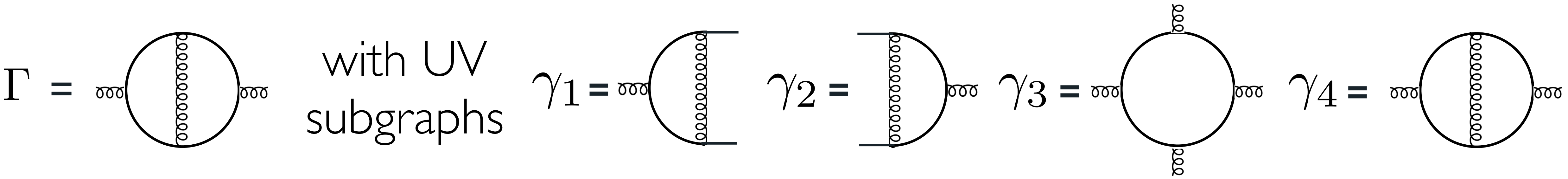
$$\boxed{k, m \rightarrow \infty \ , l \ \text{finite}}$$

$l, m \rightarrow \infty$, k finite

$k, l \rightarrow \infty$, m finite

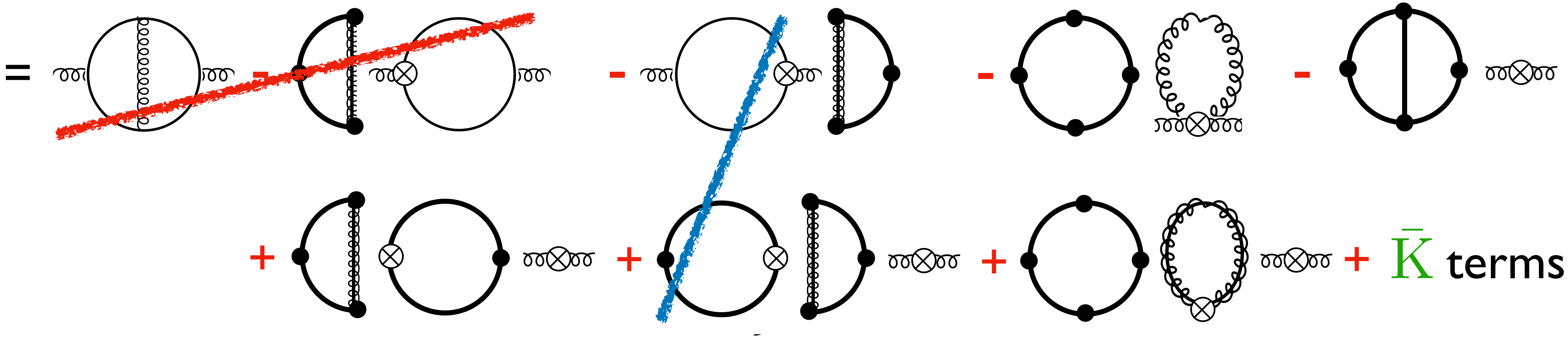
$k, l, m \rightarrow \infty$

R-OPERATOR UNFOLDING



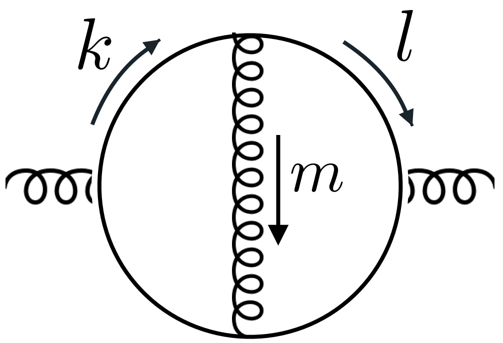
$$R(\Gamma) = \Gamma - T_0(\gamma_1)*\Gamma\backslash\gamma_1 - T_0(\gamma_2)*\Gamma\backslash\gamma_2 - T_0(\gamma_3)*\Gamma\backslash\gamma_3 - T_2(\gamma_4)*\Gamma\backslash\gamma_4$$

$$+T_2\big(T_0(\gamma_1)*\Gamma\backslash\gamma_1\big) + T_2\big(T_0(\gamma_2)*\Gamma\backslash\gamma_2\big) + T_2\big(T_0(\gamma_3)*\Gamma\backslash\gamma_3\big) + \bar{K}\text{ terms}$$



$+ \bar{K}$ terms

The four different types of UV limits are now **finite** !



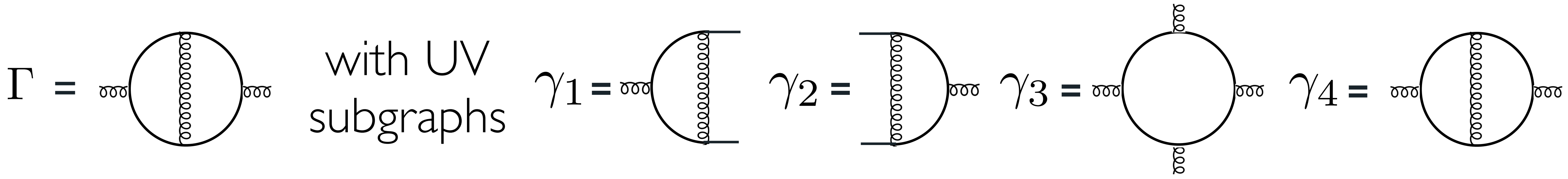
$\boxed{k, m \to \infty\ , l\ \text{ finite}}$

$l, m \to \infty\ , k\ \text{ finite}$

$k, l \to \infty\ , m\ \text{finite}$

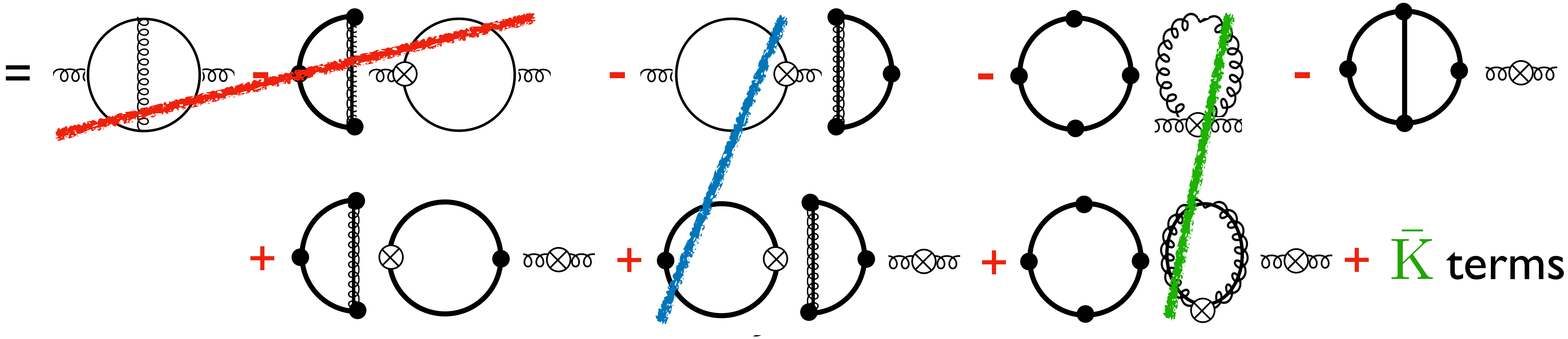
$k, l, m \to \infty$

R-OPERATOR UNFOLDING

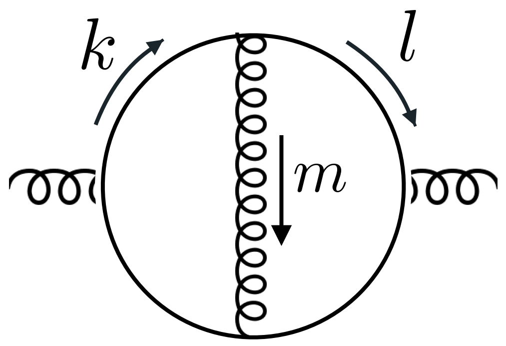


$$R(\Gamma) = \Gamma - T_0(\gamma_1)*\Gamma\backslash\gamma_1 - T_0(\gamma_2)*\Gamma\backslash\gamma_2 - T_0(\gamma_3)*\Gamma\backslash\gamma_3 - T_2(\gamma_4)*\Gamma\backslash\gamma_4$$

$$+T_2(T_0(\gamma_1)*\Gamma\backslash\gamma_1) + T_2(T_0(\gamma_2)*\Gamma\backslash\gamma_2) + T_2(T_0(\gamma_3)*\Gamma\backslash\gamma_3) + \bar{K} \text{ terms}$$



The four different types of UV limits are now **finite** !



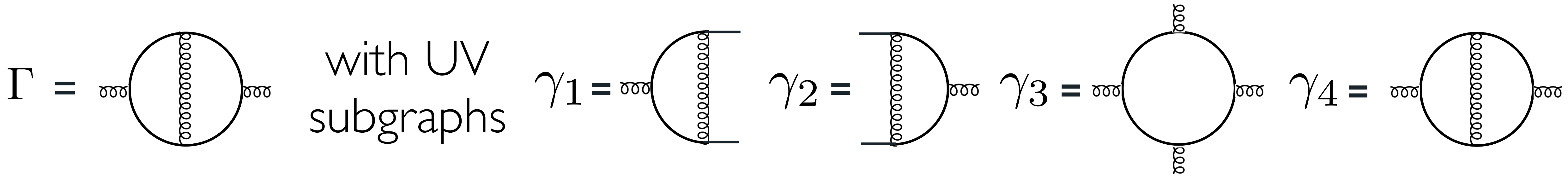
$\boxed{k, m \to \infty \;, l \;\; \text{finite}}$

$l, m \to \infty \;, k \;\; \text{finite}$

$k, l \to \infty \;, m \; \text{finite}$

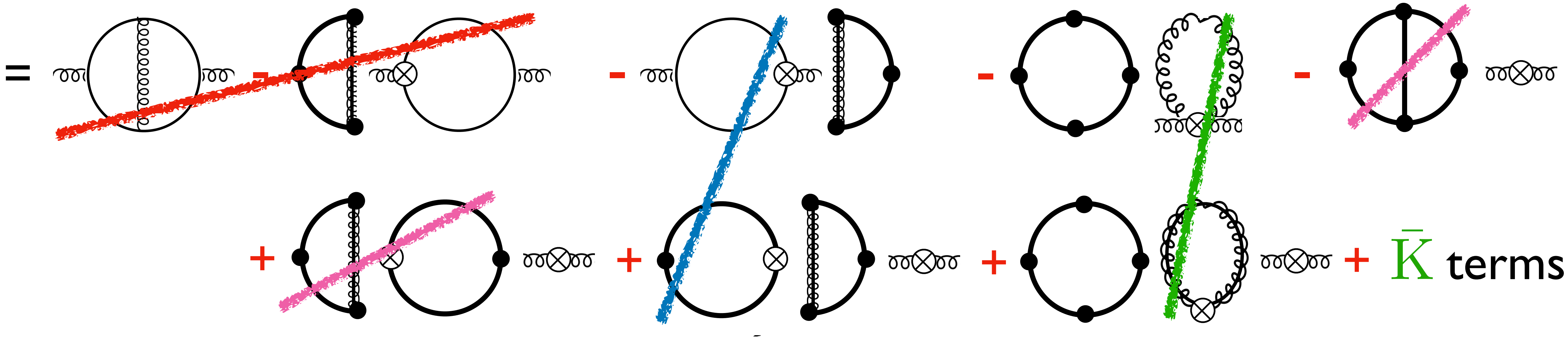
$k, l, m \to \infty$

R-OPERATOR UNFOLDING

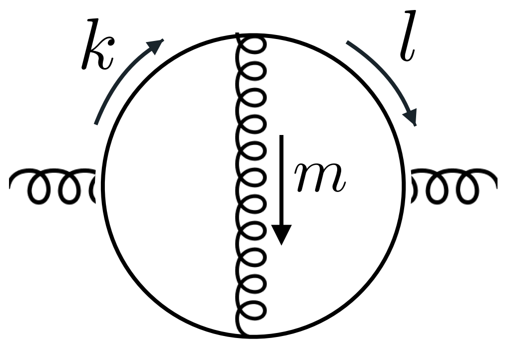


$$R(\Gamma) = \Gamma - T_0(\gamma_1)*\Gamma\backslash\gamma_1 - T_0(\gamma_2)*\Gamma\backslash\gamma_2 - T_0(\gamma_3)*\Gamma\backslash\gamma_3 - T_2(\gamma_4)*\Gamma\backslash\gamma_4$$

$$+T_2(T_0(\gamma_1)*\Gamma\backslash\gamma_1) + T_2(T_0(\gamma_2)*\Gamma\backslash\gamma_2) + T_2(T_0(\gamma_3)*\Gamma\backslash\gamma_3) + \bar{K}\ \text{terms}$$



The four different types of UV limits are now **finite** !



$\boxed{k, m \to \infty \ , l \ \text{finite}}$

$l, m \to \infty \ , k \ \text{finite}$

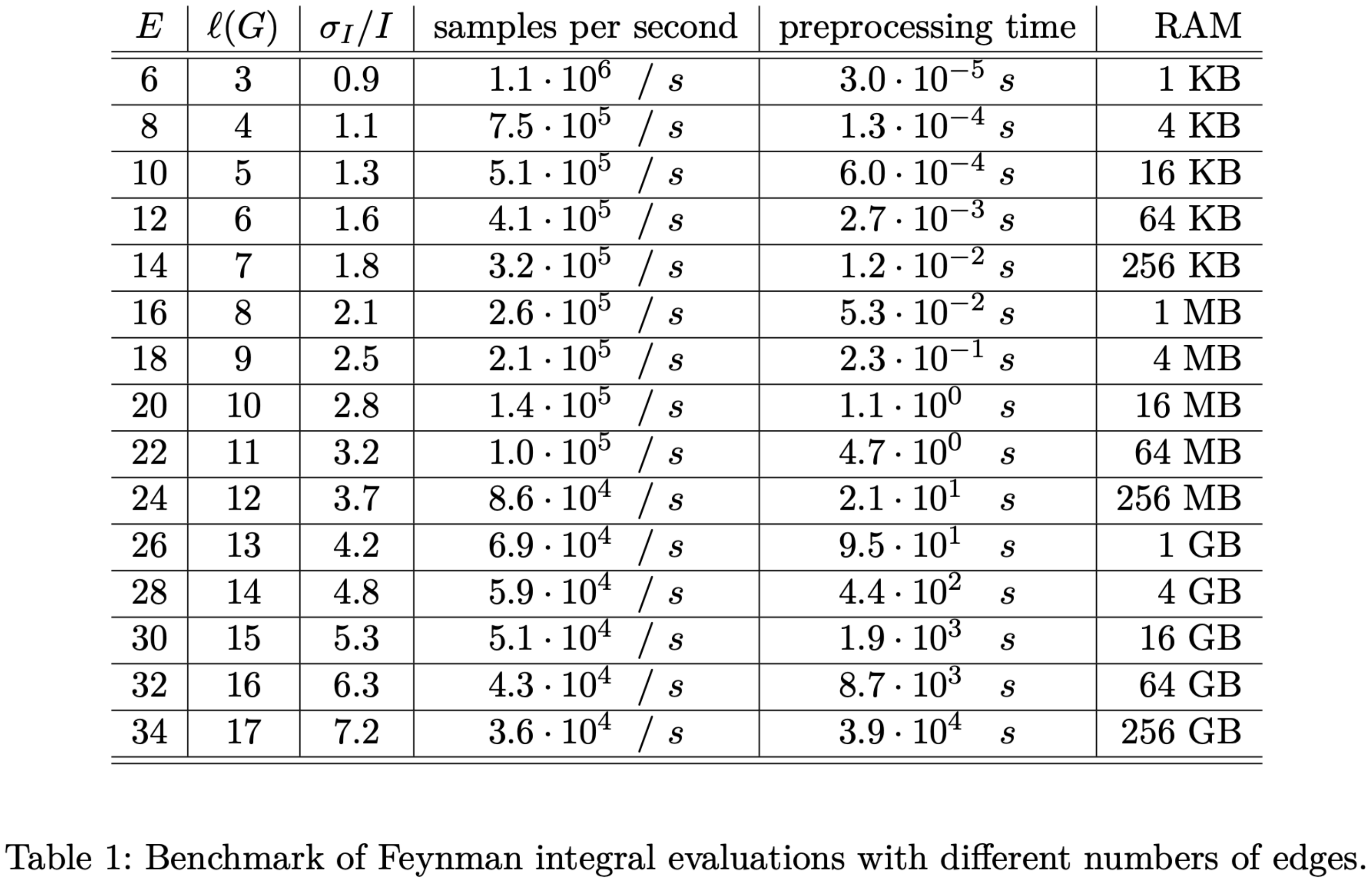
$k, l \to \infty \ , m \ \text{finite}$

$k, l, m \to \infty$

TROPICAL SAMPLING

FOR TAMING INTEGRABLE SINGULARITIES

TROPICAL SAMPLING OF EUCLIDEAN FEYNMAN INTEGRALS

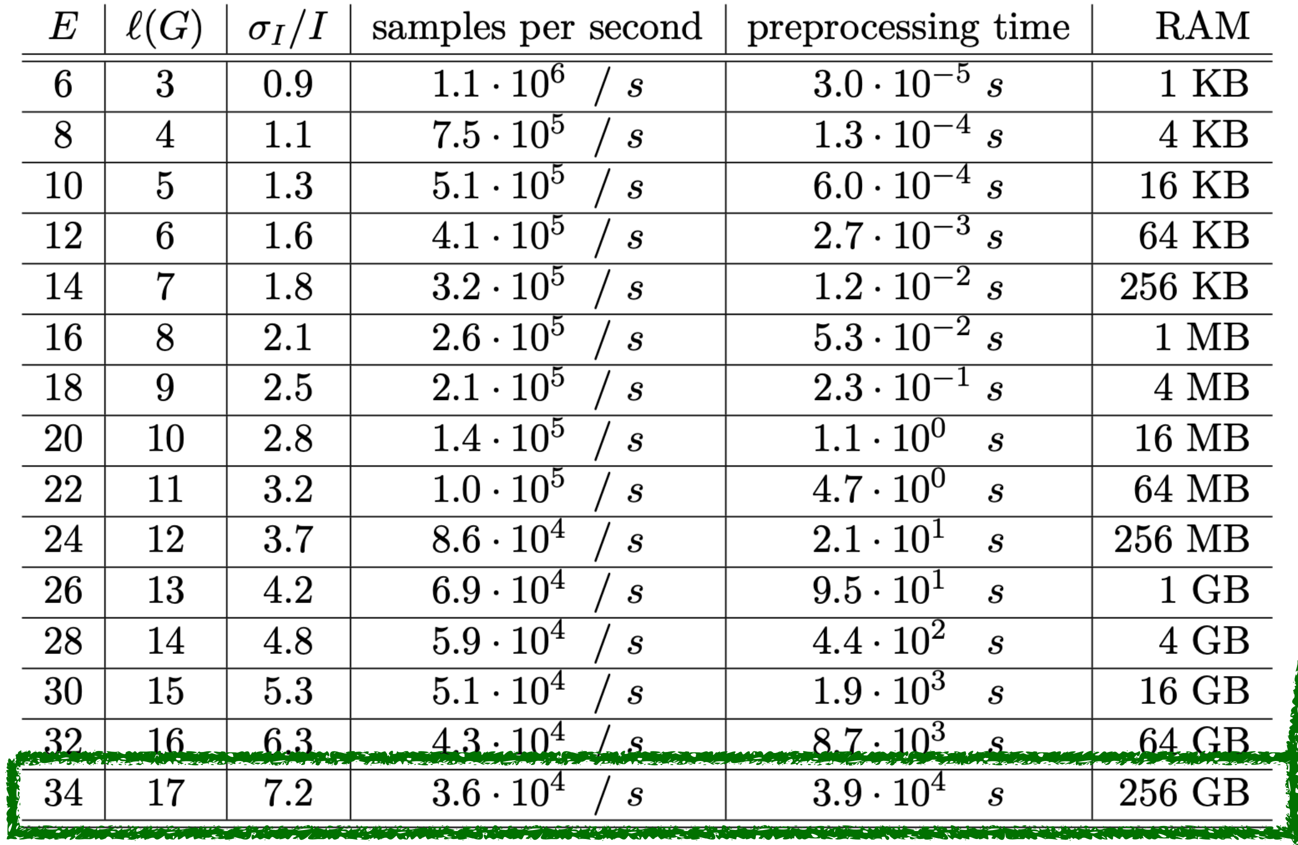


E	$\ell(G)$	σ_I/I	samples per second	preprocessing time	RAM
6	3	0.9	$1.1 \cdot 10^{6}\ / s$	$3.0 \cdot 10^{-5}\ s$	1 KB
8	4	1.1	$7.5 \cdot 10^{5}\ / s$	$1.3 \cdot 10^{-4}\ s$	4 KB
10	5	1.3	$5.1 \cdot 10^{5}\ / s$	$6.0 \cdot 10^{-4}\ s$	16 KB
12	6	1.6	$4.1 \cdot 10^{5}\ / s$	$2.7 \cdot 10^{-3}\ s$	64 KB
14	7	1.8	$3.2 \cdot 10^{5}\ / s$	$1.2 \cdot 10^{-2}\ s$	256 KB
16	8	2.1	$2.6 \cdot 10^{5}\ / s$	$5.3 \cdot 10^{-2}\ s$	1 MB
18	9	2.5	$2.1 \cdot 10^{5}\ / s$	$2.3 \cdot 10^{-1}\ s$	4 MB
20	10	2.8	$1.4 \cdot 10^{5}\ / s$	$1.1 \cdot 10^{0}\ s$	16 MB
22	11	3.2	$1.0 \cdot 10^{5}\ / s$	$4.7 \cdot 10^{0}\ s$	64 MB
24	12	3.7	$8.6 \cdot 10^{4}\ / s$	$2.1 \cdot 10^{1}\ s$	256 MB
26	13	4.2	$6.9 \cdot 10^{4}\ / s$	$9.5 \cdot 10^{1}\ s$	1 GB
28	14	4.8	$5.9 \cdot 10^{4}\ / s$	$4.4 \cdot 10^{2}\ s$	4 GB
30	15	5.3	$5.1 \cdot 10^{4}\ / s$	$1.9 \cdot 10^{3}\ s$	16 GB
32	16	6.3	$4.3 \cdot 10^{4}\ / s$	$8.7 \cdot 10^{3}\ s$	64 GB
34	17	7.2	$3.6 \cdot 10^{4}\ / s$	$3.9 \cdot 10^{4}\ s$	256 GB

Table 1: Benchmark of Feynman integral evaluations with different numbers of edges.

[M. Borinsky, arXiv : 2008.12310]

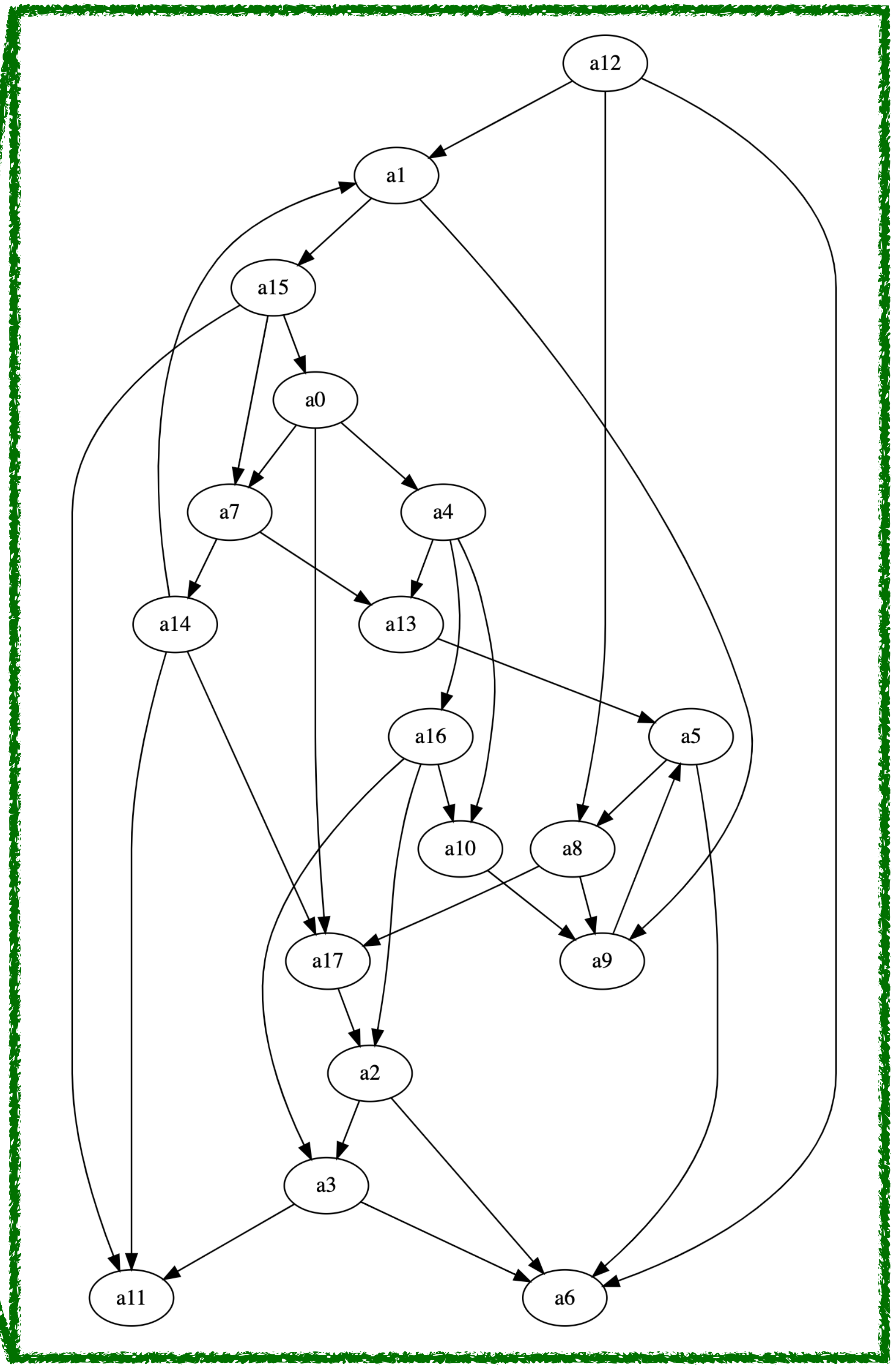
TROPICAL SAMPLING OF EUCLIDEAN FEYNMAN INTEGRALS



E	$\ell(G)$	σ_I/I	samples per second	preprocessing time	RAM
6	3	0.9	$1.1 \cdot 10^6 \;/\; s$	$3.0 \cdot 10^{-5}\; s$	1 KB
8	4	1.1	$7.5 \cdot 10^5 \;/\; s$	$1.3 \cdot 10^{-4}\; s$	4 KB
10	5	1.3	$5.1 \cdot 10^5 \;/\; s$	$6.0 \cdot 10^{-4}\; s$	16 KB
12	6	1.6	$4.1 \cdot 10^5 \;/\; s$	$2.7 \cdot 10^{-3}\; s$	64 KB
14	7	1.8	$3.2 \cdot 10^5 \;/\; s$	$1.2 \cdot 10^{-2}\; s$	256 KB
16	8	2.1	$2.6 \cdot 10^5 \;/\; s$	$5.3 \cdot 10^{-2}\; s$	1 MB
18	9	2.5	$2.1 \cdot 10^5 \;/\; s$	$2.3 \cdot 10^{-1}\; s$	4 MB
20	10	2.8	$1.4 \cdot 10^5 \;/\; s$	$1.1 \cdot 10^{0}\; s$	16 MB
22	11	3.2	$1.0 \cdot 10^5 \;/\; s$	$4.7 \cdot 10^{0}\; s$	64 MB
24	12	3.7	$8.6 \cdot 10^4 \;/\; s$	$2.1 \cdot 10^{1}\; s$	256 MB
26	13	4.2	$6.9 \cdot 10^4 \;/\; s$	$9.5 \cdot 10^{1}\; s$	1 GB
28	14	4.8	$5.9 \cdot 10^4 \;/\; s$	$4.4 \cdot 10^{2}\; s$	4 GB
30	15	5.3	$5.1 \cdot 10^4 \;/\; s$	$1.9 \cdot 10^{3}\; s$	16 GB
32	16	6.3	$4.3 \cdot 10^4 \;/\; s$	$8.7 \cdot 10^{3}\; s$	64 GB
34	17	7.2	$3.6 \cdot 10^4 \;/\; s$	$3.9 \cdot 10^{4}\; s$	256 GB

Table 1: Benchmark of Feynman integral evaluations with different numbers of edges.

[M. Borinsky, arXiv : 2008.12310]

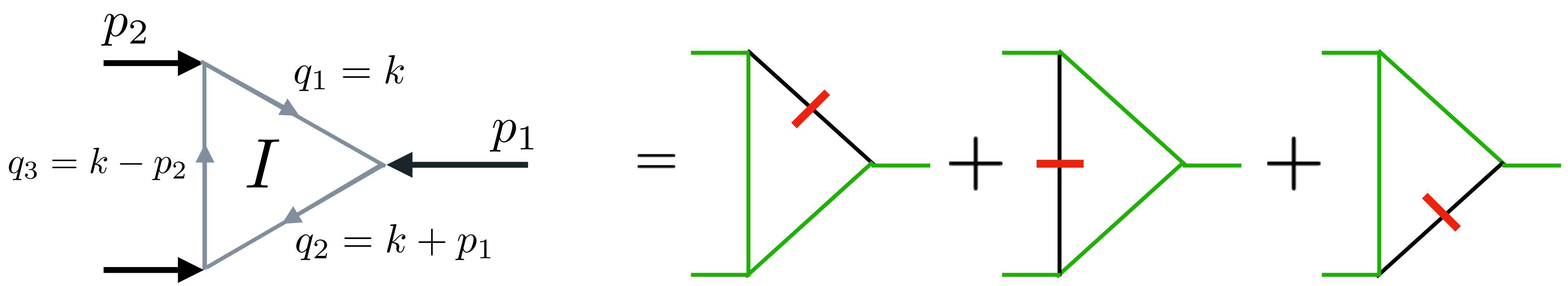


TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

[Inspired from and in collaboration with M. Borinsky, arXiv : 2008.12310]

[Investigation from **Mathijs Fraije**]

Recall:

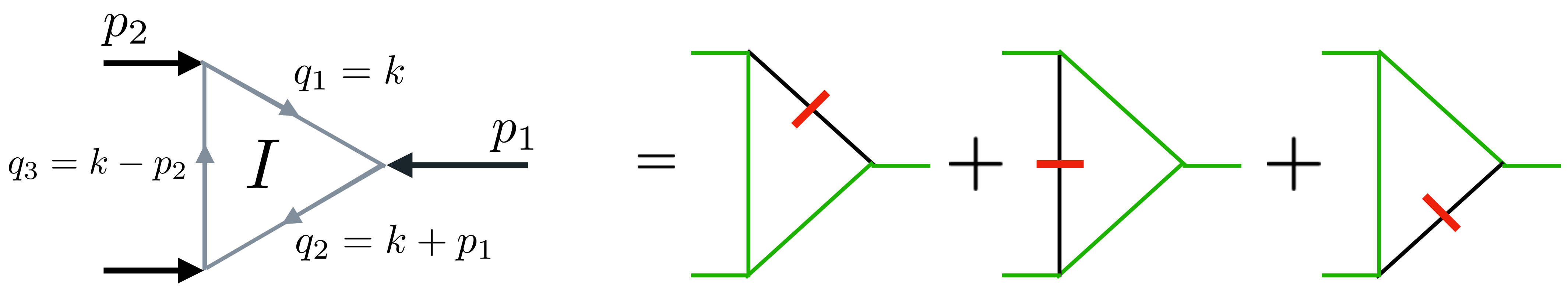


TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

[Inspired from and in collaboration with M. Borinsky, arXiv : 2008.12310]

[Investigation from **Mathijs Fraije**]

Recall:



$$= \int \mathrm{d}^4k \frac{1}{D_1D_2D_3}(D_1\delta^{(+)}(D_1) + D_2\delta^{(+)}(D_2) + D_3\delta^{(+)}(D_3)) \supset \int d^3\vec{k}\frac{1}{E_1E_2E_3}\frac{1}{E_1+E_2-p_1^0}\frac{1}{E_1+E_2+p_1^0}$$

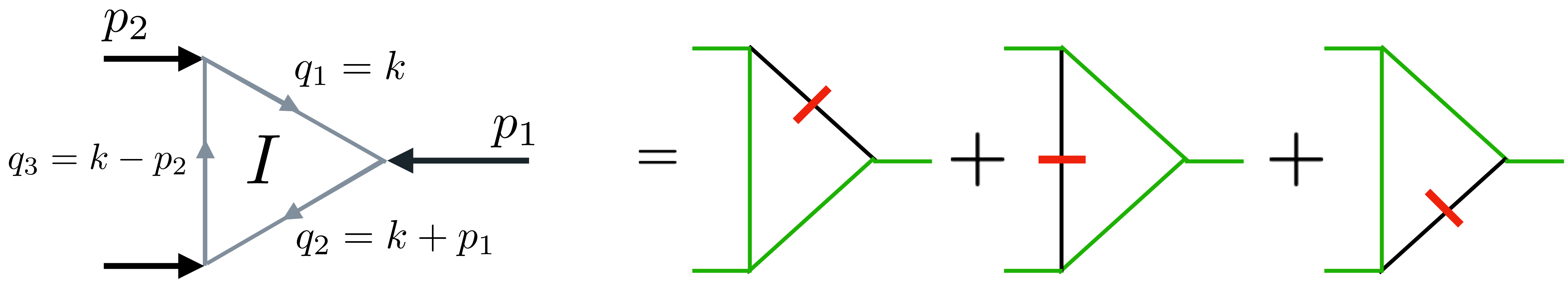
With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i\cdot\vec{q}_i + m_i^2} \overset{m=0}{=} |\vec{q}_i|$.

TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

[Inspired from and in collaboration with M. Borinsky, arXiv : 2008.12310]

[Investigation from **Mathijs Fraije**]

Recall:

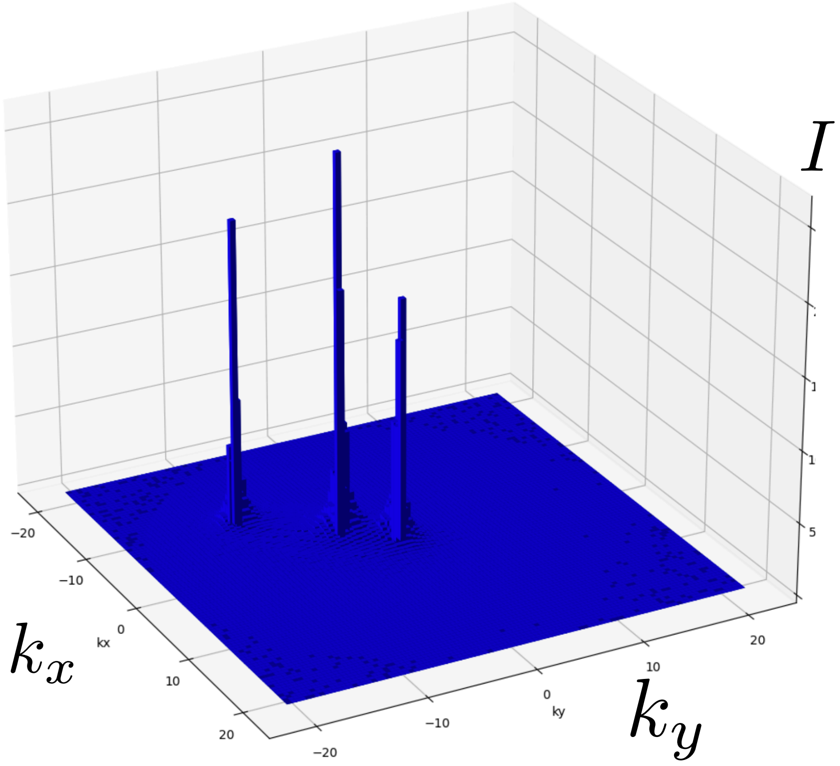


$$= \int \mathrm{d}^4k \frac{1}{D_1D_2D_3}(D_1\delta^{(+)}(D_1) + D_2\delta^{(+)}(D_2) + D_3\delta^{(+)}(D_3)) \supset \int d^3\vec{k} \frac{1}{E_1E_2E_3} \frac{1}{E_1 + E_2 - p_1^0} \frac{1}{E_1 + E_2 + p_1^0}$$

With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \overset{m=0}{=} |\vec{q}_i|$. The prefactor $\frac{1}{E_1E_2E_3}$ contains point-like integrable singularities :

$$\frac{1}{E_1E_2E_3} = \frac{1}{|\vec{k}||\vec{k}+\vec{p}_1||\vec{k}-\vec{p}_2|}$$

2D example:



TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

- Multi-channeling is the canonical approach, but better would be to build a parameterisation whose Jacobian vanishes at *all three points* $\vec{k} = \left\{\vec{0}, -\vec{p}_1, \vec{p}_2\right\}$. This is hard!

TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

- Multi-channeling is the canonical approach, but better would be to build a parameterisation whose Jacobian vanishes at *all three points* $\vec{k} = \left\{\vec{0}, -\vec{p}_1, \vec{p}_2\right\}$. This is hard!
- Tropical sampling idea: let $p(x,y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\mathrm{tr}}(x,y) = \max\left\{x^2, y^3, xy\right\}$
- Then one can show in general that the ratio $\dfrac{p^{\mathrm{tr}}}{p}$ is bounded from below and above.

TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

- Multi-channeling is the canonical approach, but better would be to build a parameterisation whose Jacobian vanishes at *all three points* $\vec{k} = \left\{\vec{0}, -\vec{p}_1, \vec{p}_2\right\}$. This is hard!
- Tropical sampling idea: let $p(x,y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\mathrm{tr}}(x,y) = \max\left\{x^2, y^3, xy\right\}$
- Then one can show in general that the ratio $\frac{p^{\mathrm{tr}}}{p}$ is bounded from below and above.
- Moreover, a systematic algorithm exists to sample according to p^{tr} (note:. sampling dimensionality increases)

TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

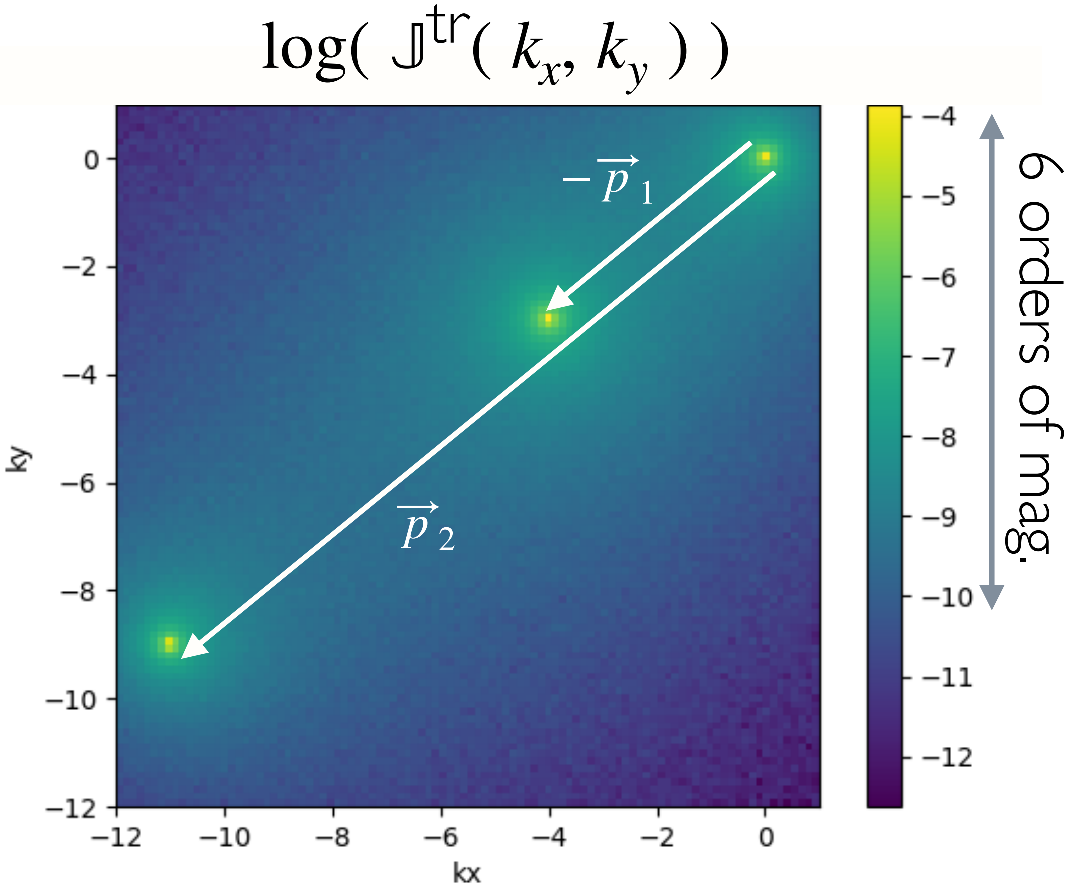
- Multi-channeling is the canonical approach, but better would be to build a parameterisation whose Jacobian vanishes at *all three points* $\vec{k} = \left\{\vec{0}, -\vec{p}_1, \vec{p}_2\right\}$. This is hard!
- Tropical sampling idea: let $p(x,y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\mathrm{tr}}(x,y) = \max\left\{x^2, y^3, xy\right\}$
- Then one can show in general that the ratio $\frac{p^{\mathrm{tr}}}{p}$ is bounded from below and above.
- Moreover, a systematic algorithm exists to sample according to p^{tr} (note:. sampling dimensionality increases)

Let's apply this approach to our example, in 2D, with a tropical sampling for removing $p := \frac{1}{|\vec{k}||\vec{k}+\vec{p}_1||\vec{k}-\vec{p}_2|}$

TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

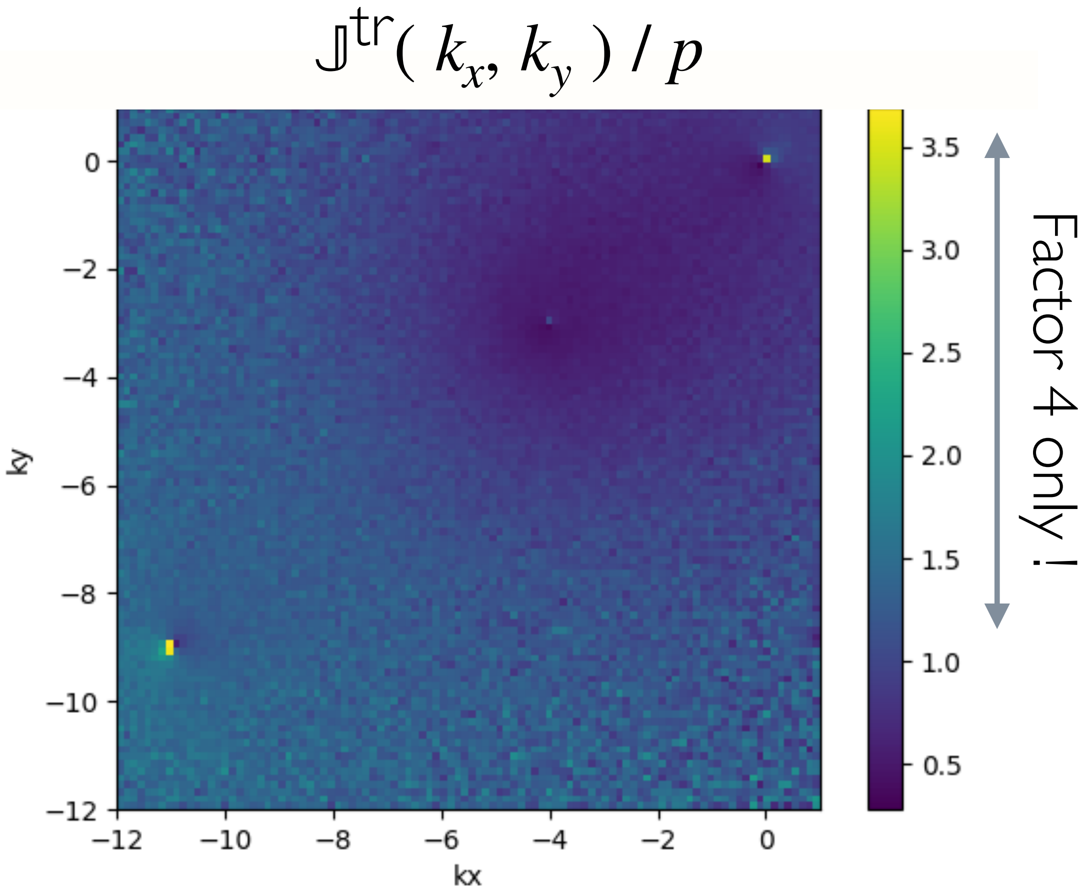
- Multi-channeling is the canonical approach, but better would be to build a parameterisation whose Jacobian vanishes at *all three points* $\vec{k} = \left\{\vec{0}, -\vec{p}_1, \vec{p}_2\right\}$. This is hard!
- Tropical sampling idea: let $p(x,y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\mathrm{tr}}(x,y) = \max\left\{x^2, y^3, xy\right\}$
- Then one can show in general that the ratio $\frac{p^{\mathrm{tr}}}{p}$ is bounded from below and above.
- Moreover, a systematic algorithm exists to sample according to p^{tr} (note:. sampling dimensionality increases)

Let's apply this approach to our example, in 2D, with a tropical sampling for removing $p := \frac{1}{|\vec{k}||\vec{k}+\vec{p}_1||\vec{k}-\vec{p}_2|}$



$\log(\, \mathbb{J}^{\mathrm{tr}}(\, k_x, k_y\,)\,)$

Log of tropical ~sampling density

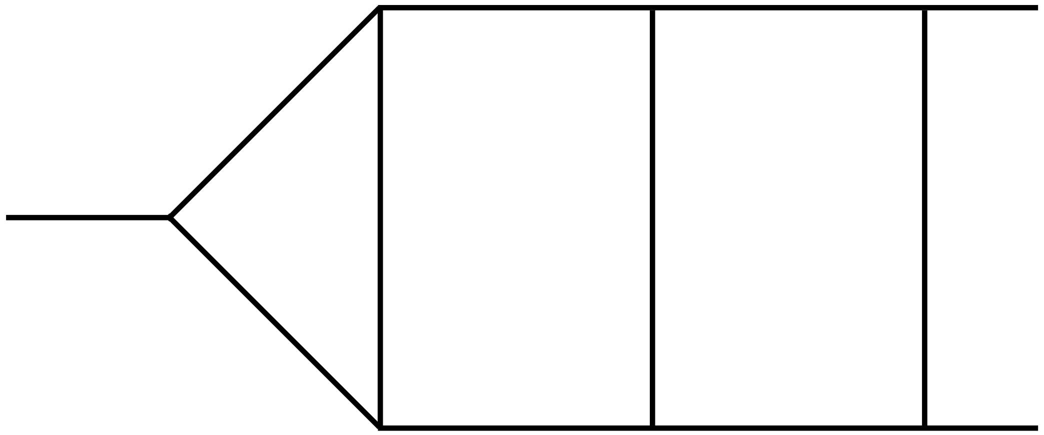


$\mathbb{J}^{\mathrm{tr}}(\, k_x, k_y\,)\, /\, p$

Integrable singularities conquered !

- Many details of this approach omitted here. In our implementation: **arbitrary numerators** supported!

TROPICAL SAMPLING RESULTS

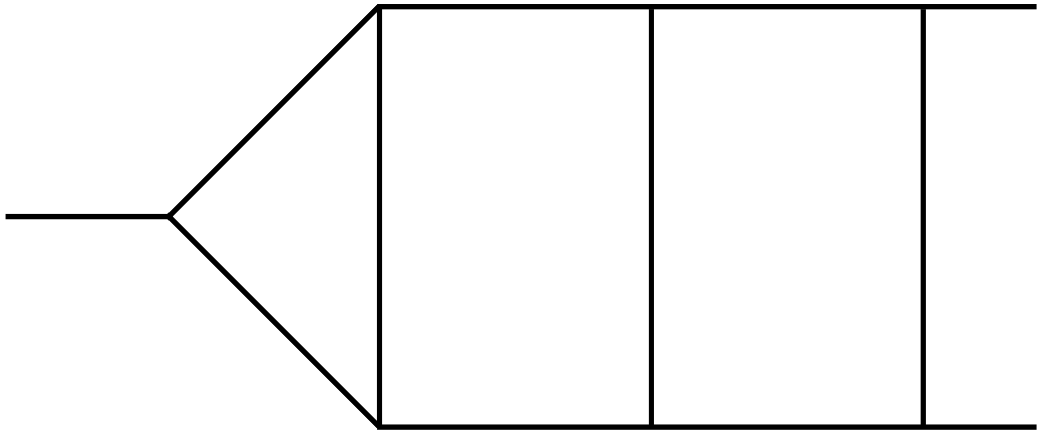


(Denominator powers: $\nu_e = 11/18$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	3.51(63)e-8 **18%**	
0.1M 0.3 s	3.78(24)e-8 **6%**	
1M 3 s	3.99(11)e-8 **2.7%**	
10M 30 s	4.045(36)e-8 **0.9%**	

Credits: Mathijs Fraaije

TROPICAL SAMPLING RESULTS

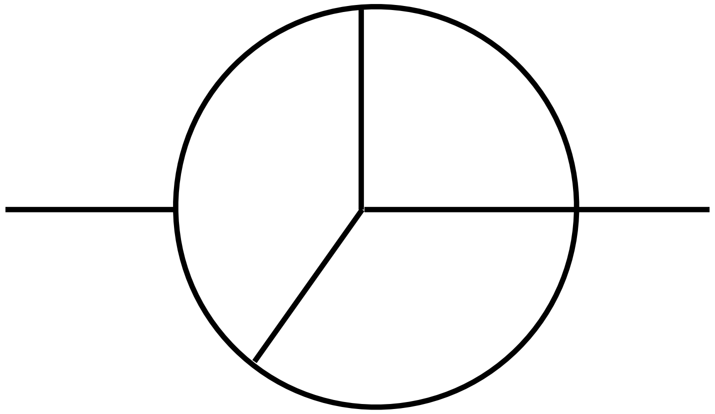


(Denominator powers: $\nu_e = 11/18$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	3.51(63)e-8 **18%**	4.050(35)e-8 **0.9%**
0.1M 0.3 s	3.78(24)e-8 **6%**	4.030(11)e-8 **0.3%**
1M 3 s	3.99(11)e-8 **2.7%**	4.0379(35)e-8 **0.09%**
10M 30 s	4.045(36)e-8 **0.9%**	4.0358(11)e-8 **0.03%**

Credits: Mathijs Fraaije

TROPICAL SAMPLING RESULTS

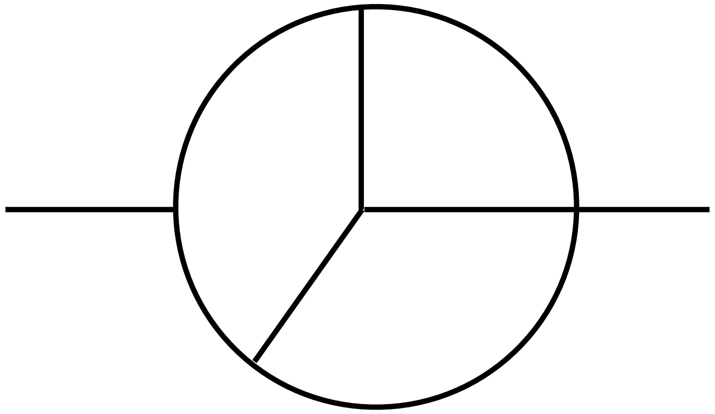


(Denominator powers: $\nu_e = 11/14$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	8.9(2.9)e-7 33%	
0.1M 0.3 s	3.5(1.0)e-6 29%	
1M 3 s	5.6(1.2)e-6 22%	
10M 30 s	1.23(41)e-5 34%	

Credits: Mathijs Fraaije

TROPICAL SAMPLING RESULTS

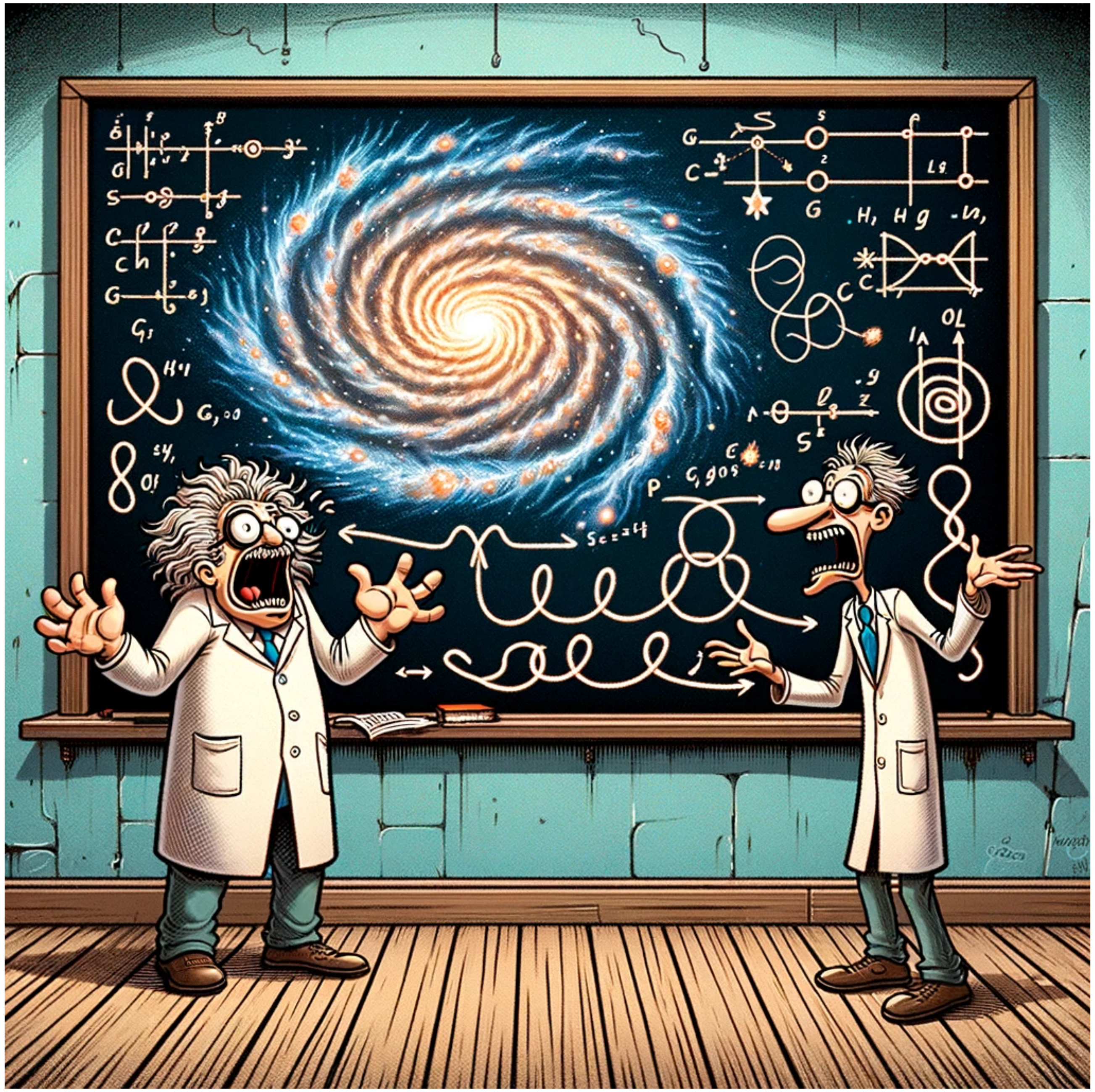


(Denominator powers: $\nu_e = 11/14$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	8.9(2.9)e-7 33%	9.403(60)e-6 0.6%
0.1M 0.3 s	3.5(1.0)e-6 29%	9.518(19)e-6 0.2%
1M 3 s	5.6(1.2)e-6 22%	9.4984(60)e-6 0.06%
10M 30 s	1.23(41)e-5 34%	9.4986(19)e-6 0.02%

Credits: Mathijs Fraaije

We heard: “ Yeah but one-loop is trivial… ”



Why? Many reasons,
but one of them called “OPP”

INTEGRAND REDUCTION

- The integrand (or OPP [Ossola, Papadopoulos, Pittau 2006]) reduction method is a purely numerical algorithm that has been automated in computer codes such as

  CutTools [G.Ossola, C.Papadopoulos, R.Pittau, 0711.3596]
  NINJA [T. Peraro, 1403.1229] (interface to MadLoop in [VH, T. Peraro, 1604.01363]
  SAMURAI [P. Mastrolia, G. Ossola, T. Reiter, F. Tramontano 1006.0710]

  to find the scalar loop coefficients

- Both OPP and Tensor Integral Reduction techniques are interfaced in MadLoop to compute loop diagrams.

How does OPP work?

INTEGRAND LEVEL

- The decomposition to scalar integrals presented before works at the level of the integrals

$$
\begin{aligned}
\mathcal{M}^{\text{1-loop}} = & \sum_{i_0<i_1<i_2<i_3} d_{i_0i_1i_2i_3} \mathrm{Box}_{i_0i_1i_2i_3} \\
& + \sum_{i_0<i_1<i_2} c_{i_0i_1i_2} \mathrm{Triangle}_{i_0i_1i_2} \\
& + \sum_{i_0<i_1} b_{i_0i_1} \mathrm{Bubble}_{i_0i_1} \\
& + \sum_{i_0} a_{i_0} \mathrm{Tadpole}_{i_0} \\
& + R + \mathcal{O}(\epsilon)
\end{aligned}
$$

If we would know a similar relation at the **integrand** level, we would be able to manipulate the integrands and extract the coefficients without doing the integrals

$$
\begin{aligned}
N(l) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \boxed{\tilde{d}_{i_0i_1i_2i_3}(l)} \right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \boxed{\tilde{c}_{i_0i_1i_2}(l)} \right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \boxed{\tilde{b}_{i_0i_1}(l)} \right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \boxed{\tilde{a}_{i_0}(l)} \right] \prod_{i\neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i
\end{aligned}
$$

Spurious term

INTEGRAND LEVEL

- The functional form of the spurious terms is known (it depends on the rank of the integral and the number of propagators in the loop) [del Aguila, Pittau 2004]
  - for example, a box coefficient from a rank 1 numerator is

$$\tilde{d}_{i_0 i_1 i_2 i_3}(l) = \tilde{d}_{i_0 i_1 i_2 i_3}\, \epsilon^{\mu\nu\rho\sigma}\, l^\mu p_1^\nu p_2^\rho p_3^\sigma$$

(remember that p_i is the sum of the momentum that has entered the loop so far, so we always have $p_0 = 0$)

  - The integral is zero

$$\int d^d l \frac{\tilde{d}_{i_0 i_1 i_2 i_3}(l)}{D_0 D_1 D_2 D_3} = \tilde{d}_{i_0 i_1 i_2 i_3} \int d^d l \frac{\epsilon^{\mu\nu\rho\sigma}\, l^\mu p_1^\nu p_2^\rho p_3^\sigma}{D_0 D_1 D_2 D_3} = 0$$

EXAMPLE - BOX COEFFICIENTS

$$N(l^{\pm}) = \left(d_{0123} + \tilde{d}_{0123}(l^{\pm})\right) \prod_{i \neq 0,1,2,3}^{m-1} D_i(l^{\pm})$$

- Two values are enough given the functional form for the spurious term. We can immediately determine the Box coefficient

$$d_{0123} = \frac{1}{2}\left[\frac{N(l^+)}{\prod_{i\neq 0,1,2,3}^{m-1} D_i(l^+)} + \frac{N(l^-)}{\prod_{i\neq 0,1,2,3}^{m-1} D_i(l^-)}\right]$$

- By choosing other values for *l*, that set other combinations of 4 "denominators" to zero, we can get all the Box coefficients

EXAMPLE - BOX COEFFICIENTS

- Compute this integral: $\int d^d l \frac{1}{D_0 D_1 D_2 D_3 D_4 D_5 D_6}$
- So we that the numerator is $N(l) = 1 \quad D_i = (l + p_i)^2 - m_i^2$
- We know that we need only Box, Triangle, Bubble (and Tadpole) contributions. Let's find the first Box integral coefficient.
- Take the two solutions of

$$D_0(l^\pm) = D_1(l^\pm) = D_2(l^\pm) = D_3(l^\pm) = 0$$

- And use the relation we found before and we directly have

$$d_{0123} = \frac{1}{2}\left[\frac{1}{D_4(l^+)D_5(l^+)D_6(l^+)} + \frac{1}{D_4(l^-)D_5(l^-)D_6(l^-)}\right]$$

OPP REDUCTION

$$
\begin{aligned}
N(l) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}(l)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}(l)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} + \tilde{b}_{i_0 i_1}(l)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i \neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i
\end{aligned}
$$

OPP REDUCTION

$$
\begin{aligned}
N(l) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}(l) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}(l) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} + \tilde{b}_{i_0 i_1}(l) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l) \right] \prod_{i \neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i
\end{aligned}
$$

To solve the OPP reduction, choosing special values for the loop momentum helps a lot

OPP REDUCTION

$$
\begin{aligned}
N(l) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \tilde{d}_{i_0i_1i_2i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \tilde{c}_{i_0i_1i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \tilde{b}_{i_0i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i
\end{aligned}
$$

To solve the OPP reduction, choosing special values for the loop momentum helps a lot

For example, choosing *l* such that

$$
\begin{aligned}
D_0(l^{\pm}) = D_1(l^{\pm}) = \\
= D_2(l^{\pm}) = D_3(l^{\pm}) = 0
\end{aligned}
$$

sets all the terms in this equation to zero except the first line

OPP REDUCTION

$$N(l) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \tilde{d}_{i_0i_1i_2i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i$$

$$+ \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \tilde{c}_{i_0i_1i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i$$

$$+ \sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \tilde{b}_{i_0i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i$$

$$+ \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i$$

$$+ \tilde{P}(l) \prod_{i}^{m-1} D_i$$

$$= 0$$

To solve the OPP reduction, choosing special values for the loop momentum helps a lot

For example, choosing *l* such that

$$D_0(l^\pm) = D_1(l^\pm) =$$
$$= D_2(l^\pm) = D_3(l^\pm) = 0$$

sets all the terms in this equation to zero except the first line

OPP REDUCTION

$$N(l) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}(l)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i$$

$$\boxed{\begin{aligned} &+ \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}(l)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\ &+ \sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} + \tilde{b}_{i_0 i_1}(l)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\ &+ \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i \neq i_0}^{m-1} D_i \\ &+ \tilde{P}(l) \prod_{i}^{m-1} D_i \end{aligned}}$$

$$= 0$$

To solve the OPP reduction, choosing special values for the loop momentum helps a lot

For example, choosing *l* such that

$$D_0(l^\pm) = D_1(l^\pm) = \\ = D_2(l^\pm) = D_3(l^\pm) = 0$$

sets all the terms in this equation to zero except the first line

There are two (complex) solutions to this equation due to the quadratic nature of the propagators

OPP REDUCTION

$$\begin{aligned} N(l) = & \boxed{\sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \tilde{d}_{i_0i_1i_2i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i} \\ & + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \tilde{c}_{i_0i_1i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\ & + \sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \tilde{b}_{i_0i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\ & + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i \\ & + \tilde{P}(l) \prod_{i}^{m-1} D_i \end{aligned}$$

$$D_0(l^i) = D_1(l^i) = D_2(l^i) = 0$$

🟪 Coefficient computed in a previous step

OPP REDUCTION

$$
\begin{aligned}
N(l) = & \boxed{\sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}(l)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i} \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}(l)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} + \tilde{b}_{i_0 i_1}(l)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i \neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i
\end{aligned}
$$

Now we choose l such that

$$D_0(l^i) = D_1(l^i) = D_2(l^i) = 0$$

sets all the terms in this equation to zero except the first and second line

🟪 Coefficient computed in a previous step

OPP REDUCTION

$$
\begin{aligned}
N(l) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \tilde{d}_{i_0i_1i_2i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \tilde{c}_{i_0i_1i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \tilde{b}_{i_0i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i \\
& = 0
\end{aligned}
$$

Now we choose l such that

$$D_0(l^i) = D_1(l^i) = D_2(l^i) = 0$$

sets all the terms in this equation to zero except the first and second line

🟪 Coefficient computed in a previous step

OPP REDUCTION

$$
\begin{aligned}
N(l) = & \boxed{\sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \tilde{d}_{i_0i_1i_2i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i} \\
& + \boxed{\sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \tilde{c}_{i_0i_1i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i} \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \tilde{b}_{i_0i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i
\end{aligned}
$$

$$D_0(l^i) = D_1(l^i) = 0$$

🟪 Coefficient computed in a previous step

OPP REDUCTION

$$
\begin{aligned}
N(l) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \tilde{d}_{i_0i_1i_2i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
+ & \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \tilde{c}_{i_0i_1i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
+ & \sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \tilde{b}_{i_0i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
+ & \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i \\
+ & \tilde{P}(l) \prod_{i}^{m-1} D_i
\end{aligned}
$$

Now, choosing *l* such that

$$D_0(l^i) = D_1(l^i) = 0$$

sets all the terms in this equation to zero except the first, second and third line

🟪 Coefficient computed in a previous step

OPP REDUCTION

$$
\begin{aligned}
N(l) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}(l)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}(l)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} + \tilde{b}_{i_0 i_1}(l)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i \neq i_0}^{m-1} D_i \\
& + \tilde{P}(l) \prod_{i}^{m-1} D_i
\end{aligned}
$$

$$= 0$$

Now, choosing *l* such that

$$D_0(l^i) = D_1(l^i) = 0$$

sets all the terms in this equation to zero except the first, second and third line

🟪 Coefficient computed in a previous step

OPP REDUCTION

$$\begin{aligned}
N(l) = & \boxed{\sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0 i_1 i_2 i_3} + \tilde{d}_{i_0 i_1 i_2 i_3}(l)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i} \\
+ & \boxed{\sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0 i_1 i_2} + \tilde{c}_{i_0 i_1 i_2}(l)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i} \\
+ & \boxed{\sum_{i_0<i_1}^{m-1} \left[b_{i_0 i_1} + \tilde{b}_{i_0 i_1}(l)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i} \\
+ & \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i \neq i_0}^{m-1} D_i \\
+ & \tilde{P}(l) \prod_{i}^{m-1} D_i
\end{aligned}$$

Now, choosing *l* such that

$$D_1(l^i) = 0$$

sets the last line to zero

OPP REDUCTION

$$N(l) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d_{i_0i_1i_2i_3} + \tilde{d}_{i_0i_1i_2i_3}(l)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i$$

$$+ \sum_{i_0<i_1<i_2}^{m-1} \left[c_{i_0i_1i_2} + \tilde{c}_{i_0i_1i_2}(l)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i$$

$$+ \sum_{i_0<i_1}^{m-1} \left[b_{i_0i_1} + \tilde{b}_{i_0i_1}(l)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i$$

$$+ \sum_{i_0}^{m-1} \left[a_{i_0} + \tilde{a}_{i_0}(l)\right] \prod_{i\neq i_0}^{m-1} D_i$$

$$+\tilde{P}(l) \prod_{i}^{m-1} D_i$$

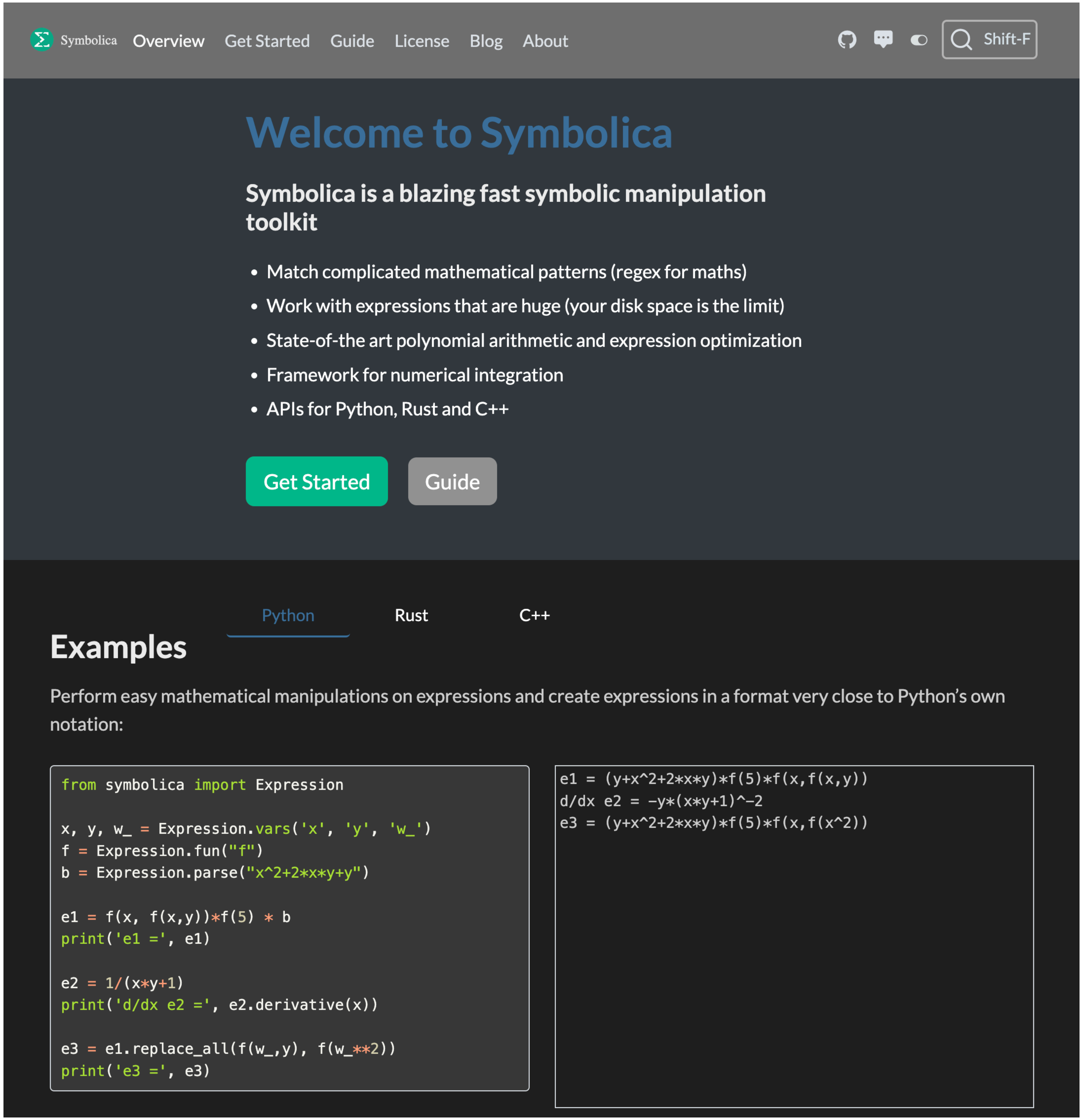
$$= 0$$

Now, choosing *l* such that

$$D_1(l^i) = 0$$

sets the last line to zero

WWW.SYMBOLICA.IO



Welcome to Symbolica

Symbolica is a blazing fast symbolic manipulation toolkit

- Match complicated mathematical patterns (regex for maths)
- Work with expressions that are huge (your disk space is the limit)
- State-of-the art polynomial arithmetic and expression optimization
- Framework for numerical integration
- APIs for Python, Rust and C++

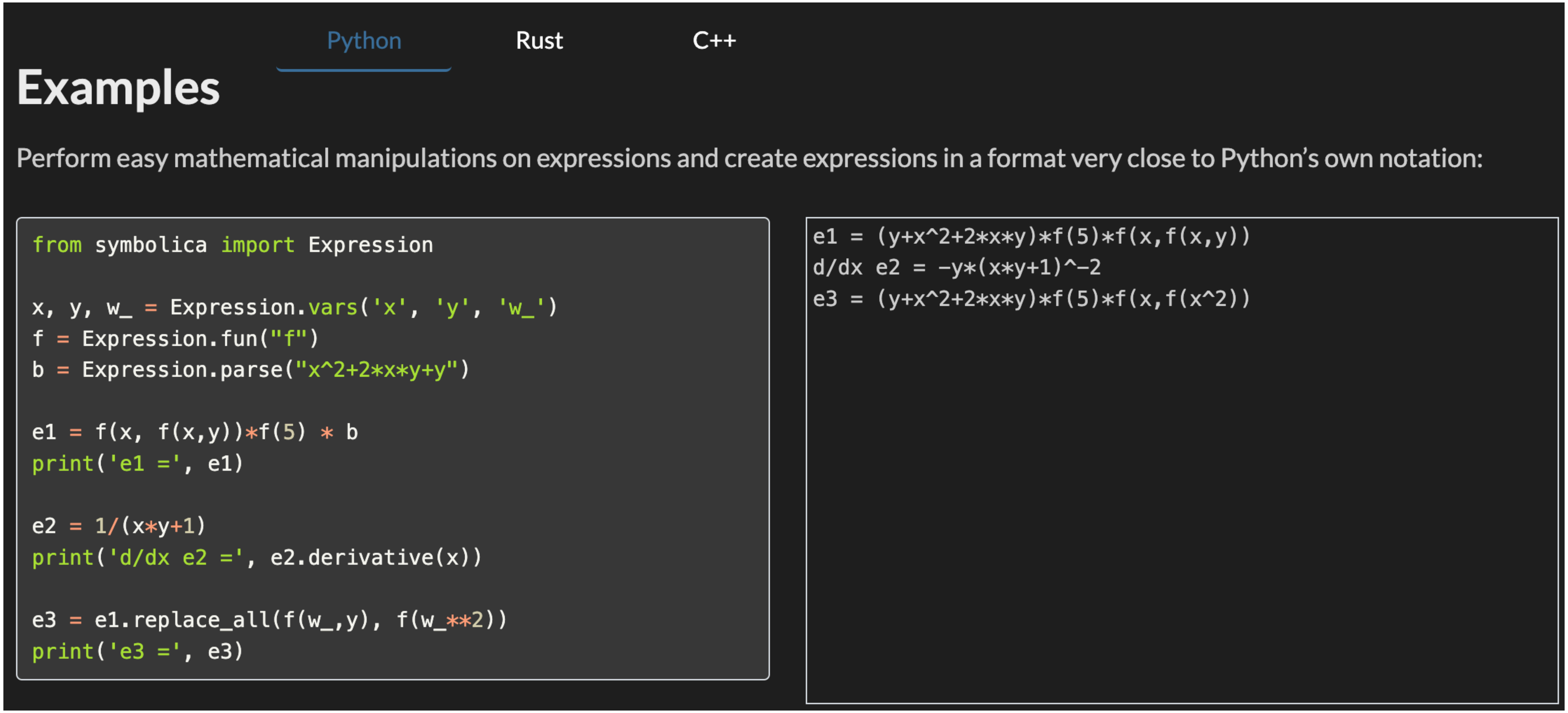
Examples

Perform easy mathematical manipulations on expressions and create expressions in a format very close to Python's own notation:

```
from symbolica import Expression

x, y, w_ = Expression.vars('x', 'y', 'w_')
f = Expression.fun("f")
b = Expression.parse("x^2+2*x*y+y")

e1 = f(x, f(x,y))*f(5) * b
print('e1 =', e1)

e2 = 1/(x*y+1)
print('d/dx e2 =', e2.derivative(x))

e3 = e1.replace_all(f(w_,y), f(w_**2))
print('e3 =', e3)
```

```
e1 = (y+x^2+2*x*y)*f(5)*f(x,f(x,y))
d/dx e2 = -y*(x*y+1)^-2
e3 = (y+x^2+2*x*y)*f(5)*f(x,f(x^2))
```

SYMBOLICA



Examples

Perform easy mathematical manipulations on expressions and create expressions in a format very close to Python's own notation:

```python
from symbolica import Expression

x, y, w_ = Expression.vars('x', 'y', 'w_')
f = Expression.fun("f")
b = Expression.parse("x^2+2*x*y+y")

e1 = f(x, f(x,y))*f(5) * b
print('e1 =', e1)

e2 = 1/(x*y+1)
print('d/dx e2 =', e2.derivative(x))

e3 = e1.replace_all(f(w_,y), f(w_**2))
print('e3 =', e3)
```

```
e1 = (y+x^2+2*x*y)*f(5)*f(x,f(x,y))
d/dx e2 = -y*(x*y+1)^-2
e3 = (y+x^2+2*x*y)*f(5)*f(x,f(x^2))
```

Try it for yourself in this colab notebook !

SUMMARY - POSSIBLE OVERLAP WITH LSS

Conceptual :

- **Falsehood :** "Analytical = solved && Numerical = black box"
- Numerics power is **genericity** and **flexibility**:

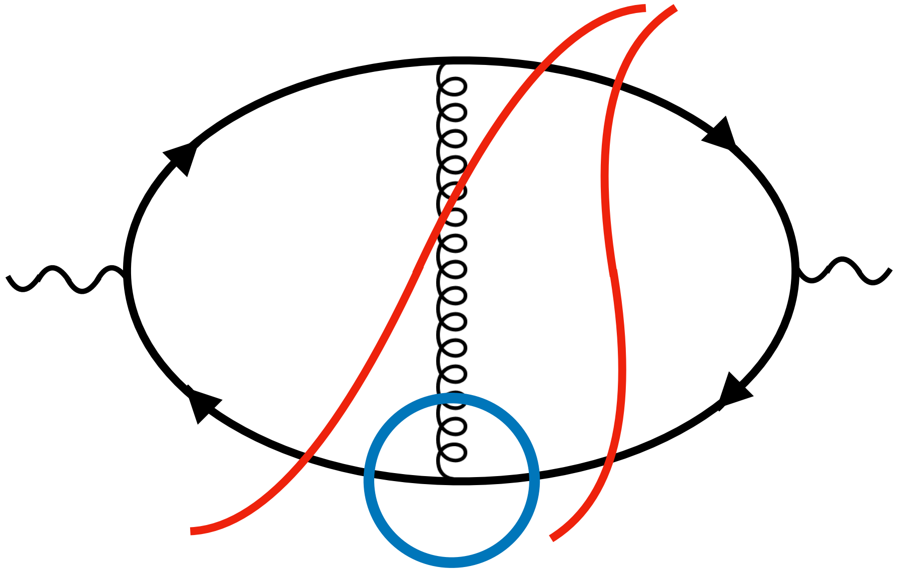
  use it so simply the construction and avoid large cancellation.
- Aim for an approach **with computation resources** as the only limiting factor

Technical

- Numerical **Tropical Sampling** may be a viable options for **higher-loops**.
- **Integrand-level reduction** may be the best for one-loop **higher-point.**
- UV treatment bears similarity with **QCD: R-formla** may be of useful.
- Method likely well-suited for deployment on GPU.

BACK-UP SLIDES

LOCAL UNITARITY



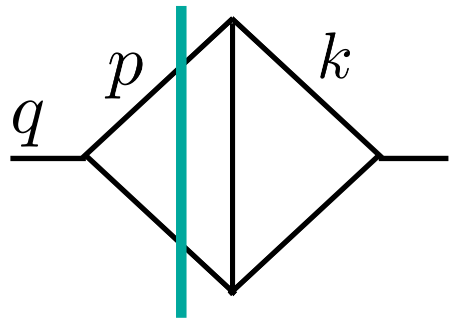
local

cancellation

LOCALITY UNITARITY

We convert the four-dimensional Minkowski loop integration measure into a three-dimensional Euclidean phase-space measure:

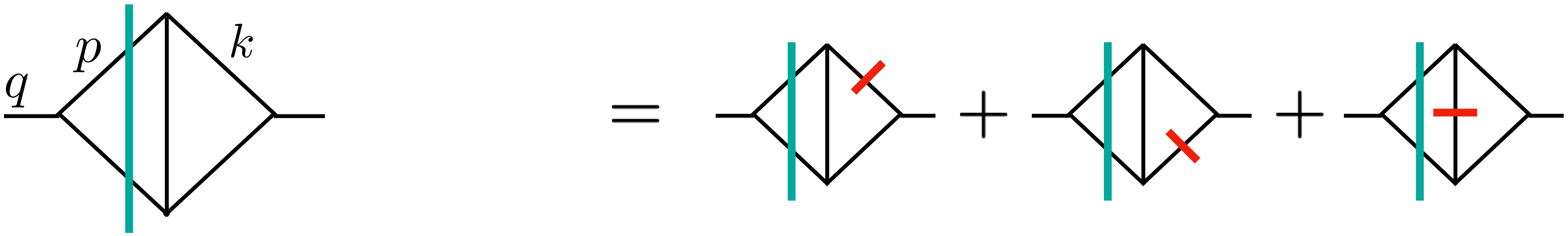
$$\frac{\mathrm{d}^3\vec{p}}{2|\vec{p}|}\mathrm{d}^4k\,\delta(|\vec{p}|+|\vec{p}-\vec{q}|-Q_0)$$



LOCALITY UNITARITY

We convert the four-dimensional Minkowski loop integration measure into a three-dimensional Euclidean phase-space measure:

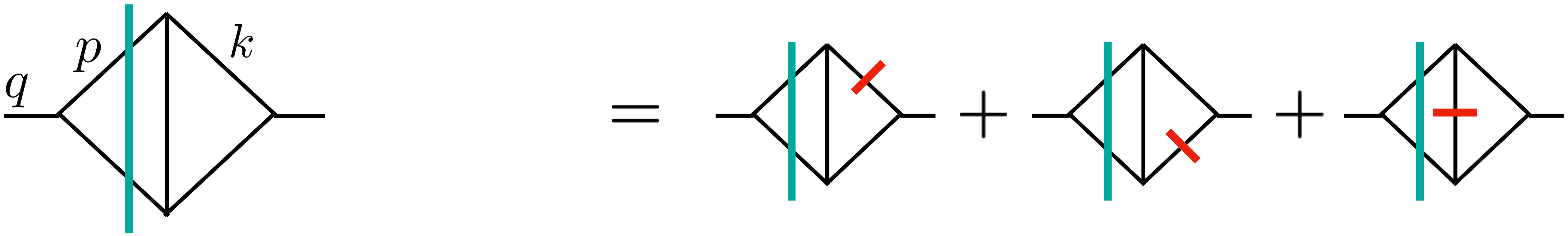
$$\frac{\mathrm{d}^3\vec{p}}{2|\vec{p}|}\mathrm{d}^4k\,\delta(|\vec{p}|+|\vec{p}-\vec{q}|-Q_0) \quad\rightarrow\quad \frac{\mathrm{d}^3\vec{p}}{2|\vec{p}|}\frac{\mathrm{d}^3\vec{k}}{2|\vec{k}|}\delta(|\vec{p}|+|\vec{p}-\vec{q}|-Q_0)$$



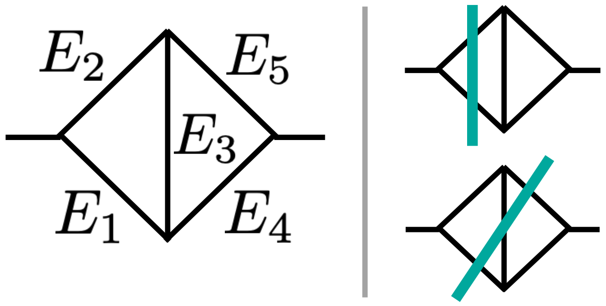
Locality Unitarity

We convert the four-dimensional Minkowski loop integration measure into a three-dimensional Euclidean phase-space measure:

$$\frac{\mathrm{d}^3\vec{p}}{2|\vec{p}|}\mathrm{d}^4k\,\delta(|\vec{p}|+|\vec{p}-\vec{q}|-Q_0) \rightarrow \frac{\mathrm{d}^3\vec{p}}{2|\vec{p}|}\frac{\mathrm{d}^3\vec{k}}{2|\vec{k}|}\delta(|\vec{p}|+|\vec{p}-\vec{q}|-Q_0)$$



But the measure is **not yet fully aligned**:

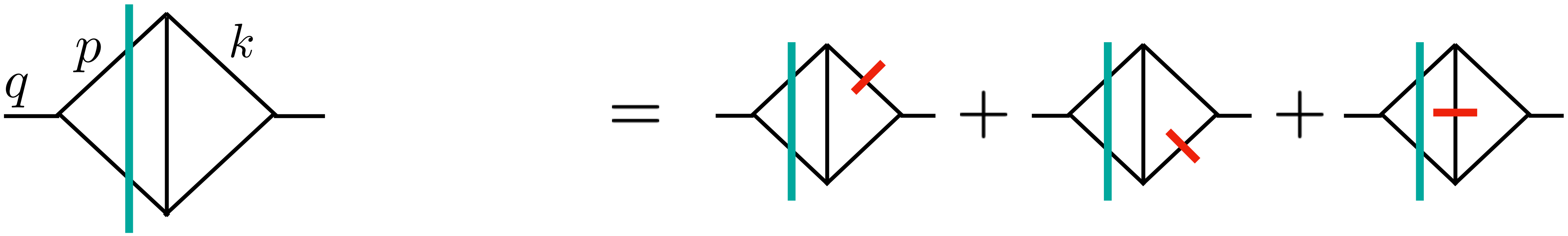


$$= \int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{p}\,\left(\delta(E_1+E_2-Q_0)f_{\mathrm{virt}}+\delta(E_1+E_3+E_5-Q_0)f_{\mathrm{real}}\right)$$

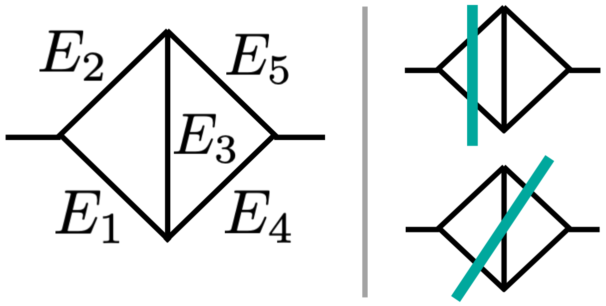
LOCALITY UNITARITY

We convert the four-dimensional Minkowski loop integration measure into a three-dimensional Euclidean phase-space measure:

$$\frac{\mathrm{d}^3\vec{p}}{2|\vec{p}|}\,\mathrm{d}^4k\,\delta(|\vec{p}|+|\vec{p}-\vec{q}|-Q_0)\;\rightarrow\;\frac{\mathrm{d}^3\vec{p}}{2|\vec{p}|}\,\frac{\mathrm{d}^3\vec{k}}{2|\vec{k}|}\,\delta(|\vec{p}|+|\vec{p}-\vec{q}|-Q_0)$$



But the measure is **not yet fully aligned**:



$$=\int \mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{p}\,\big(\delta(\underbrace{E_1+E_2-Q_0}_{\eta_{\mathrm{v}}(\vec{k},\vec{p})})f_{\mathrm{virt}}+\delta(\underbrace{E_1+E_3+E_5-Q_0}_{\eta_{\mathrm{r}}(\vec{k},\vec{p})})f_{\mathrm{real}}\big)$$

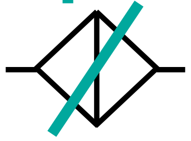
(on-shell energies: $E_i(\vec{k}_i)=\sqrt{\vec{k}_i^2+m_i^2-i\delta}$)

CAUSAL FLOW

CAUSAL FLOW

The measure now differs only in the **delta enforcing on shell energy conservation**

$$\sim \delta(E_1 + E_2 - Q_0)$$



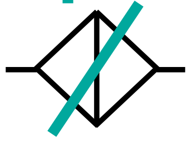
$$\sim \delta(E_1 + E_3 + E_5 - Q_0)$$

Objective: find a common variable to solve both deltas.

CAUSAL FLOW

The measure now differs only in the **delta enforcing on shell energy conservation**

$$\sim \delta(E_1 + E_2 - Q_0)$$



$$\sim \delta(E_1 + E_3 + E_5 - Q_0)$$

Objective: find a common variable to solve both deltas.

A different perspective on the usual phase space mapping problem

CAUSAL FLOW

The measure now differs only in the **delta enforcing on shell energy conservation**

$$\sim \delta(E_1 + E_2 - Q_0)$$

$$\sim \delta(E_1 + E_3 + E_5 - Q_0)$$

Objective: find a common variable to solve both deltas.

A different perspective on the usual phase space mapping problem

Solution: introduce an auxiliary variable in which to solve the delta

$$\delta(|\vec{k}| - Q_0) \underset{\vec{k} \to t\vec{k}}{\longrightarrow} \delta(t|\vec{k}| - Q_0) \longrightarrow t = \frac{Q_0}{|\vec{k}|}$$

Soper,
arXiv: 9804454 (1998)

Soper,
arXiv: 0102031 (2001 @ RADCOR)

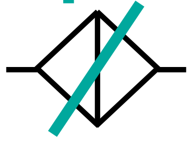
ZC, Hirschi, Pelloni, Ruijl
arXiv: 2010.01068 (2020)

General FSR cancellations
For N to M N^kLO processes

CAUSAL FLOW

The measure now differs only in the **delta enforcing on shell energy conservation**

$$\sim \delta(E_1 + E_2 - Q_0)$$



$$\sim \delta(E_1 + E_3 + E_5 - Q_0)$$

Objective: find a common variable to solve both deltas.

A different perspective on the usual phase space mapping problem

Solution: introduce an auxiliary variable in which to solve the delta

$$\delta(|\vec{k}| - Q_0) \underset{\vec{k} \to t\vec{k}}{\longrightarrow} \delta(t|\vec{k}| - Q_0) \longrightarrow t = \frac{Q_0}{|\vec{k}|}$$

Soper,
arXiv: 9804454 (1998)

Soper,
arXiv: 0102031 (2001 @ RADCOR)

ZC, Hirschi, Pelloni, Ruijl
arXiv: 2010.01068 (2020)

General FSR cancellations
For N to M N^{k}LO processes

A toy example:

$$\int \mathrm{d}^3\vec{k}\, \delta(|\vec{k}| - Q_0) f(\vec{k})$$

CAUSAL FLOW : TOY INTEGRAL

CAUSAL FLOW : TOY INTEGRAL

$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0) f(\vec{k})$$ using $$1 = \int \mathrm{d}t\, h(t)$$

Causal flow : toy integral

$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0) f(\vec{k})$$ using $$1 = \int \mathrm{d}t\, h(t)$$

$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, t^3\, h(t)\, \delta(t|\vec{k}| - Q_0) f(t\vec{k})$$ using $$\vec{k} \to t\vec{k}$$

CAUSAL FLOW : TOY INTEGRAL

$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0) f(\vec{k})$$ using $$1 = \int \mathrm{d}t\, h(t)$$

$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, t^3\, h(t)\, \delta(t|\vec{k}| - Q_0) f(t\vec{k})$$ using $$\vec{k} \to t\vec{k}$$

$$= \int \mathrm{d}^3\vec{k}\, \frac{Q_0^3}{|\vec{k}|^4}\, h(Q_0/|\vec{k}|)\, f(Q_0\vec{k}/|\vec{k}|)$$ with $$t^\star = Q_0/|\vec{k}|$$

Solve all deltas in the common scaling variable. This completes the alignment of the measure!

CAUSAL FLOW : TOY INTEGRAL

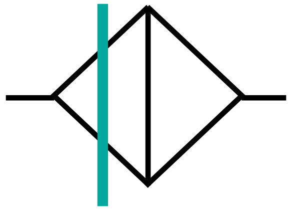
$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0) f(\vec{k})$$ using $$1 = \int \mathrm{d}t\, h(t)$$

$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, t^3\, h(t)\, \delta(t|\vec{k}| - Q_0) f(t\vec{k})$$ using $$\vec{k} \to t\vec{k}$$

$$= \int \mathrm{d}^3\vec{k}\, \frac{Q_0^3}{|\vec{k}|^4}\, h(Q_0/|\vec{k}|)\, f(Q_0\vec{k}/|\vec{k}|)$$ with $$t^\star = Q_0/|\vec{k}|$$

Solve all deltas in the common scaling variable. This completes the alignment of the measure!

When applying this construction to LU we get:



$$= \int \mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}\delta(E_1 + E_2 - Q_0) f_{\mathrm{virt}} = \int \mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}\, g_{\mathrm{v}}(t^\star_{\mathrm{v}})$$

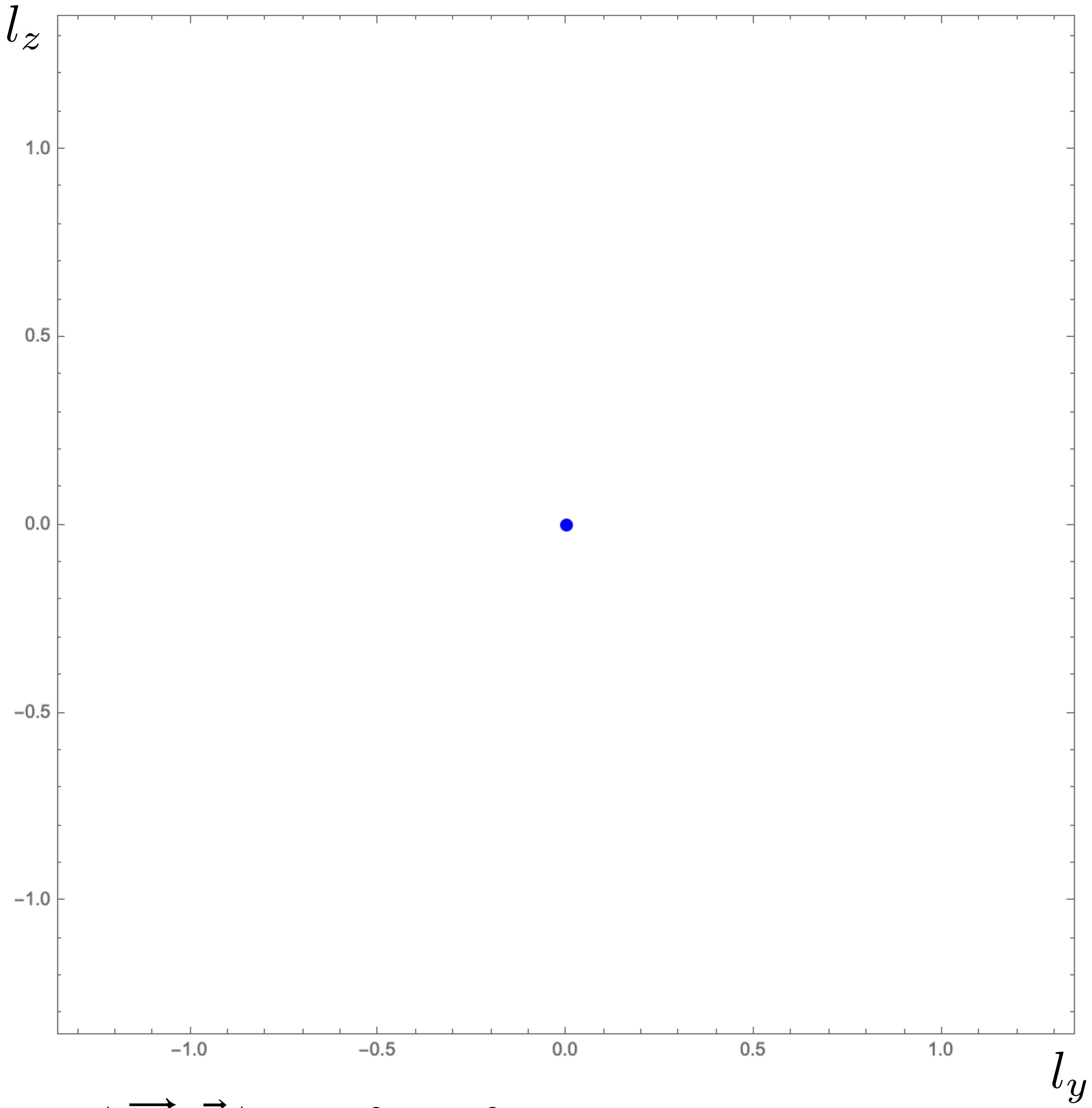
where $t^\star_{\mathrm{v}} = t^\star_{\mathrm{v}}(\vec{k},\vec{p}) = \dfrac{Q_0}{E_1 + E_2}$

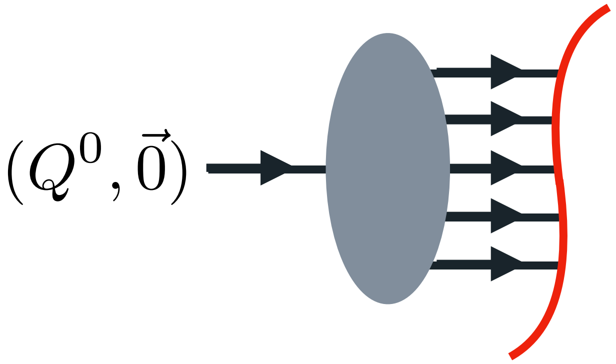
$$(\vec{p},\vec{k}) \to \vec{\phi}(t,(\vec{p},\vec{k}))$$

"Causal flow" is called like this because it is the generalisation of the Soper, derive from a contour deformation field satisfying the causal constraints.

$$\begin{cases} \partial_t \vec{\phi} = \vec{\kappa} \circ \vec{\phi} \\ \vec{\phi}(0,(\vec{k},\vec{l})) = (\vec{k},\vec{l}) \end{cases}$$

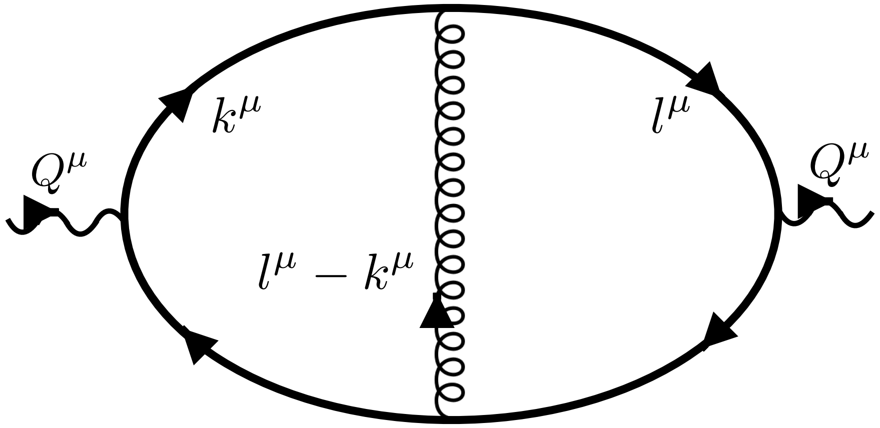
LOCALITY UNITARITY: VISUALISATION





$$Q^\mu = (2, 0, 0, 0)$$

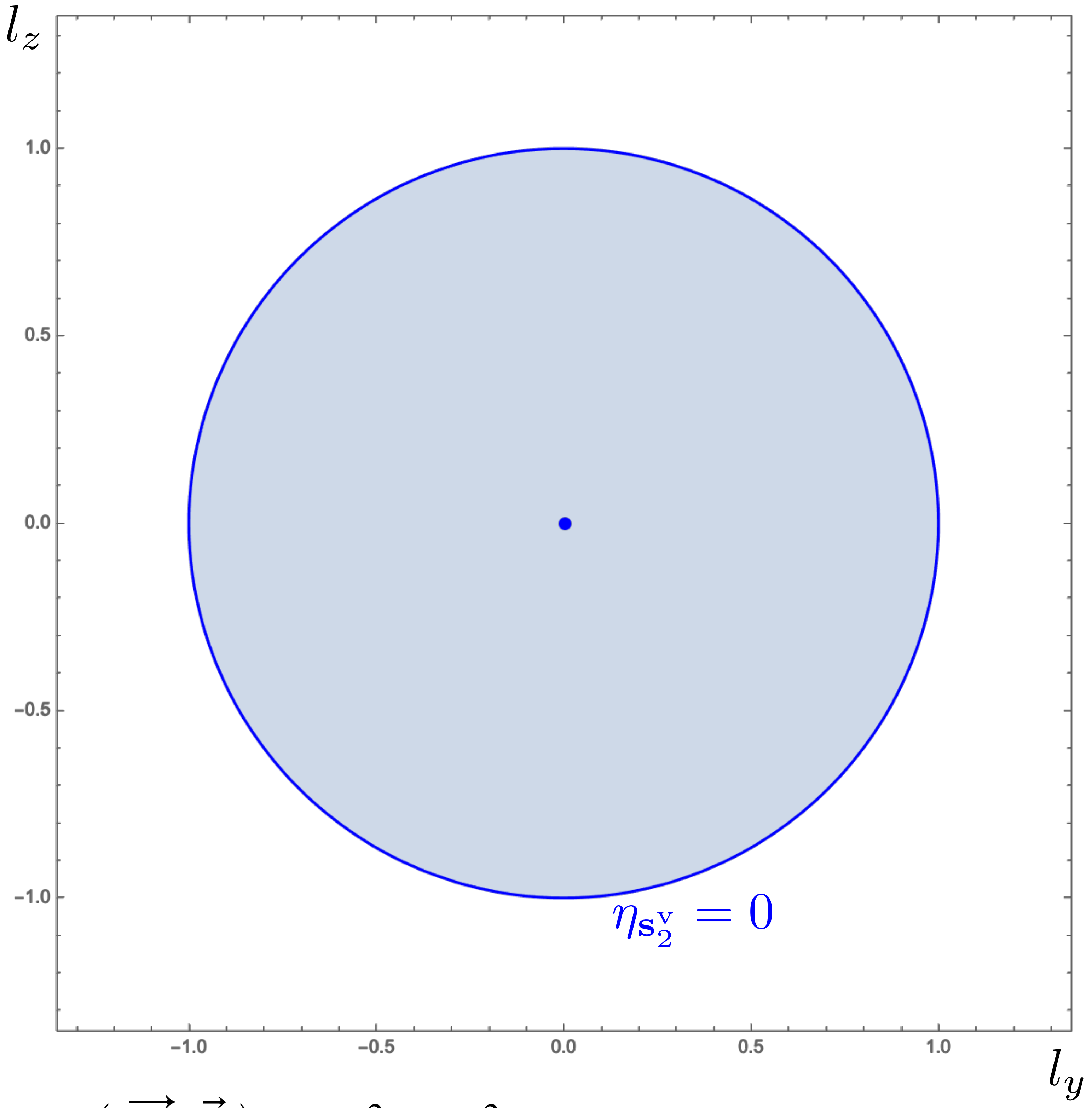
$$(\vec{k}, \vec{l}) = (\ (0, 0.5, 0.5),\ (0, l_y, l_z)\ )$$



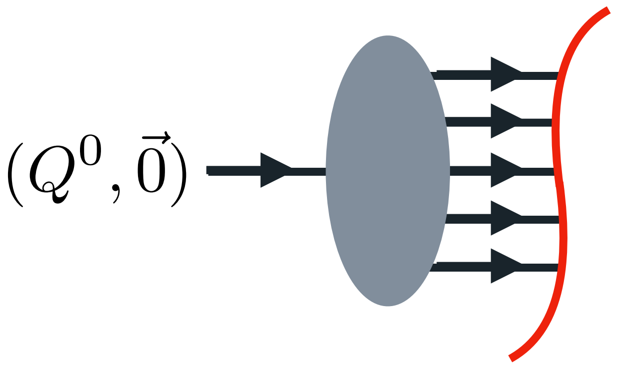
$(\ \vec{k}, \vec{l}\ ) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(\ l_y, l_z\ ) \in \mathbb{R}^2$

— = Cutkosky cut ≡ threshold

LOCALITY UNITARITY: VISUALISATION



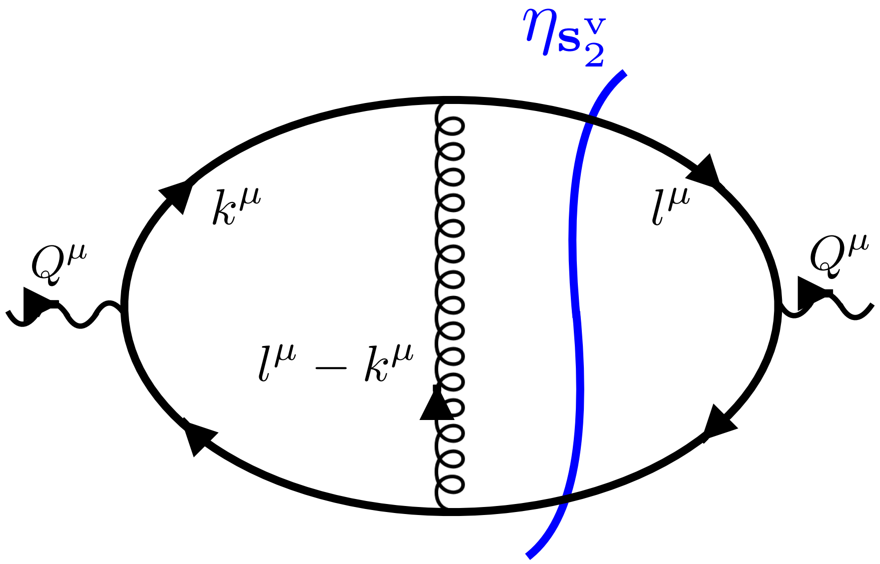
$(\vec{k}, \vec{l}) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$



$$Q^\mu = (2, 0, 0, 0)$$

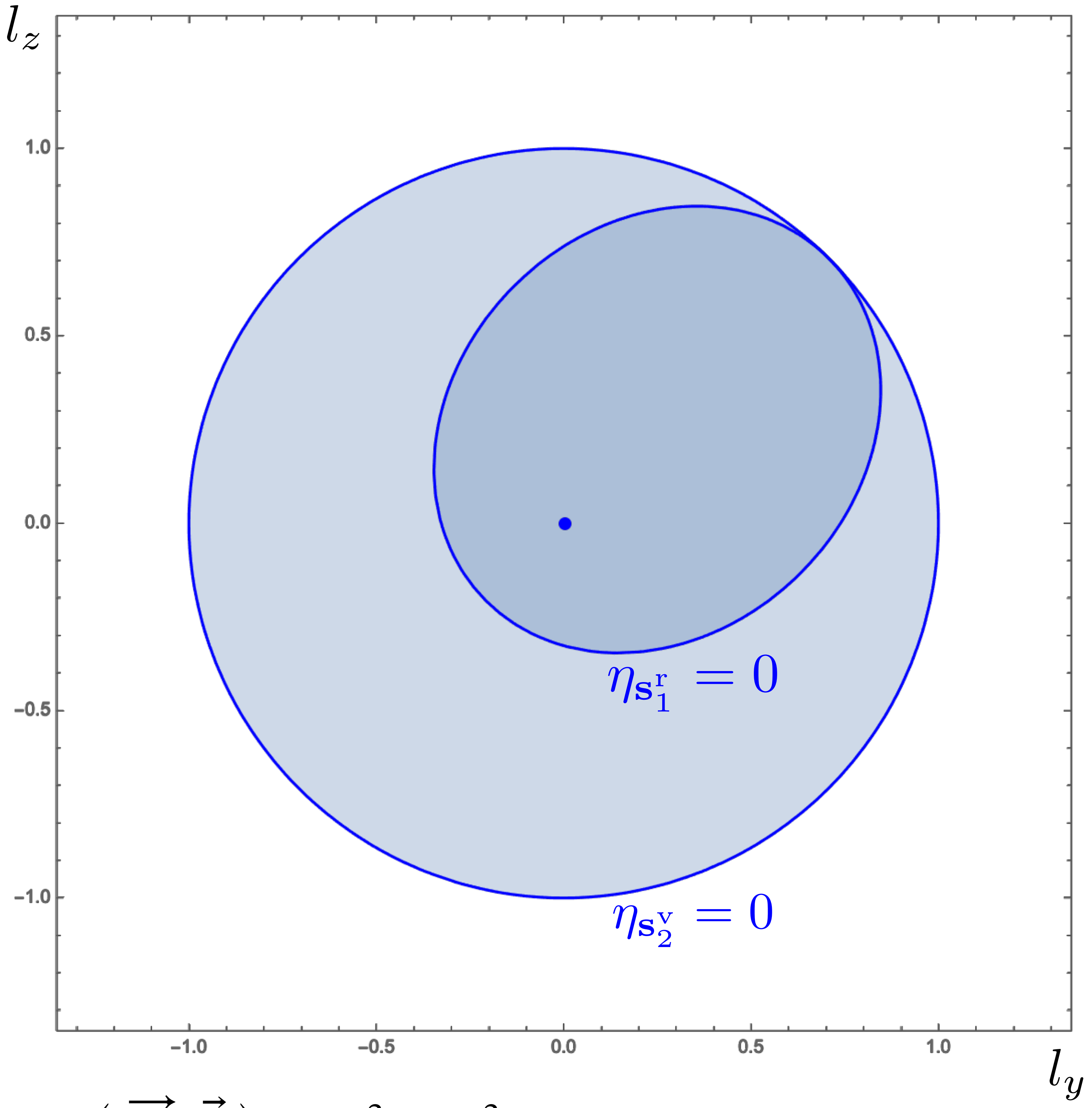
$$(\vec{k}, \vec{l}) = (\ (0, 0.5, 0.5),\ (0, l_y, l_z)\ )$$

$$\eta_{\mathbf{s}_2^{\mathrm{v}}} \to 2|\vec{l}\,'| = Q^0$$

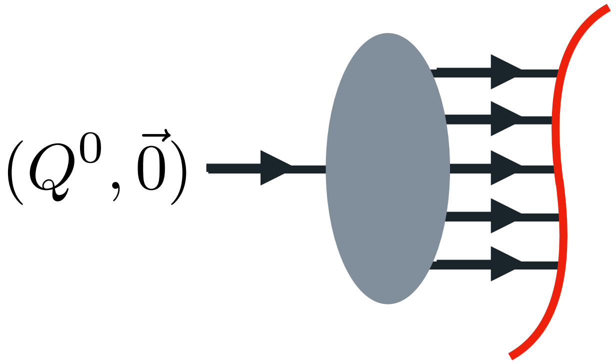


— = Cutkosky cut ≡ threshold

LOCALITY UNITARITY: VISUALISATION



$(\vec{k}, \vec{l}) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$

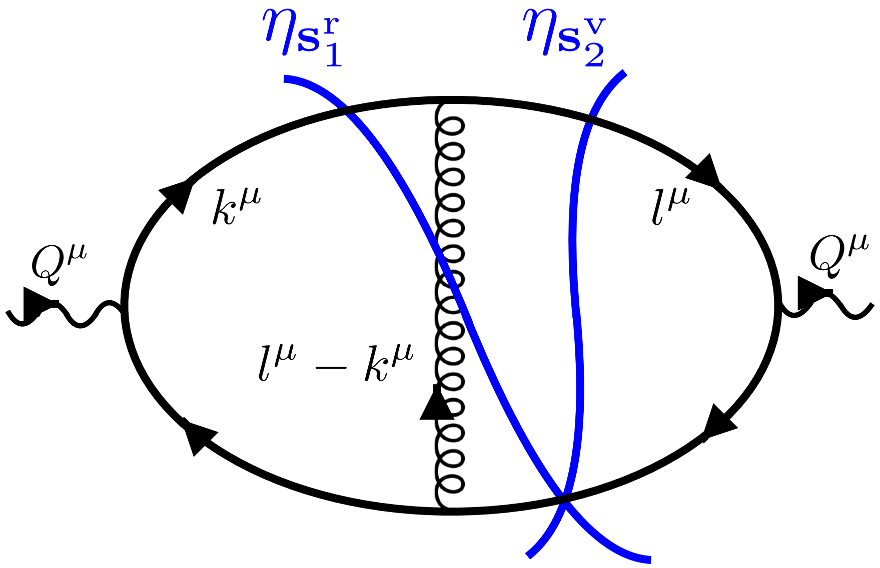


$$Q^\mu = (2,0,0,0)$$

$$(\vec{k}, \vec{l}) = (\ (0,0.5,0.5),\ (0,l_y,l_z)\ )$$

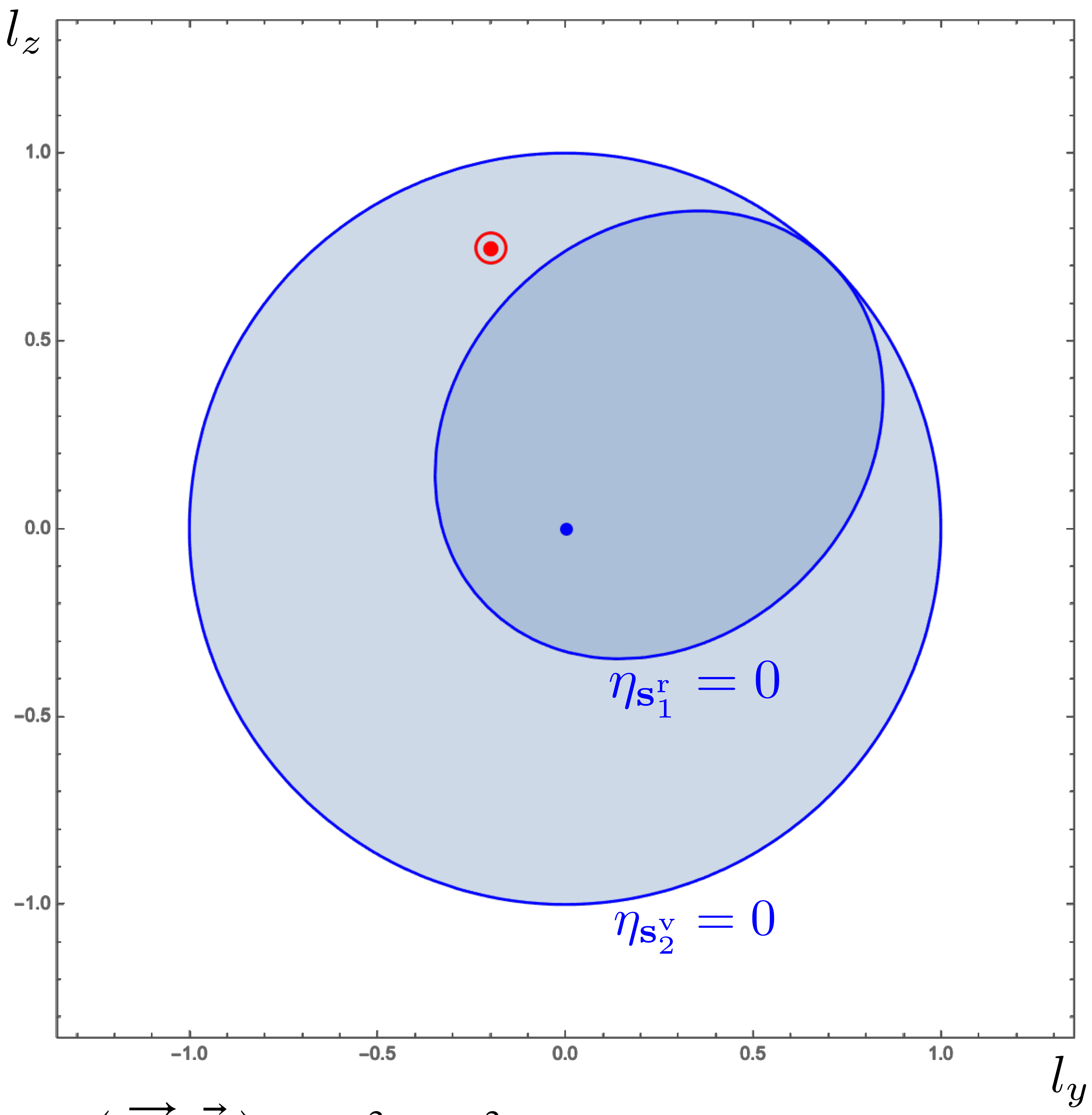
$$\eta_{\mathbf{s}_2^{\mathrm{v}}} \to 2|\vec{l}\,'| = Q^0$$

$$\eta_{\mathbf{s}_1^{\mathrm{r}}} \to |\vec{l}\,'| + |\vec{l}\,' - \vec{k}'| = Q^0 - |\vec{k}'|$$

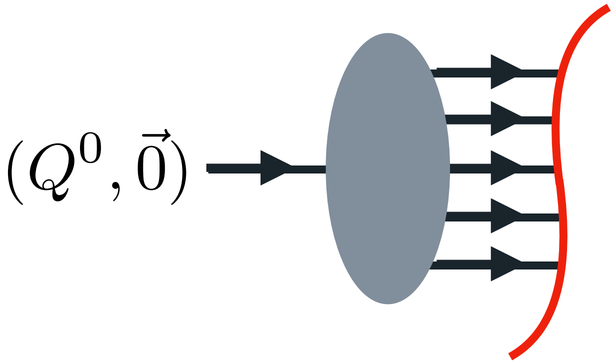


— = Cutkosky cut ≡ threshold

LOCALITY UNITARITY: VISUALISATION



$\left(\vec{k}, \vec{l} \right) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $\left(l_y, l_z \right) \in \mathbb{R}^2$

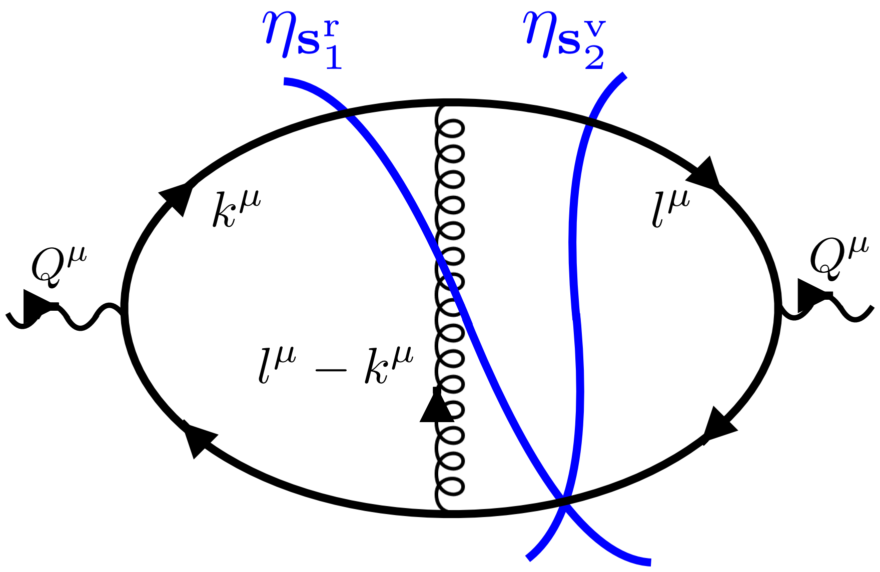


$$Q^\mu = (2, 0, 0, 0)$$

$$(\vec{k}, \vec{l}) = (\ (0, 0.5, 0.5),\ (0, l_y, l_z)\ )$$

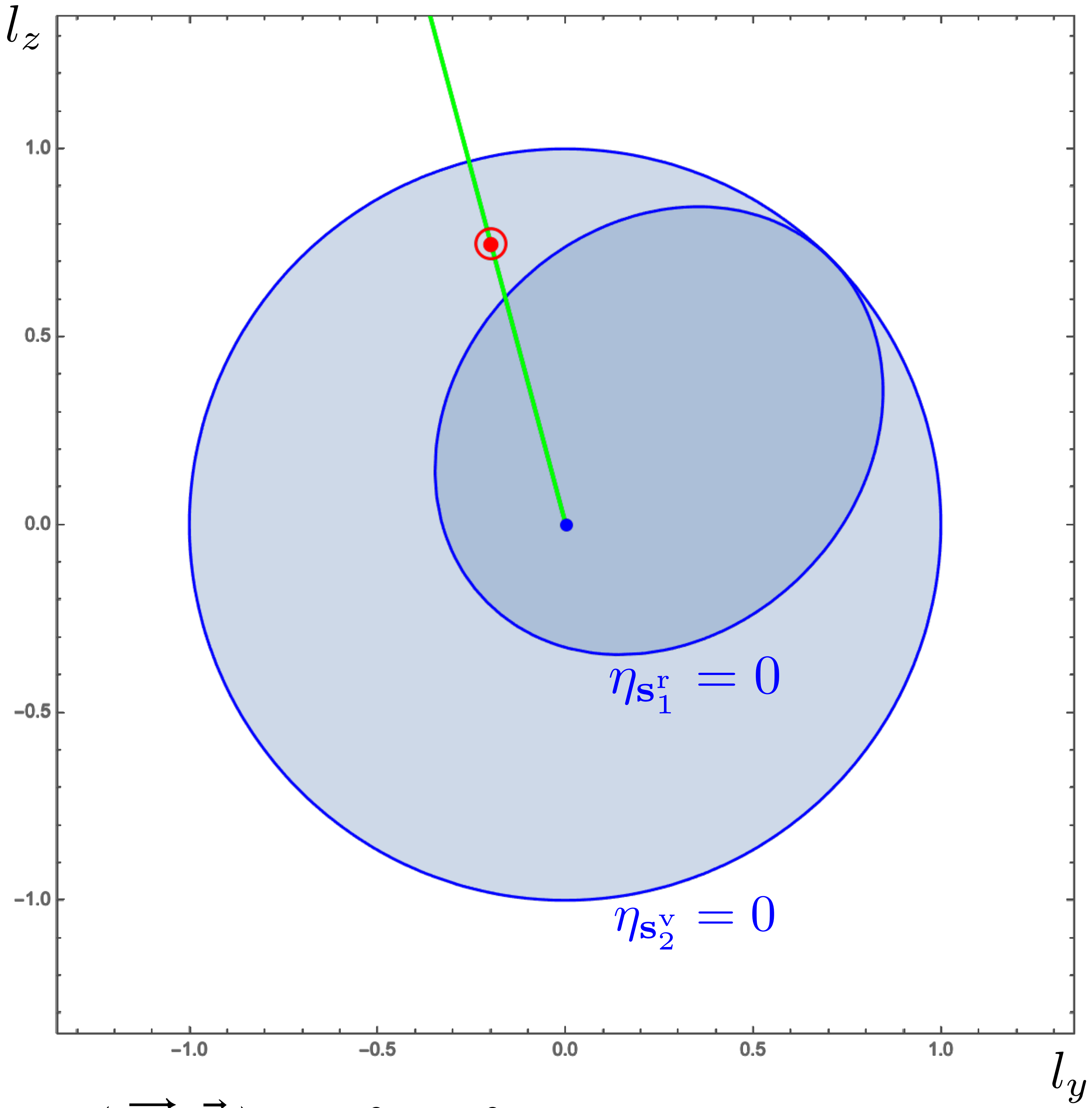
$$\eta_{\mathbf{s}_2^{\mathrm{v}}} \to 2|\vec{l}\,'| = Q^0$$

$$\eta_{\mathbf{s}_1^{\mathrm{r}}} \to |\vec{l}\,'| + |\vec{l}\,' - \vec{k}'| = Q^0 - |\vec{k}'|$$

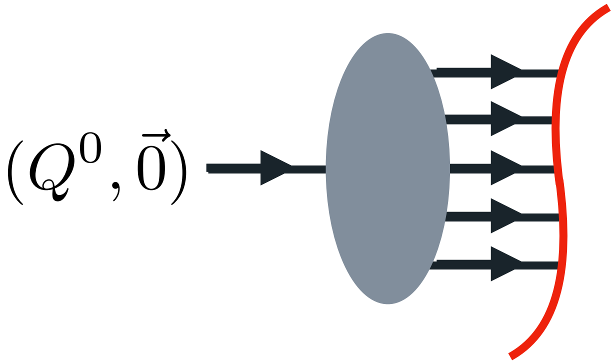


— = Cutkosky cut ≡ threshold

LOCALITY UNITARITY: VISUALISATION



$(\vec{k}, \vec{l}) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$

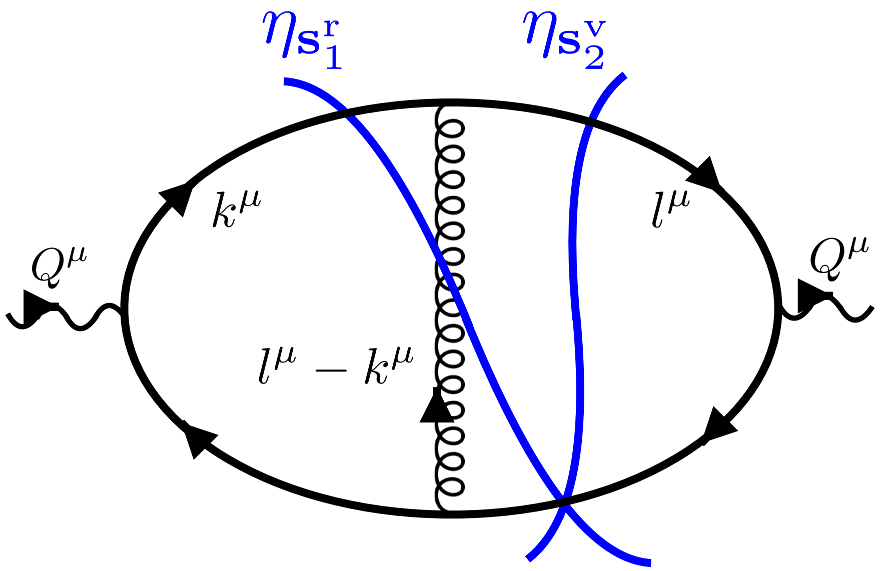


$$Q^\mu = (2,0,0,0)$$

$$(\vec{k}, \vec{l}) = (\ (0, 0.5, 0.5),\ (0, l_y, l_z)\ )$$

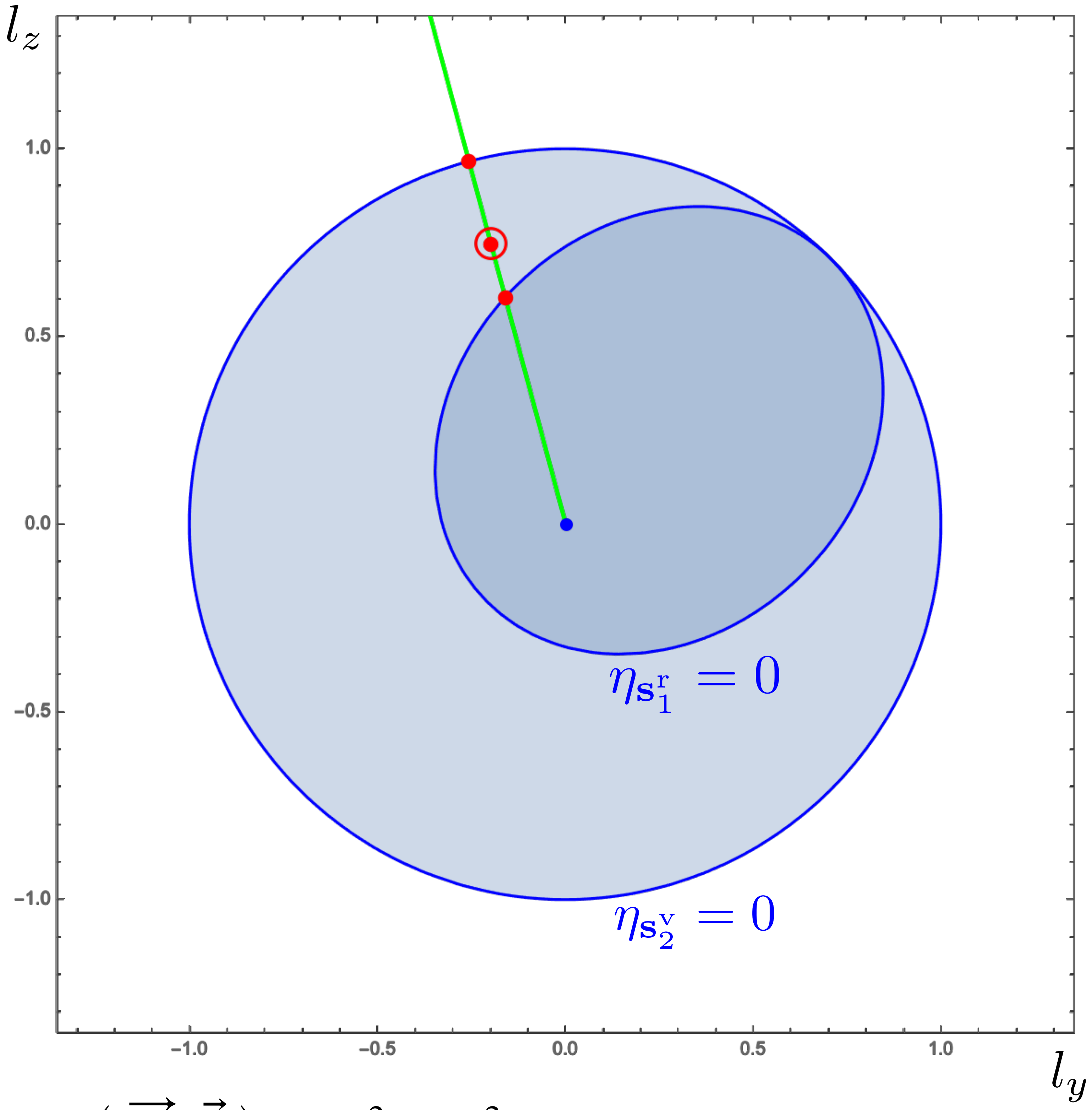
$$\eta_{\mathbf{s}_2^{\mathrm{v}}} \to 2|\vec{l}\,'| = Q^0$$

$$\eta_{\mathbf{s}_1^{\mathrm{r}}} \to |\vec{l}\,'| + |\vec{l}\,' - \vec{k}'| = Q^0 - |\vec{k}'|$$

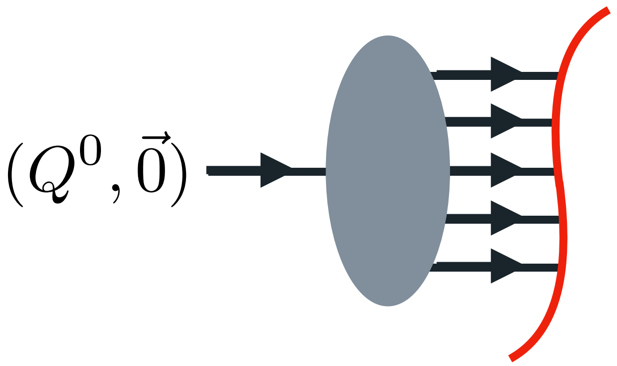


— = Cutkosky cut ≡ threshold

LOCALITY UNITARITY: VISUALISATION



$(\vec{k}, \vec{l}) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$

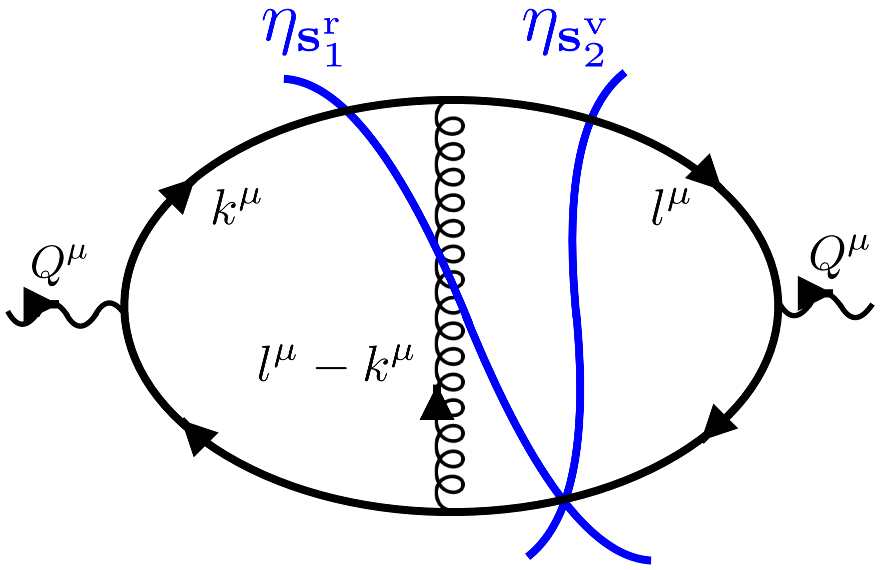


$$Q^\mu = (2, 0, 0, 0)$$

$$(\vec{k}, \vec{l}) = (\ (0, 0.5, 0.5),\ (0, l_y, l_z)\ )$$

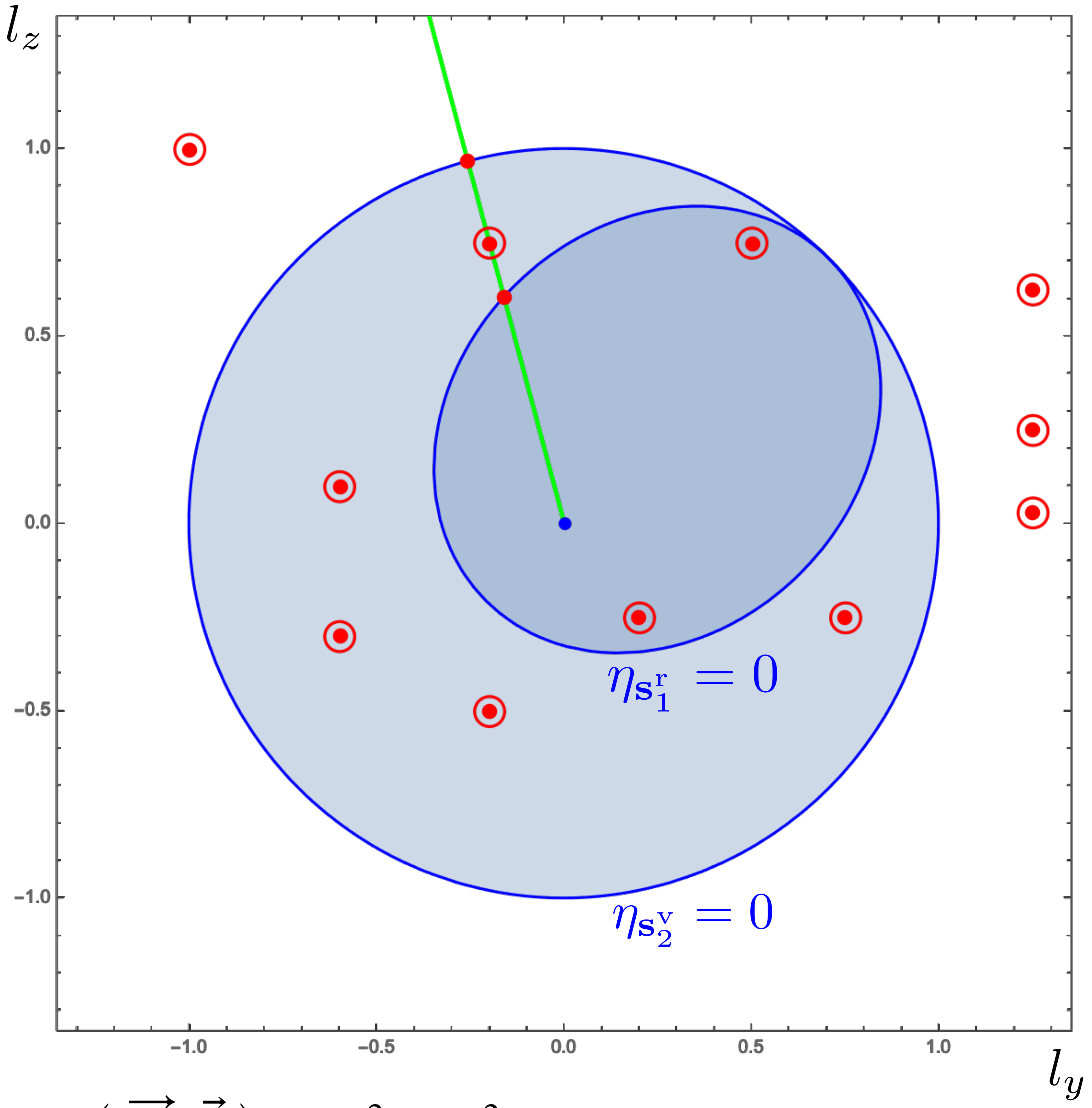
$$\eta_{\mathbf{s}_2^{\mathrm{v}}} \to 2|\vec{l}\,'| = Q^0$$

$$\eta_{\mathbf{s}_1^{\mathrm{r}}} \to |\vec{l}\,'| + |\vec{l}\,' - \vec{k}'| = Q^0 - |\vec{k}'|$$

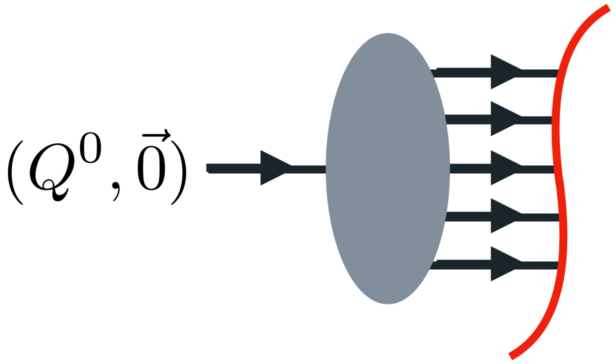


— = Cutkosky cut ≡ threshold

LOCALITY UNITARITY: VISUALISATION



$(\vec{k}, \vec{l}) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$

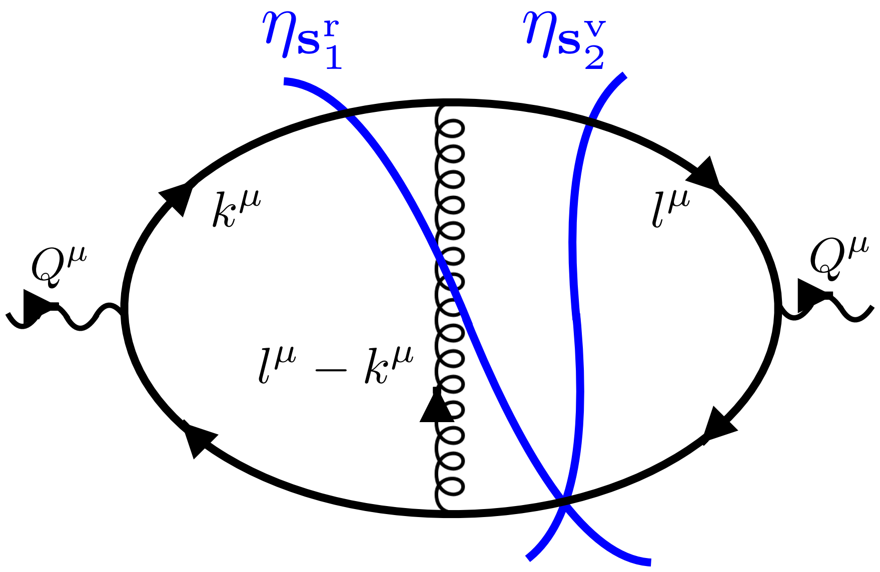


$$Q^\mu = (2, 0, 0, 0)$$

$$(\vec{k}, \vec{l}) = (\ (0, 0.5, 0.5), \ (0, l_y, l_z) \ )$$

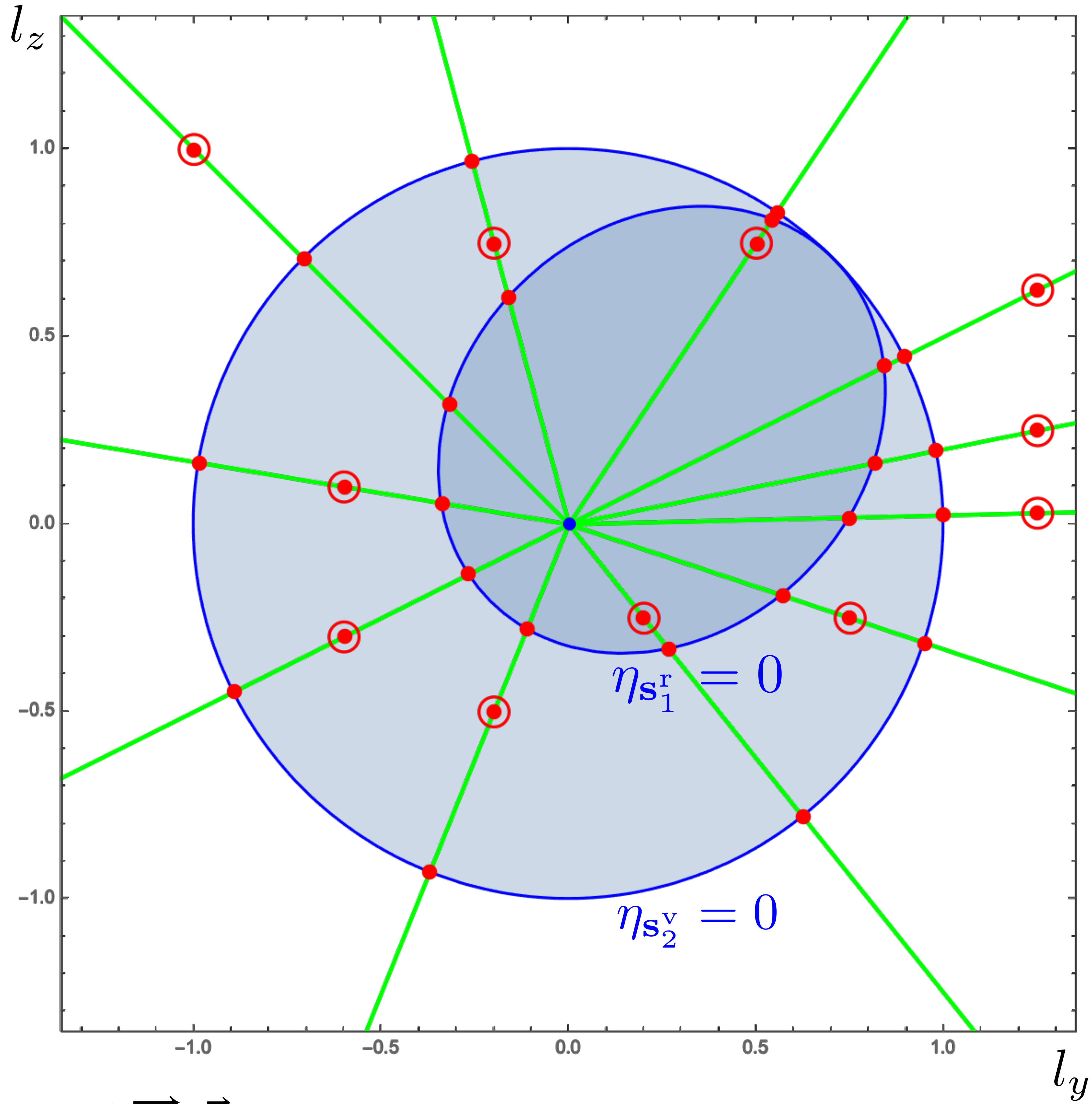
$$\eta_{\mathbf{s}_2^{\mathrm{v}}} \to 2|\vec{l}\,'| = Q^0$$

$$\eta_{\mathbf{s}_1^{\mathrm{r}}} \to |\vec{l}\,'| + |\vec{l}\,' - \vec{k}'| = Q^0 - |\vec{k}'|$$

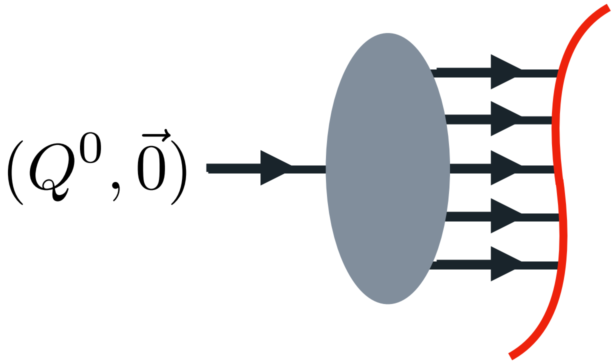


— = Cutkosky cut ≡ threshold

Locality Unitarity: visualisation



$(\vec{k}, \vec{l}\,) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$

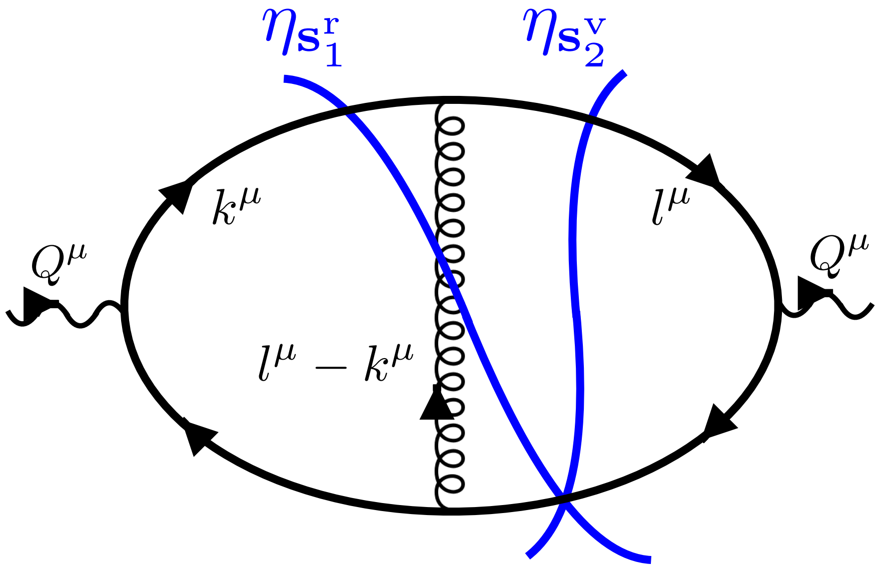


$$Q^\mu = (2, 0, 0, 0)$$

$$(\vec{k}, \vec{l}) = (\ (0, 0.5, 0.5),\ (0, l_y, l_z)\ )$$

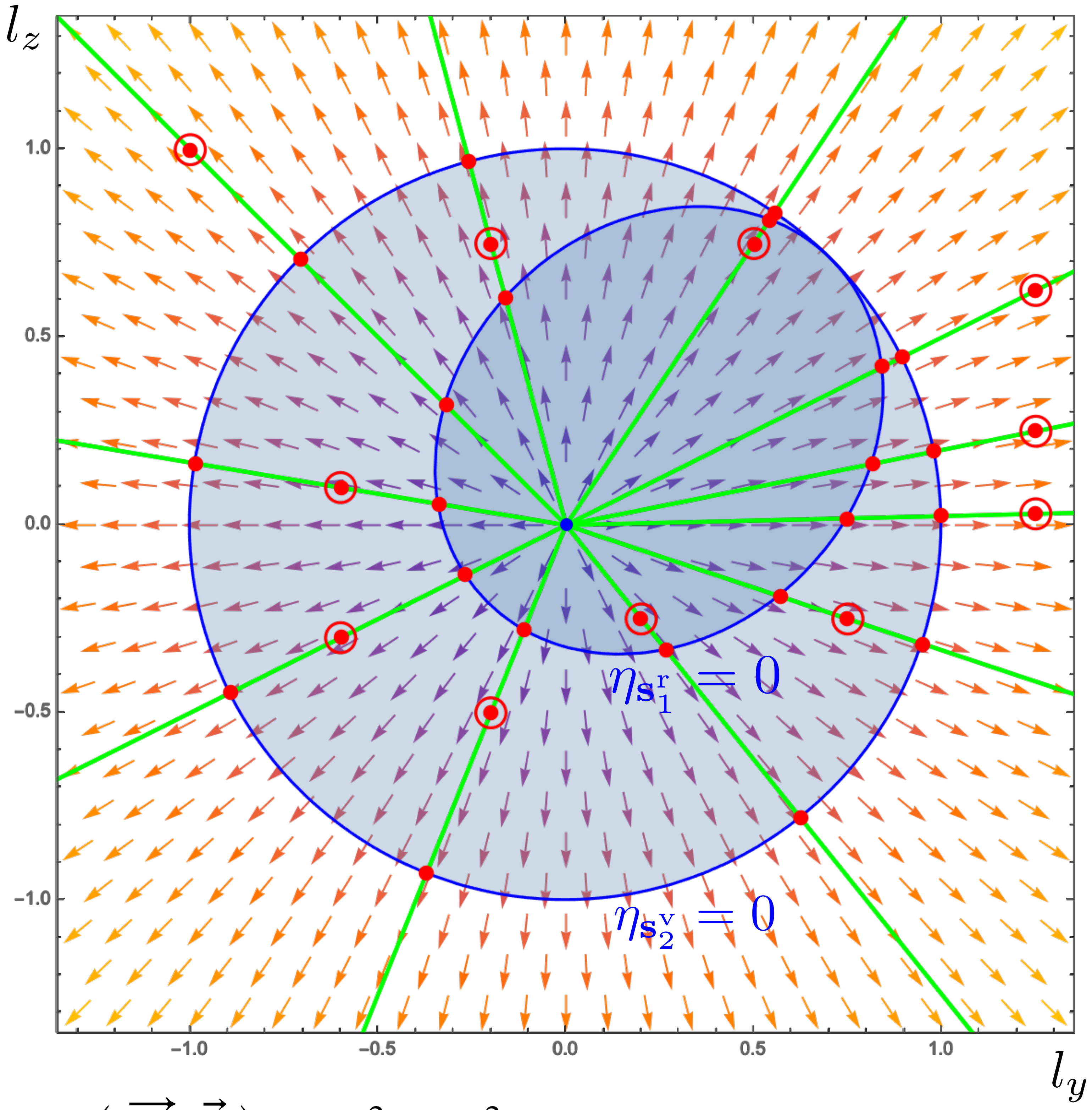
$$\eta_{\mathbf{s}_2^{\mathrm{v}}} \to 2|\vec{l}\,'| = Q^0$$

$$\eta_{\mathbf{s}_1^{\mathrm{r}}} \to |\vec{l}\,'| + |\vec{l}\,' - \vec{k}'| = Q^0 - |\vec{k}'|$$

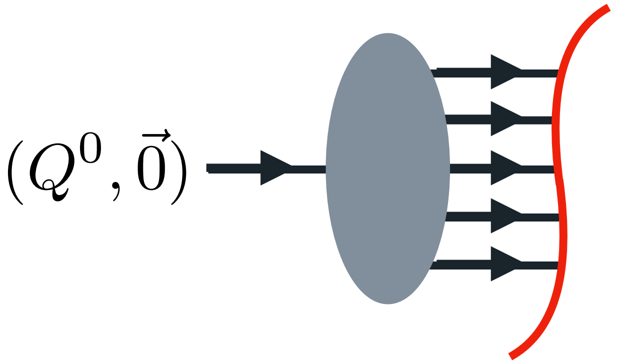


— = Cutkosky cut ≡ threshold

LOCALITY UNITARITY: VISUALISATION



$(\vec{k}, \vec{l}) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$

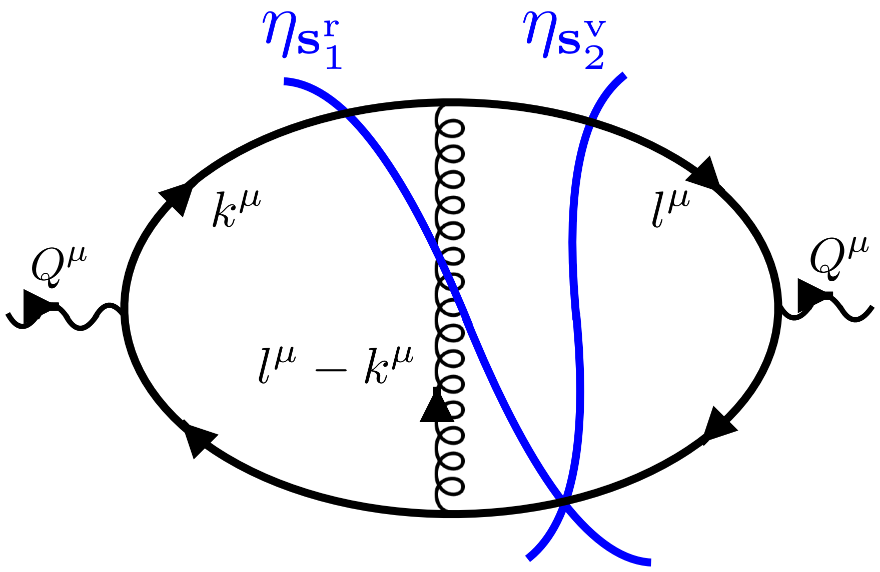


$$Q^\mu = (2, 0, 0, 0)$$

$$(\vec{k}, \vec{l}) = (\ (0, 0.5, 0.5),\ (0, l_y, l_z)\ )$$

$$\eta_{\mathbf{s}_2^{\mathrm{v}}} \to 2|\vec{l}\,'| = Q^0$$

$$\eta_{\mathbf{s}_1^{\mathrm{r}}} \to |\vec{l}\,'| + |\vec{l}\,' - \vec{k}\,'| = Q^0 - |\vec{k}\,'|$$



— = Cutkosky cut ≡ threshold

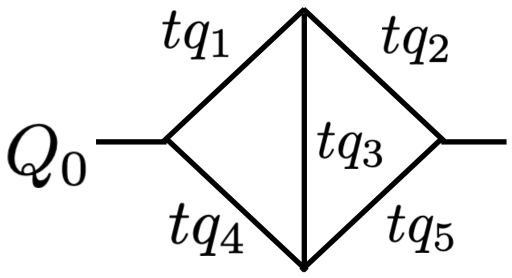
LOCALITY UNITARITY: ALL-ORDERS PROOF

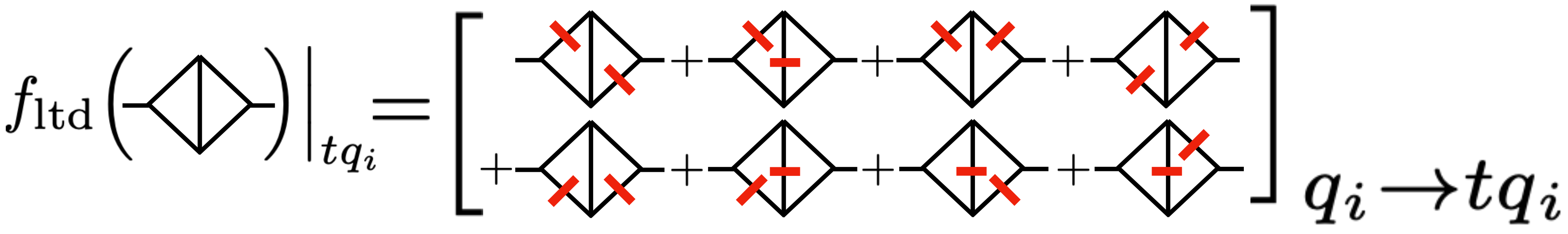
[Z. Capatti, VH, A. Pelloni, B. Ruijl, arXiv : 2010.01068]   [Summary in proceedings, arXiv : 2110.15662]

LOCALITY UNITARITY: ALL-ORDERS PROOF

[Z. Capatti, VH, A. Pelloni, B. Ruijl, arXiv : 2010.01068]   [Summary in proceedings, arXiv : 2110.15662]

The **LTD representation** of the double triangle with rescaled momenta is

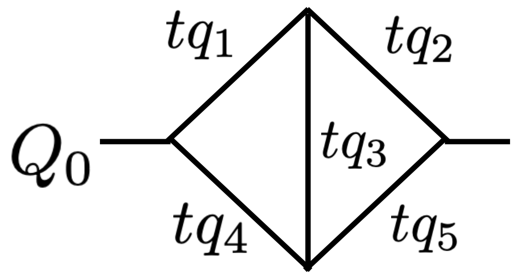


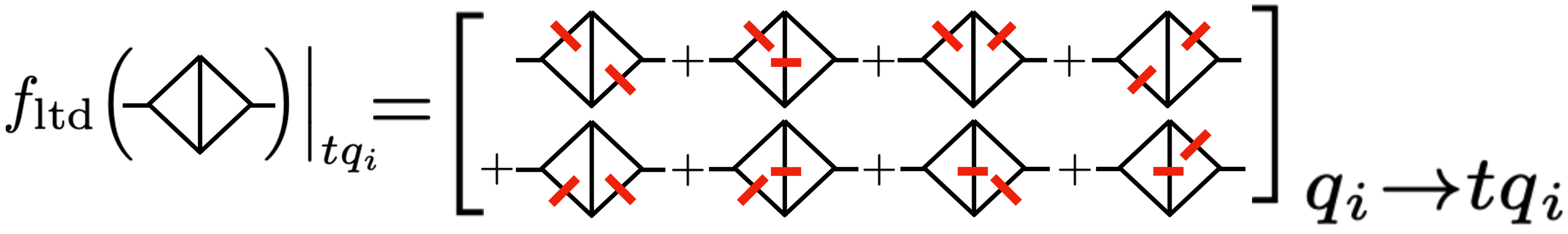


LOCALITY UNITARITY: ALL-ORDERS PROOF

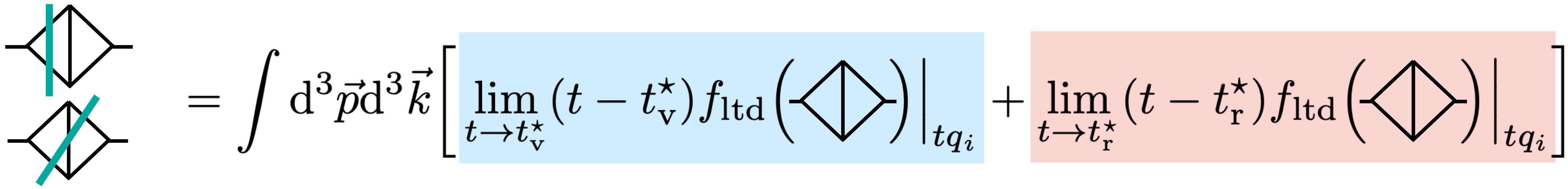
[Z. Capatti, VH, A. Pelloni, B. Ruijl, arXiv : 2010.01068] [Summary in proceedings, arXiv : 2110.15662]

The **LTD representation** of the double triangle with rescaled momenta is





Then one can capture the **thresholds** of this forward-scattering graphs with

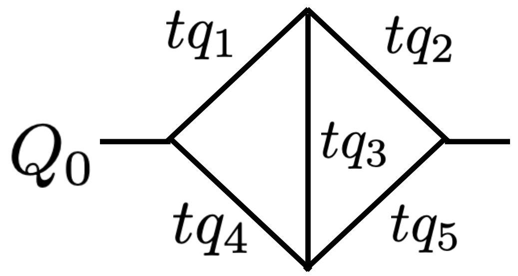


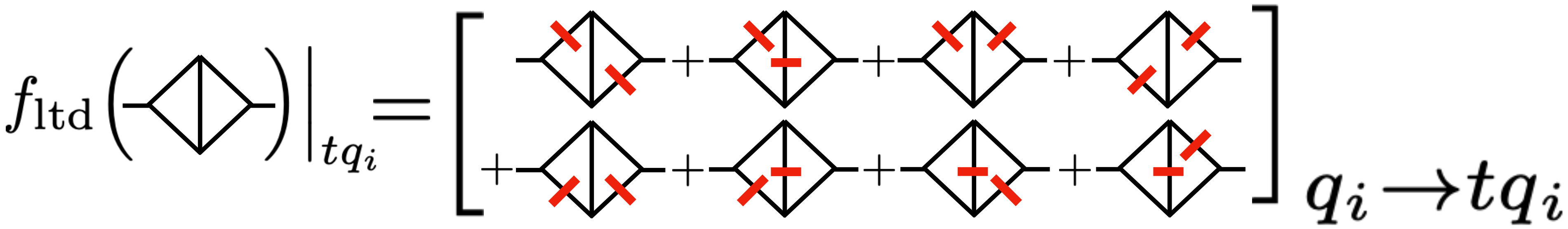
$g_{\rm v}$, $g_{\rm r}$ can be written as different limits of the same function!

LOCALITY UNITARITY: ALL-ORDERS PROOF

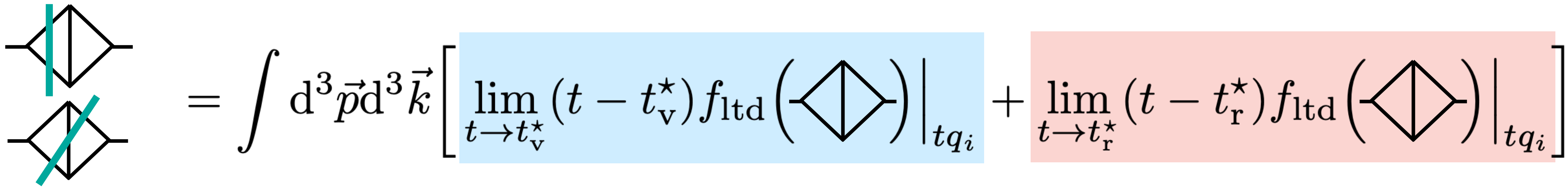
[Z. Capatti, VH, A. Pelloni, B. Ruijl, arXiv : 2010.01068]   [Summary in proceedings, arXiv : 2110.15662]

The **LTD representation** of the double triangle with rescaled momenta is



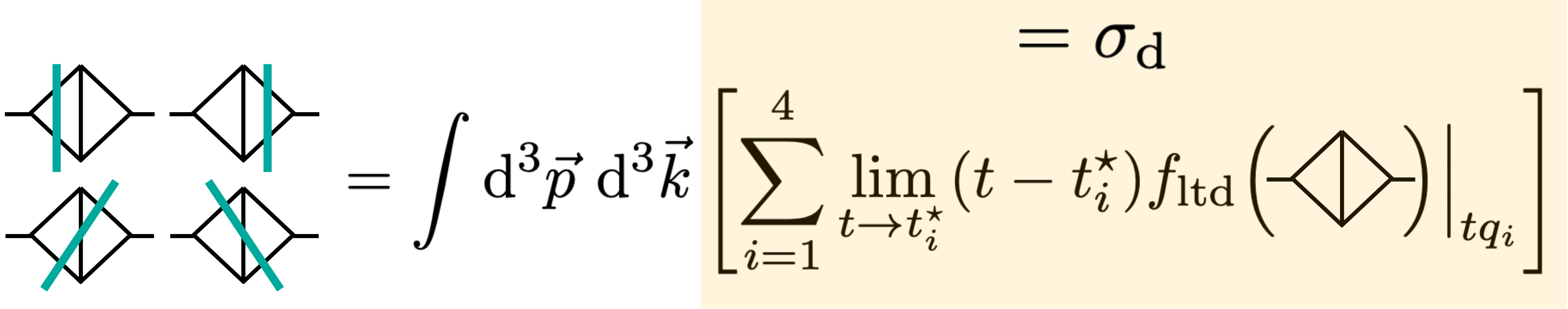


Then one can capture the **thresholds** of this forward-scattering graphs with



$g_{\rm v}$, $g_{\rm r}$ can be written as different limits of the same function!

Solving delta in the scaling variable $\Longrightarrow$ **1d residue theorem along the line** $\gamma(t) = (t\vec{k}, t\vec{p})$



LU representation

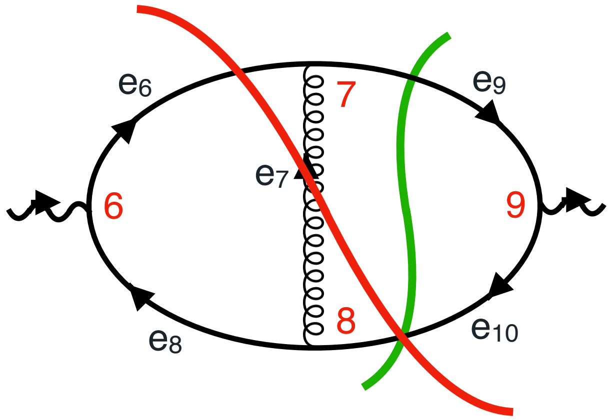
Cutkosky, but at the local level! We prove cancellations by studying the limit $t^\star_r \to t^\star_{\rm v}$

LOCALITY UNITARITY

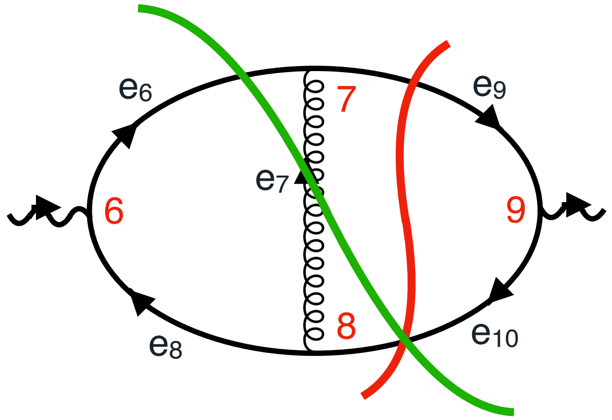
[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

🟥 = Cutkosky cut 🟩 = threshold singularity



cancels

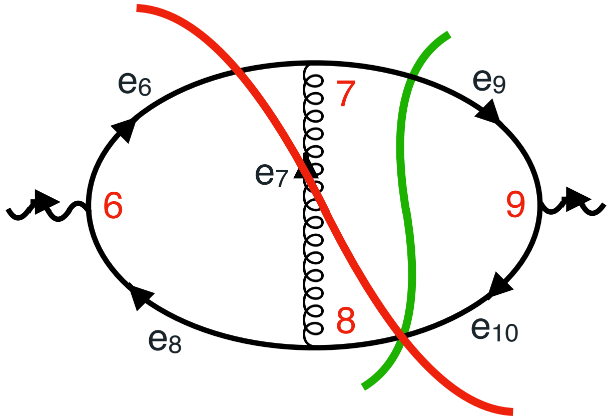


LOCALITY UNITARITY

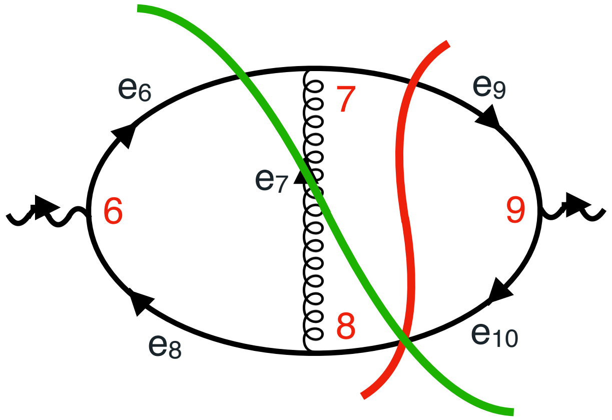
[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

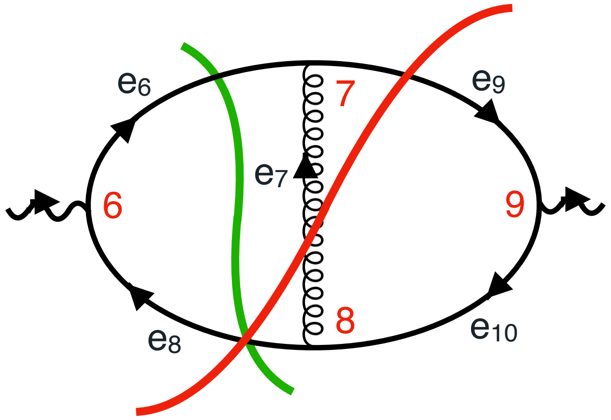
This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

🟥 = Cutkosky cut 🟩 = threshold singularity

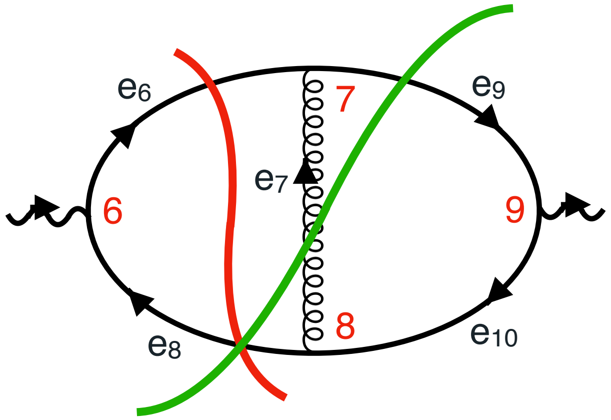


cancels





cancels

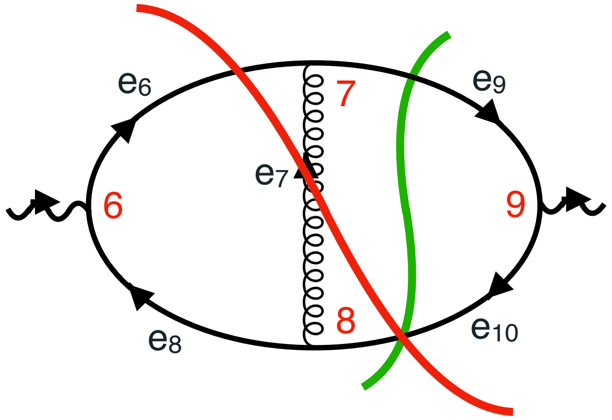


LOCALITY UNITARITY

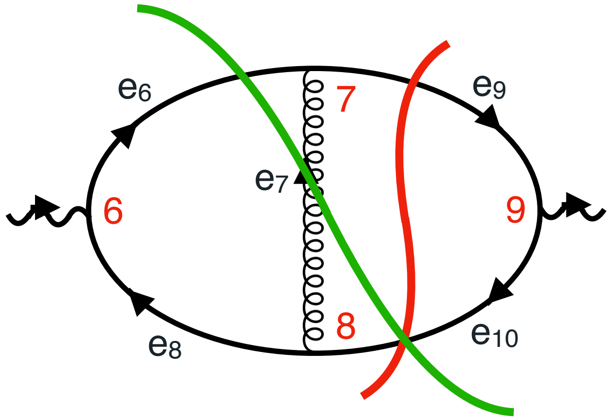
[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

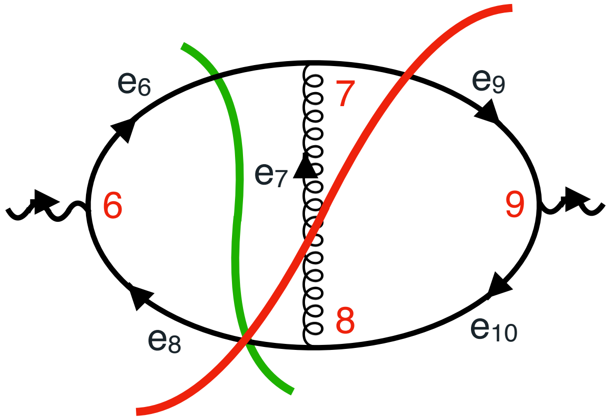
This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

🟥 = Cutkosky cut 🟩 = threshold singularity

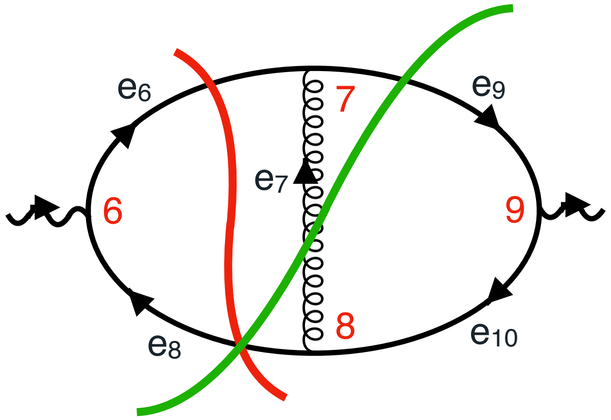


cancels

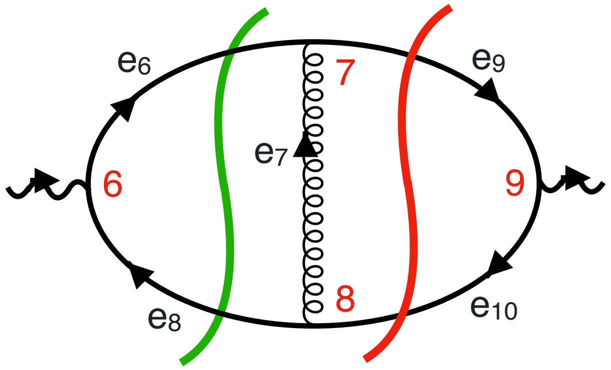




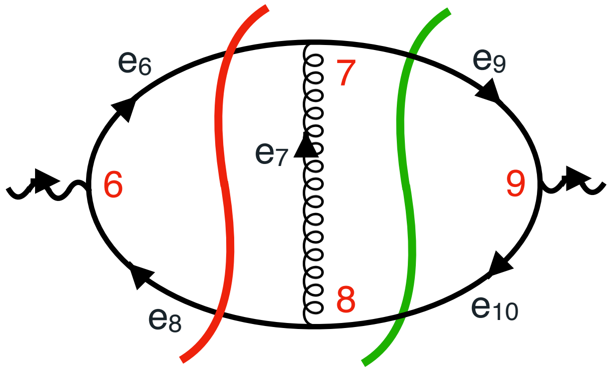
cancels



Even for **non-pinched singular threshold** ! (when $\mathcal{O}_{\mathbf{s}} \equiv 1$) :



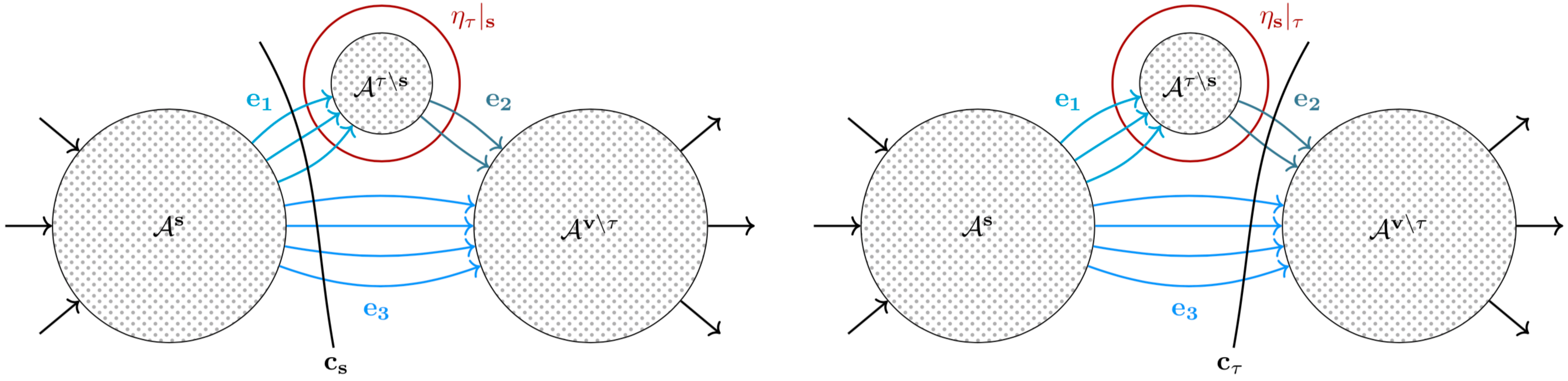
cancels



LOCALITY UNITARITY

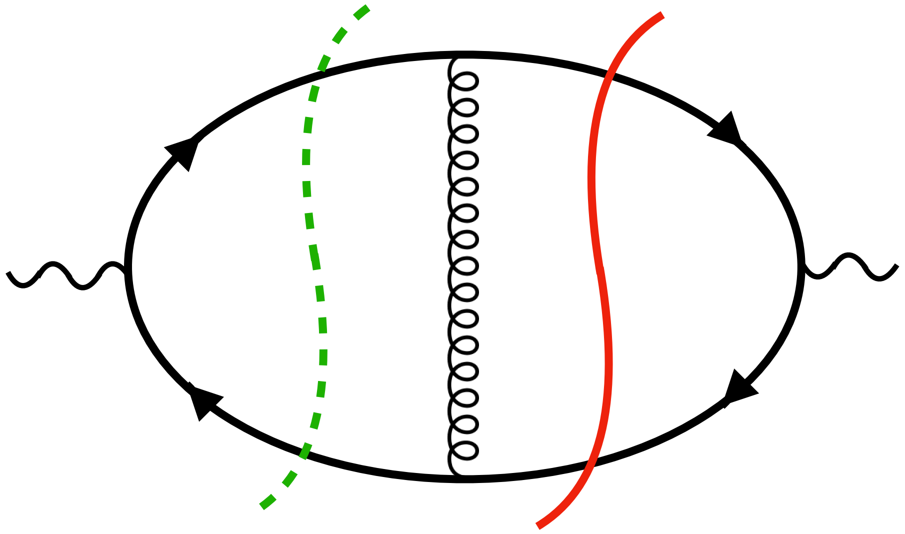
[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :



THRESHOLDS

$$E_1 = \sqrt{|\vec{k}|^2 + m^2 - i\epsilon}$$



$$q = (Q^0, \vec{q})$$

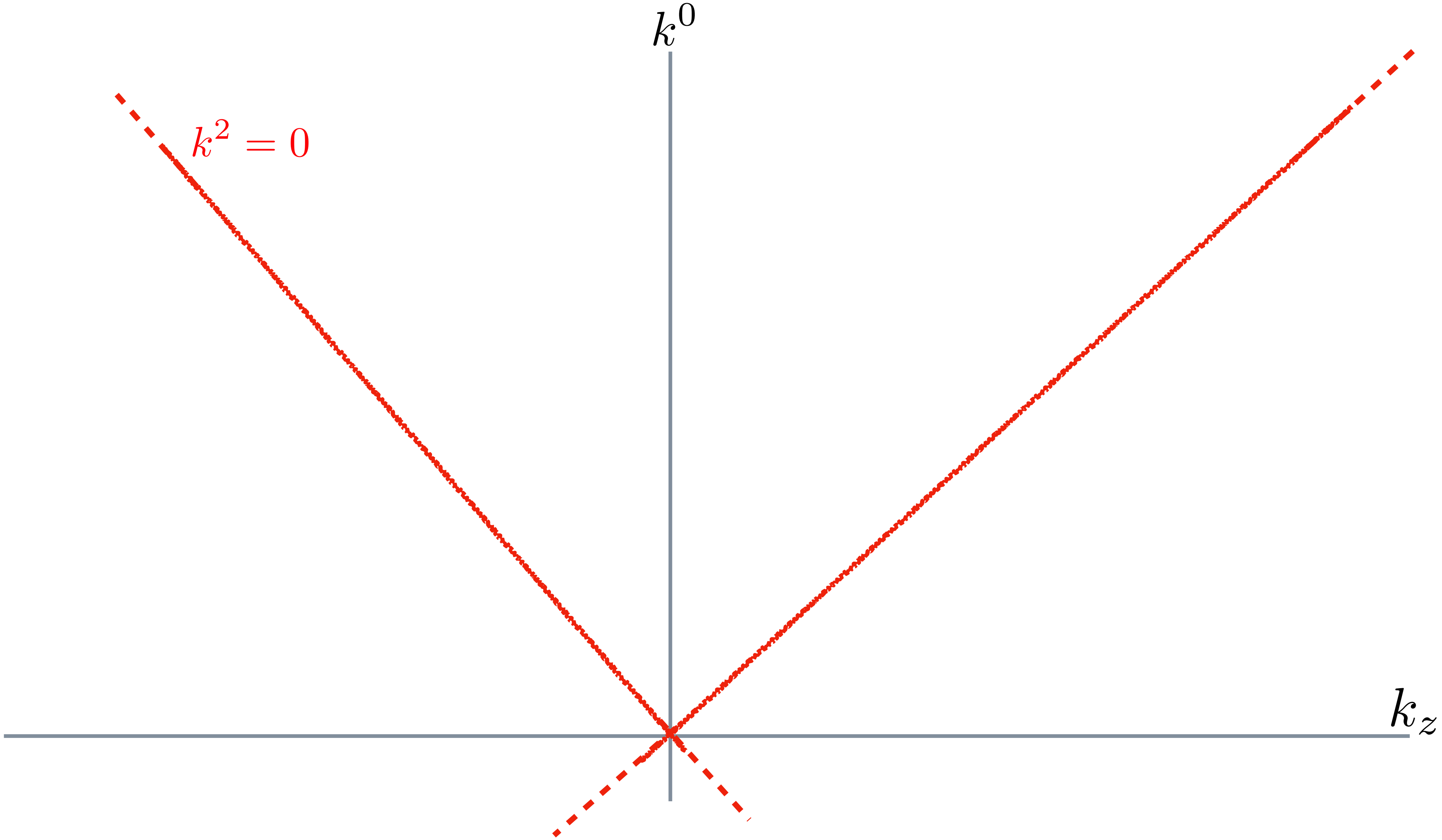
$$E_2 = \sqrt{|\vec{k} - \vec{q}|^2 + m^2 - i\epsilon}$$

$$\int d^3\vec{k}\; I^{(\text{Local Unitarity})} \supset \int d^3\vec{k}\; \frac{1}{E_1 E_2 E_3}\left(\frac{1}{(E_1 + E_2 - Q^0)(E_1 + E_2 + Q^0)}\right)$$

$$\eta(\vec{k}) = E_1 + E_2 - Q^0 \;\overset{\vec{Q}=0\ \ m=0}{\overbrace{=}}\; 2|\vec{k}| - Q^0$$

SINGULAR SURFACES IN MINKOWSKI SPACE

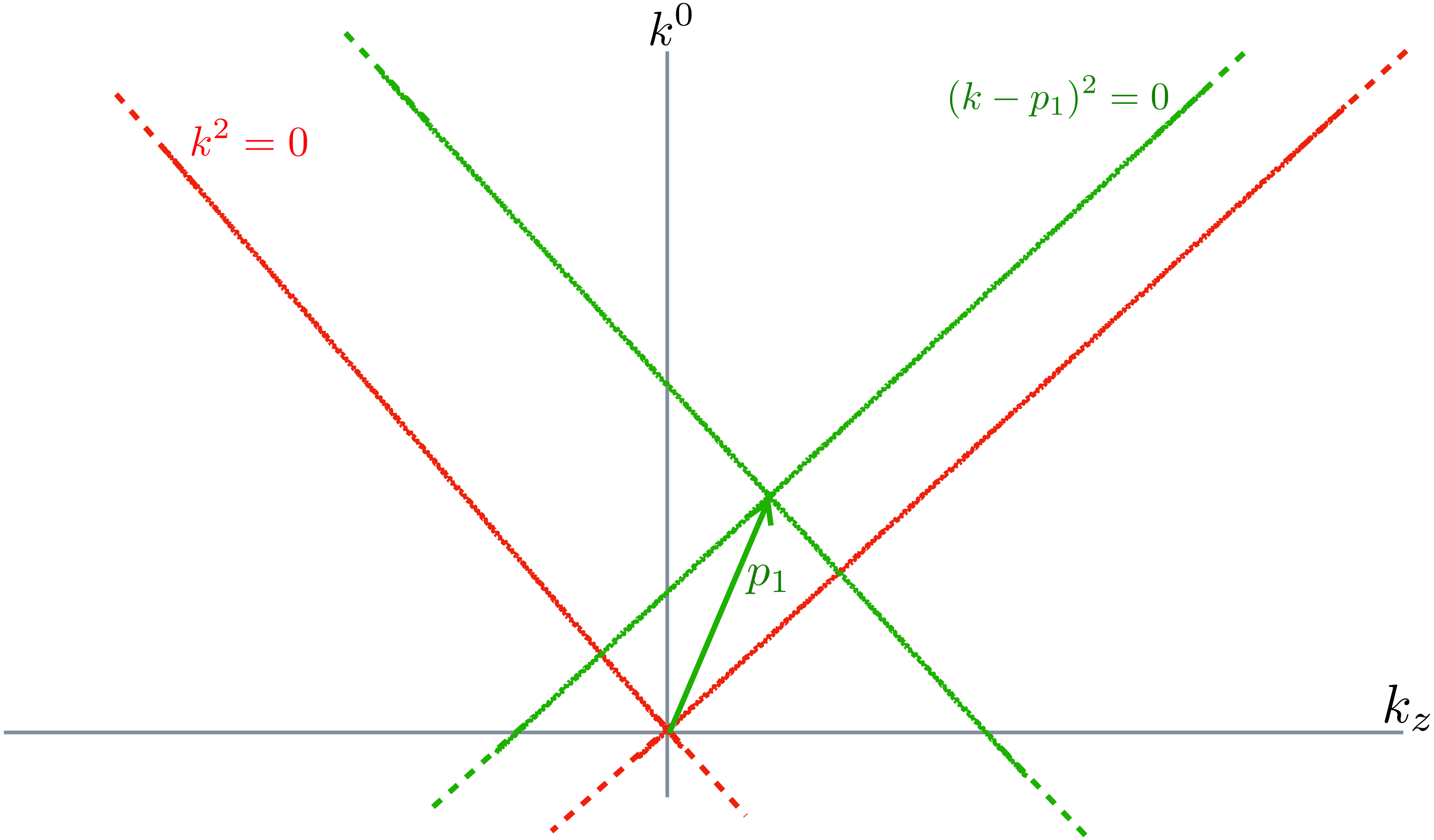
$$\int d^4k \frac{1}{k^2 (k-p_1)^2 (k-p_1-p_2)^2}$$



The integrand is singular along each of the coloured surface

SINGULAR SURFACES IN MINKOWSKI SPACE

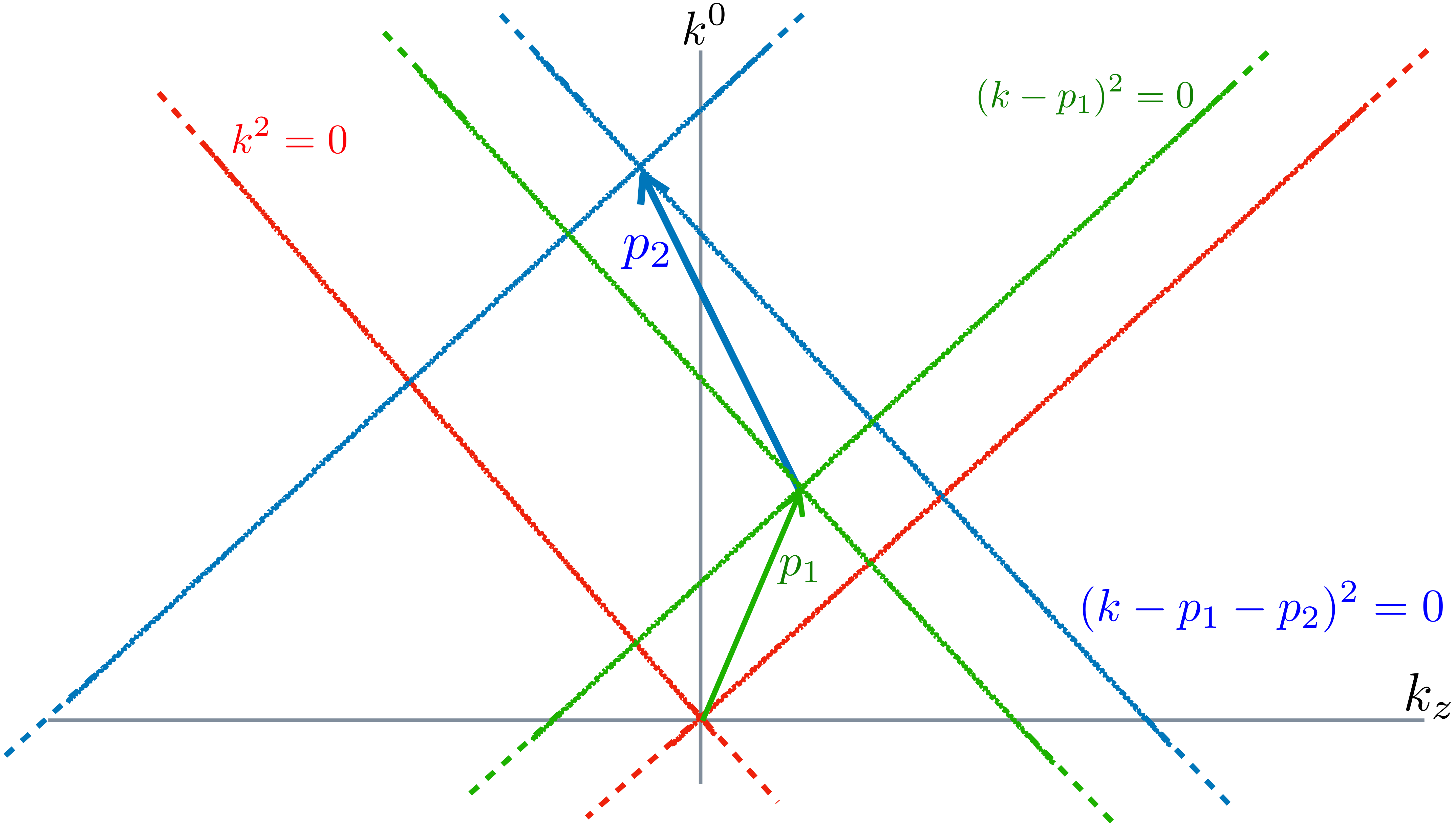
$$\int d^4k \frac{1}{k^2 (k-p_1)^2 (k-p_1-p_2)^2}$$



The integrand is singular along each of the coloured surface

SINGULAR SURFACES IN MINKOWSKI SPACE

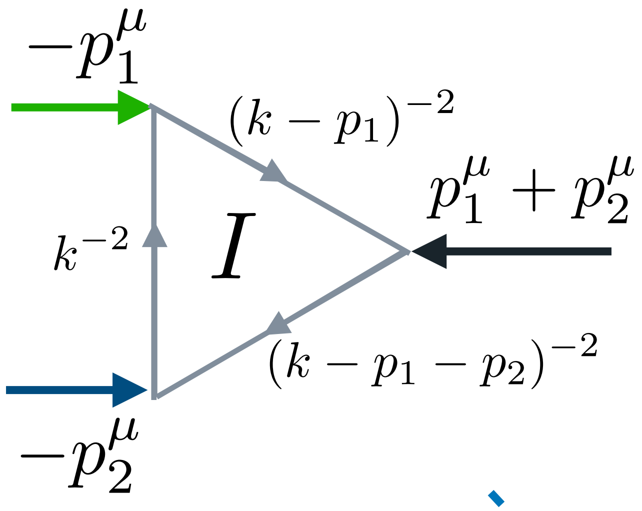
$$\int d^4k \frac{1}{k^2 (k-p_1)^2 (k-p_1-p_2)^2}$$

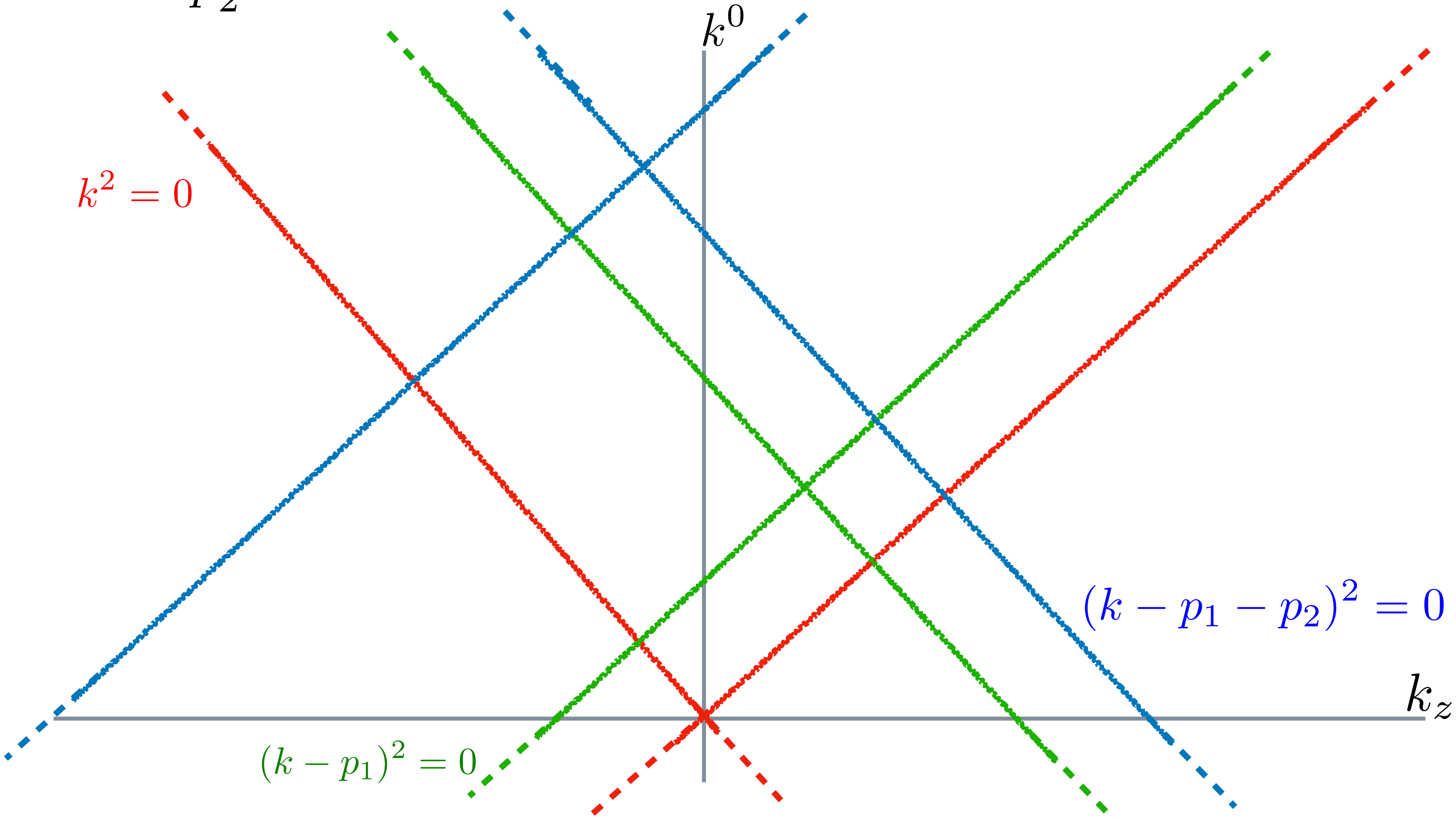


The integrand is singular along each of the coloured surface

SINGULAR SURFACES OF THE LTD REPRESENTATION

Analytically integrate over the loop energies using Cauchy's theorem (LTD):

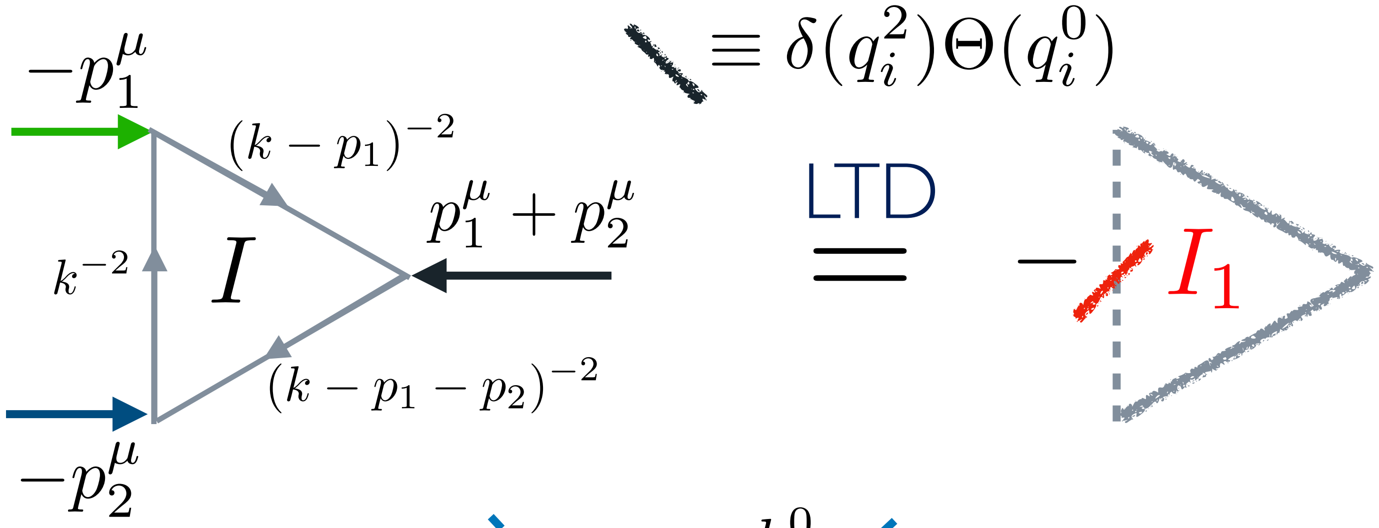


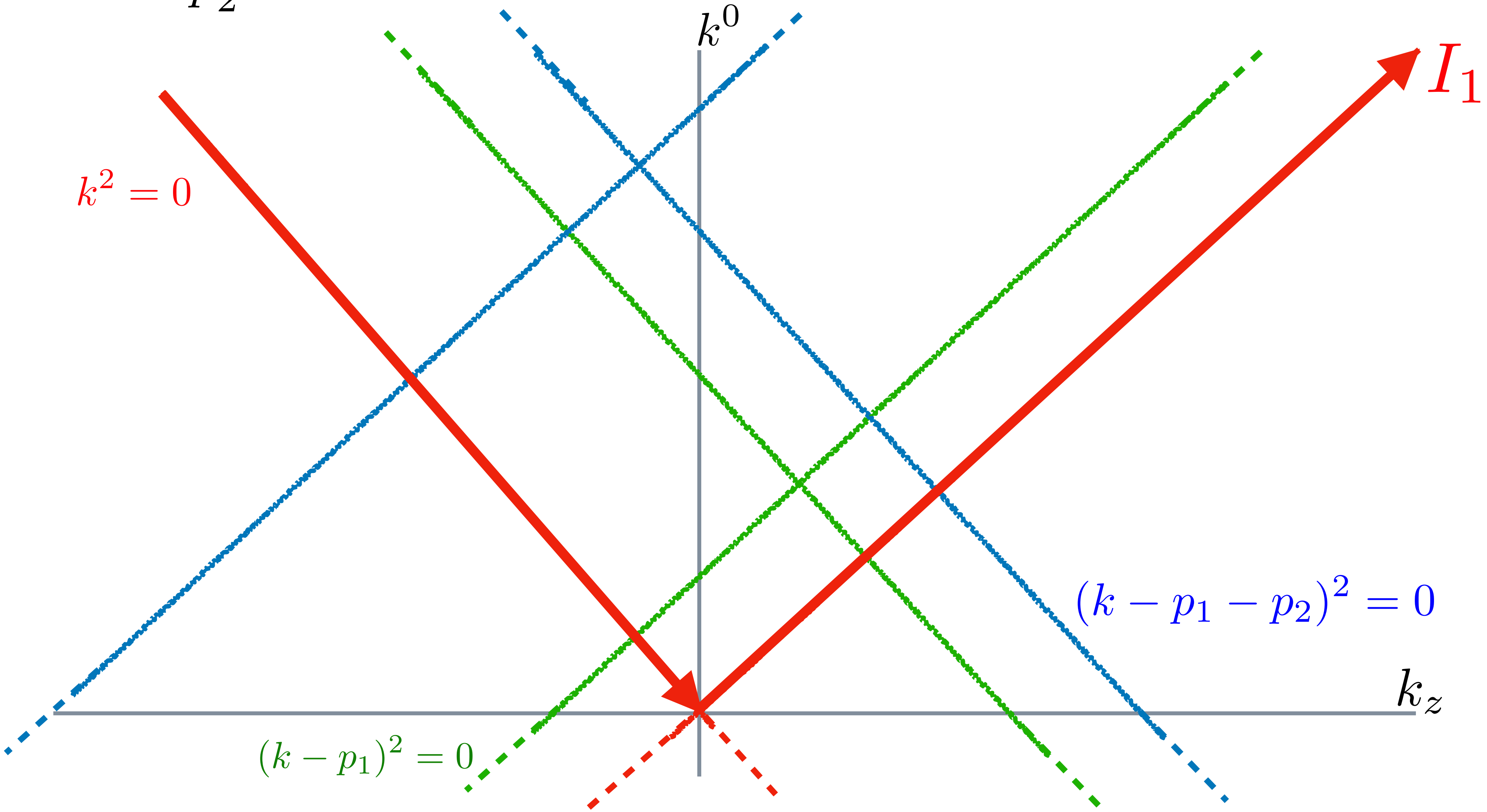


$p_i{}^2 > 0 \quad \forall i$

SINGULAR SURFACES OF THE LTD REPRESENTATION

Analytically integrate over the loop energies using Cauchy's theorem (LTD):

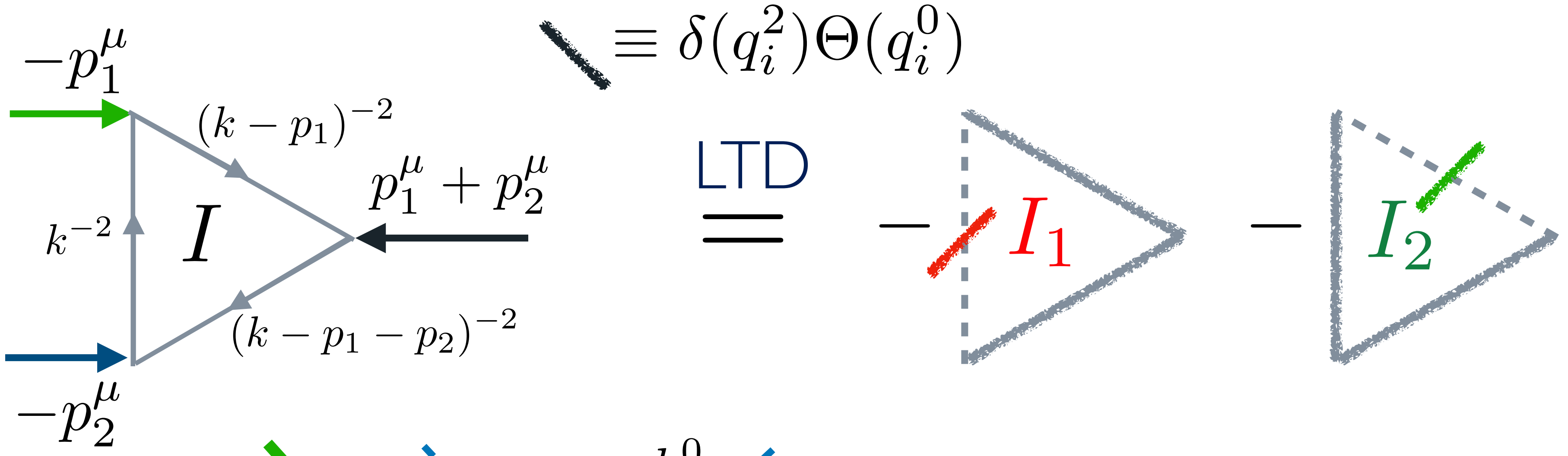


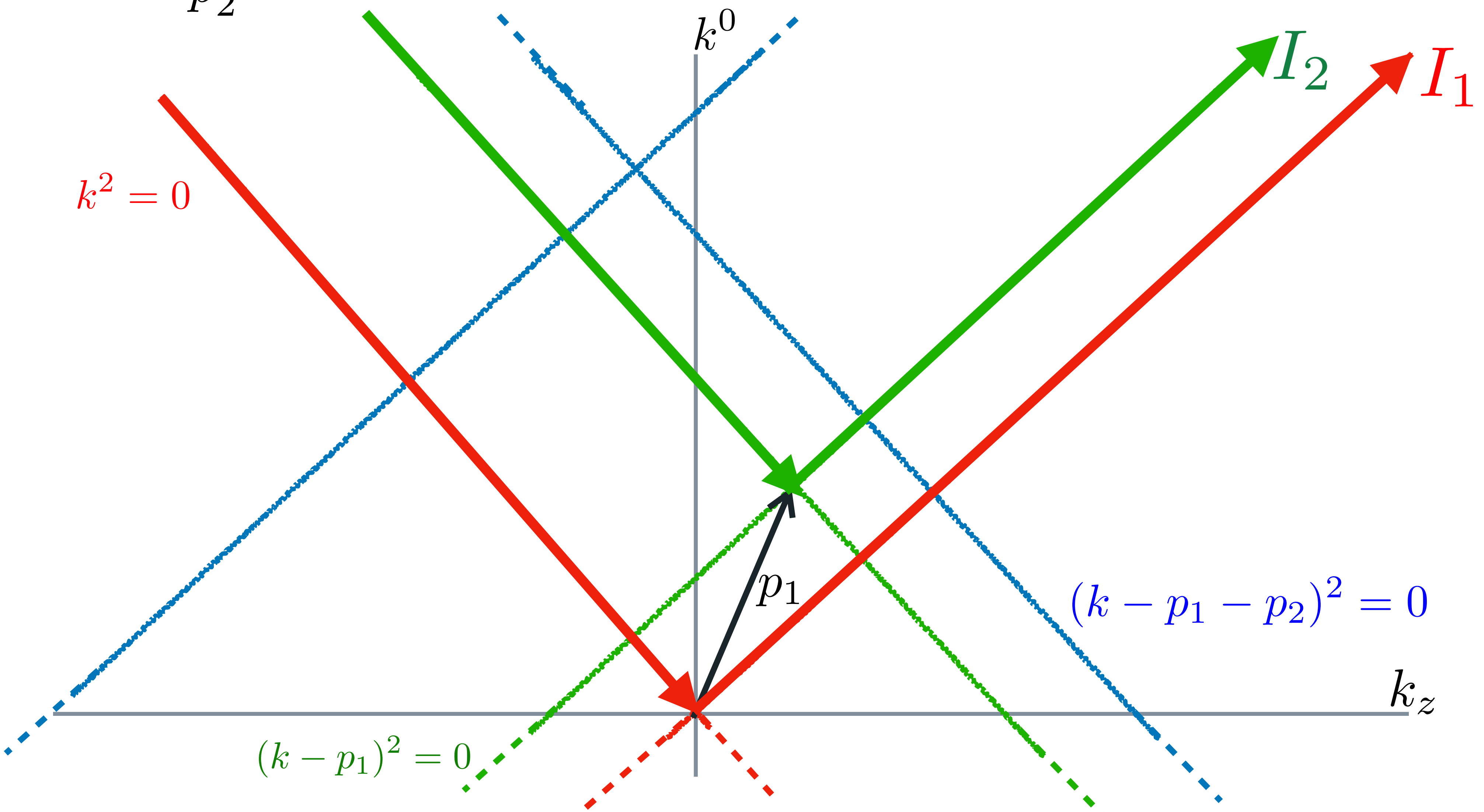


$$p_i{}^2 > 0 \quad \forall i$$

SINGULAR SURFACES OF THE LTD REPRESENTATION

Analytically integrate over the loop energies using Cauchy's theorem (LTD):

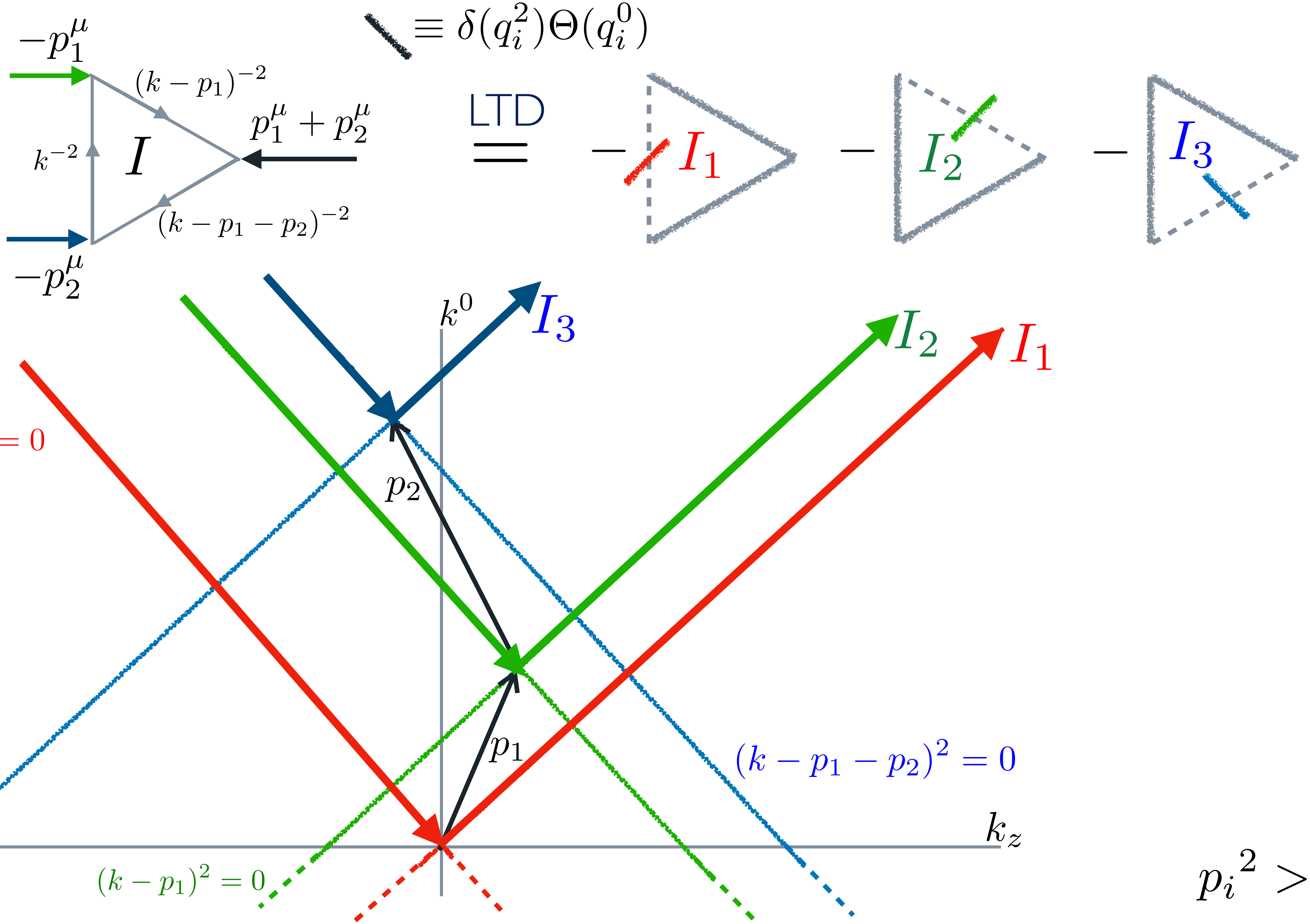




$p_i{}^2 > 0 \quad \forall i$

SINGULAR SURFACES OF THE LTD REPRESENTATION

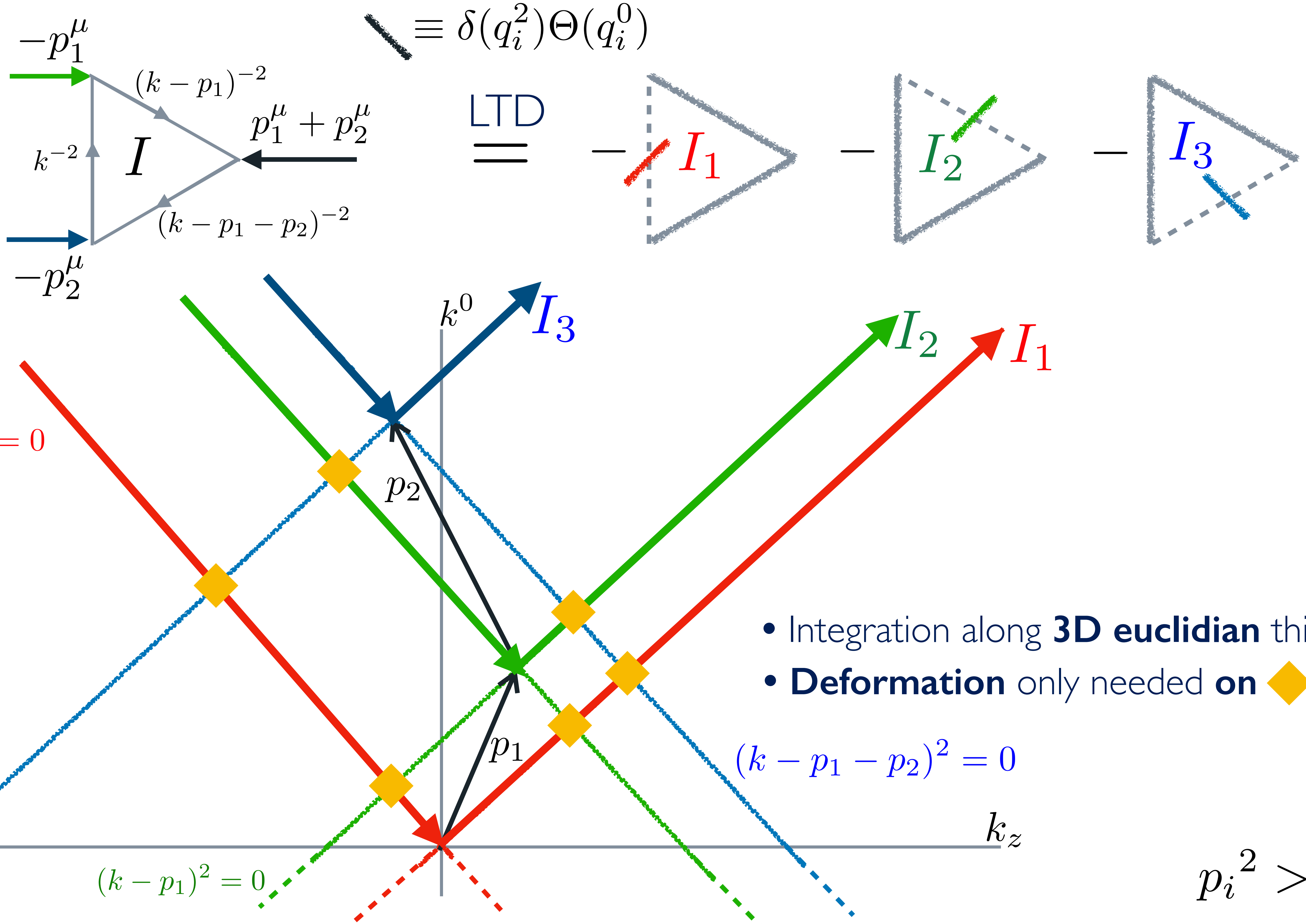
Analytically integrate over the loop energies using Cauchy's theorem (LTD):



$p_i{}^2 > 0 \quad \forall i$

SINGULAR SURFACES OF THE LTD REPRESENTATION

Analytically integrate over the loop energies using Cauchy's theorem (LTD):

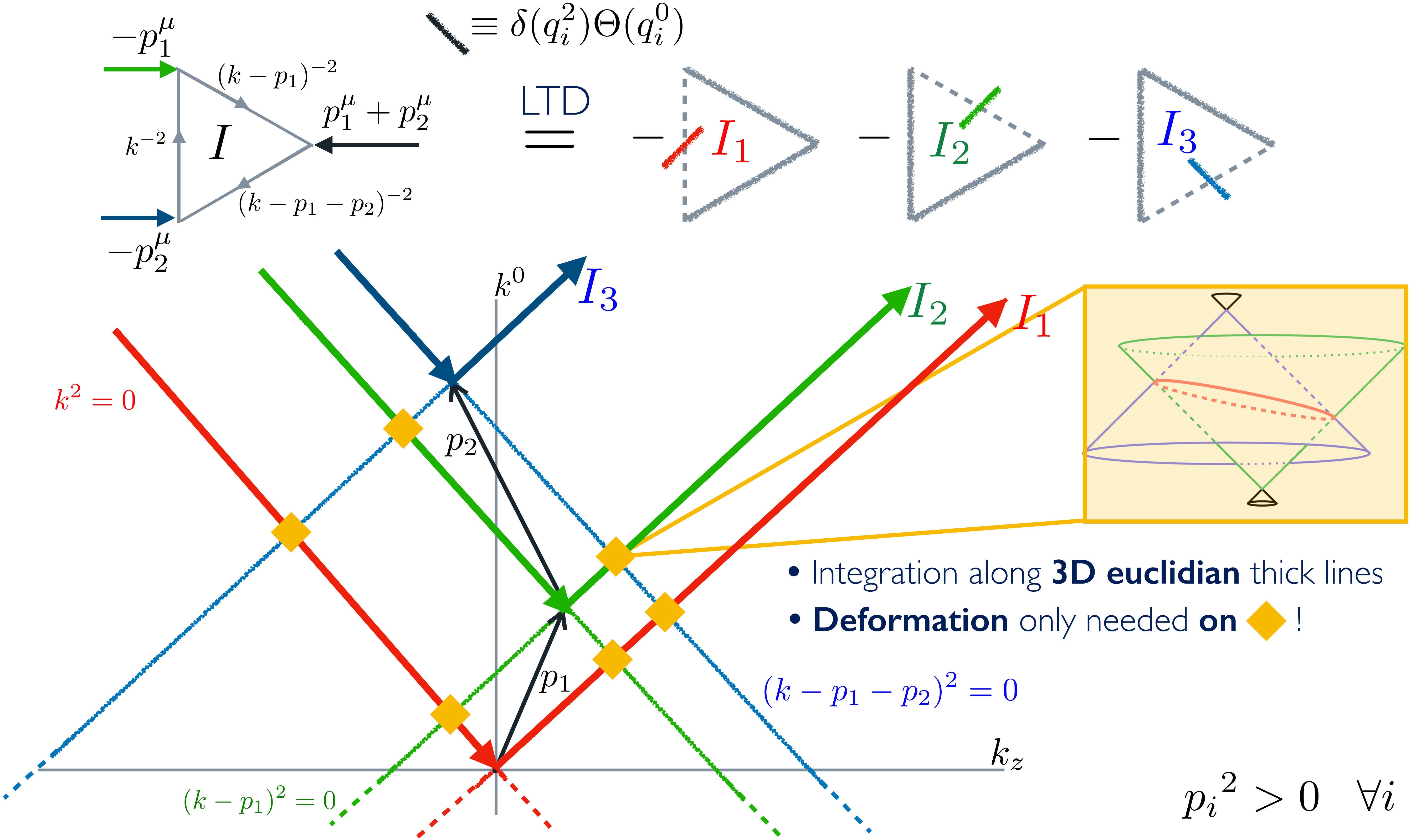


- Integration along **3D euclidian** thick lines
- **Deformation** only needed **on** ◆ !

$$p_i{}^2 > 0 \quad \forall i$$

SINGULAR SURFACES OF THE LTD REPRESENTATION

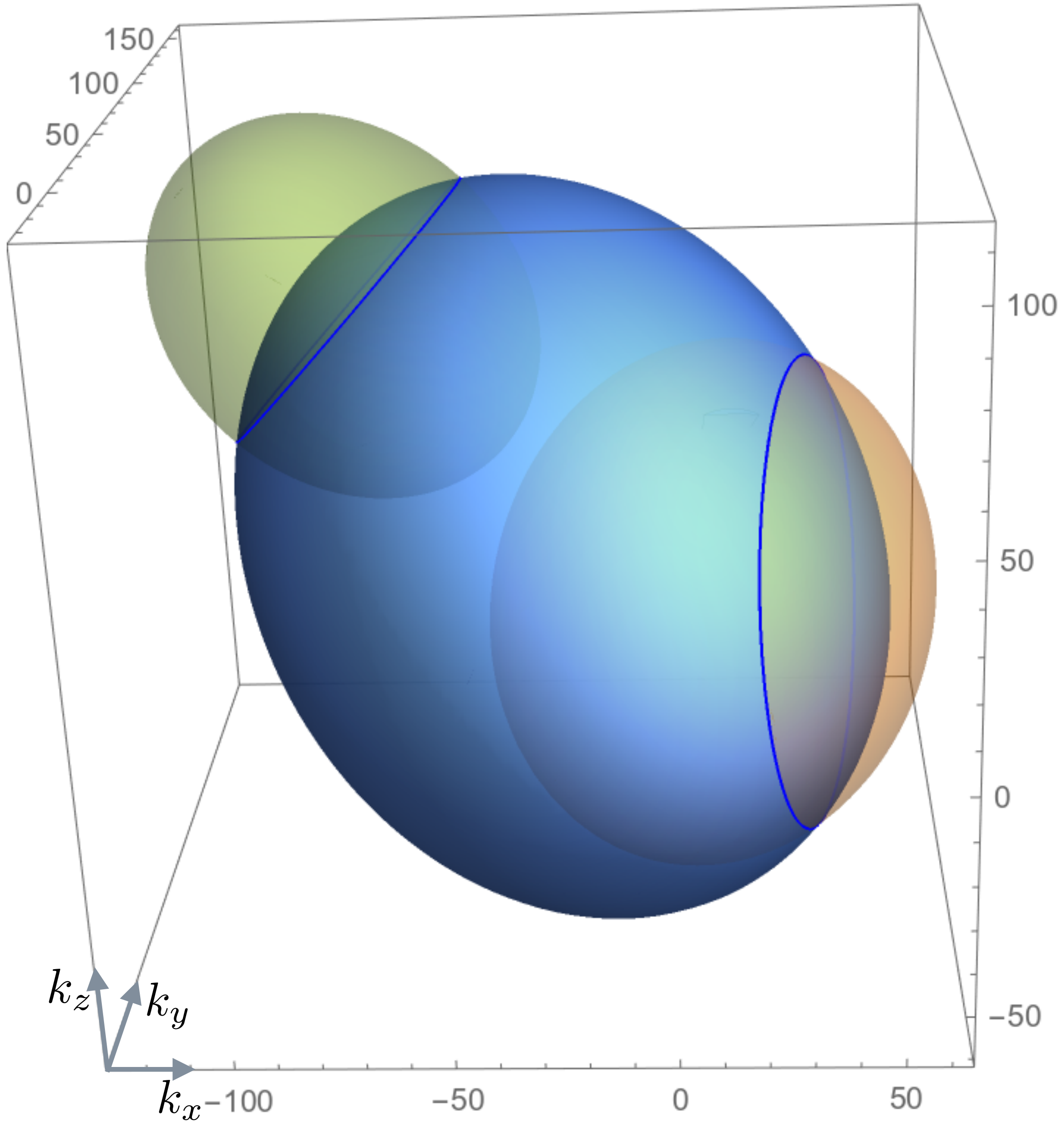
Analytically integrate over the loop energies using Cauchy's theorem (LTD):



- Integration along **3D euclidian** thick lines
- **Deformation** only needed **on** ◆ !

$$p_i{}^2 > 0 \quad \forall i$$

SINGULAR SURFACES - 2D ELLIPSOIDS



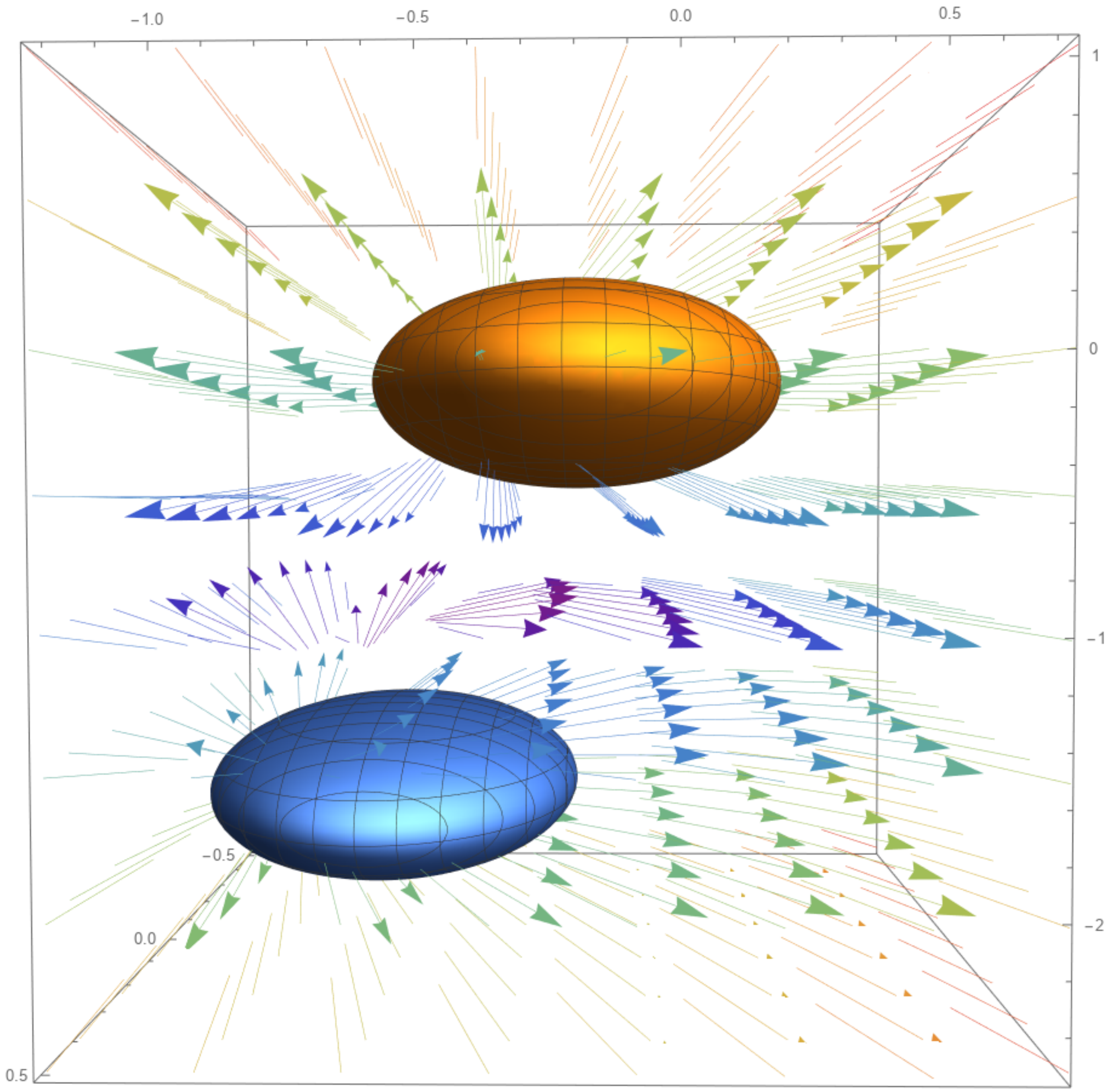
- General **one-loop** ellipsoid **expression**:

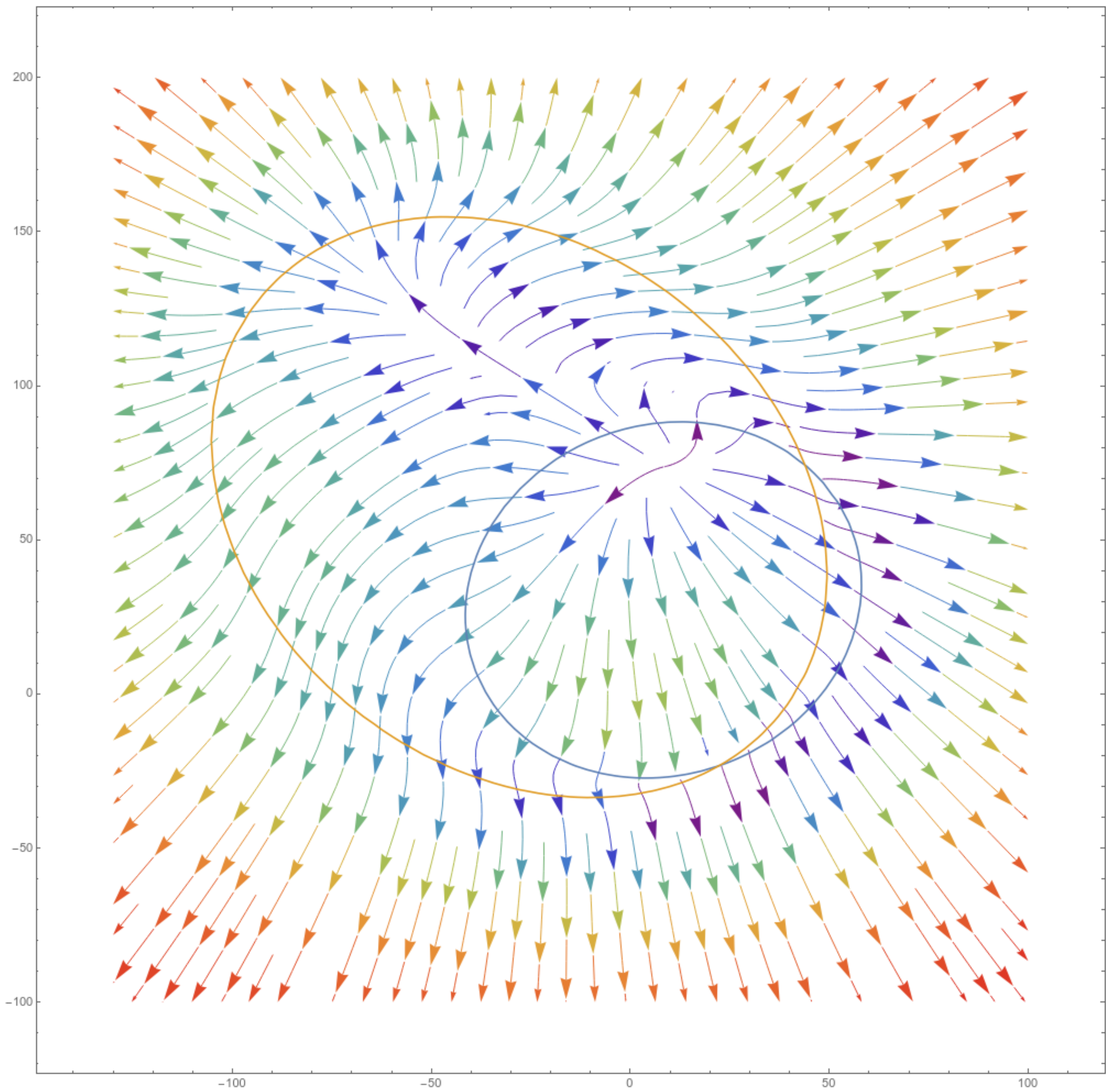
$$E_{ij}(\vec{k}) = \sqrt{\left(\vec{k}+\vec{p_i}\right)^2 + m_i^2 - i\delta} + \sqrt{\left(\vec{k}+\vec{p_j}\right)^2 + m_j^2 - i\delta} - p_i^0 + p_j^0$$

DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

Deformation: $\vec{k} \rightarrow \vec{k} - i\vec{\kappa}$

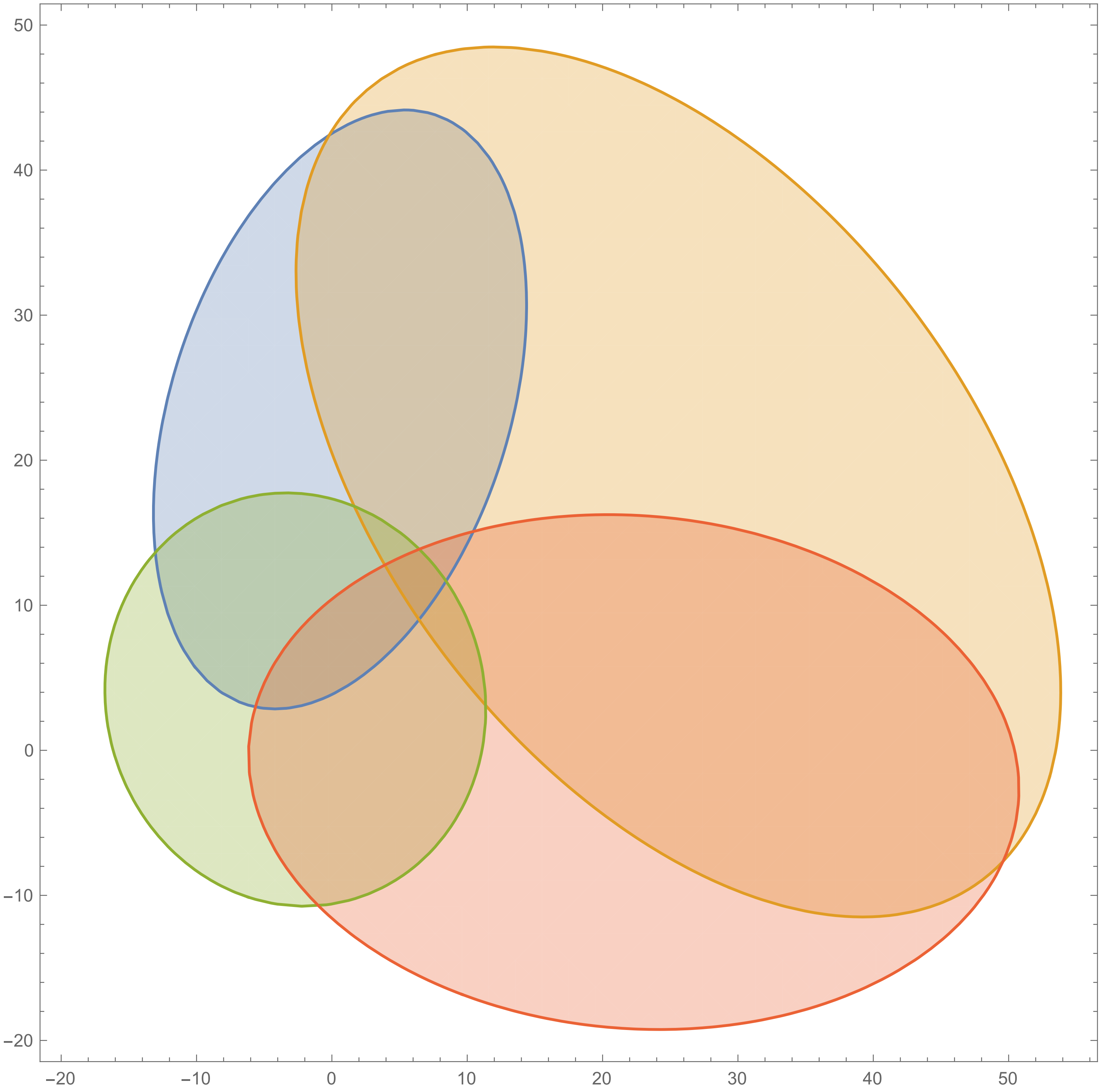
Causal prescription imposes: $\vec{\kappa} \cdot \vec{n}_{E_{ij}} > 0$





DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

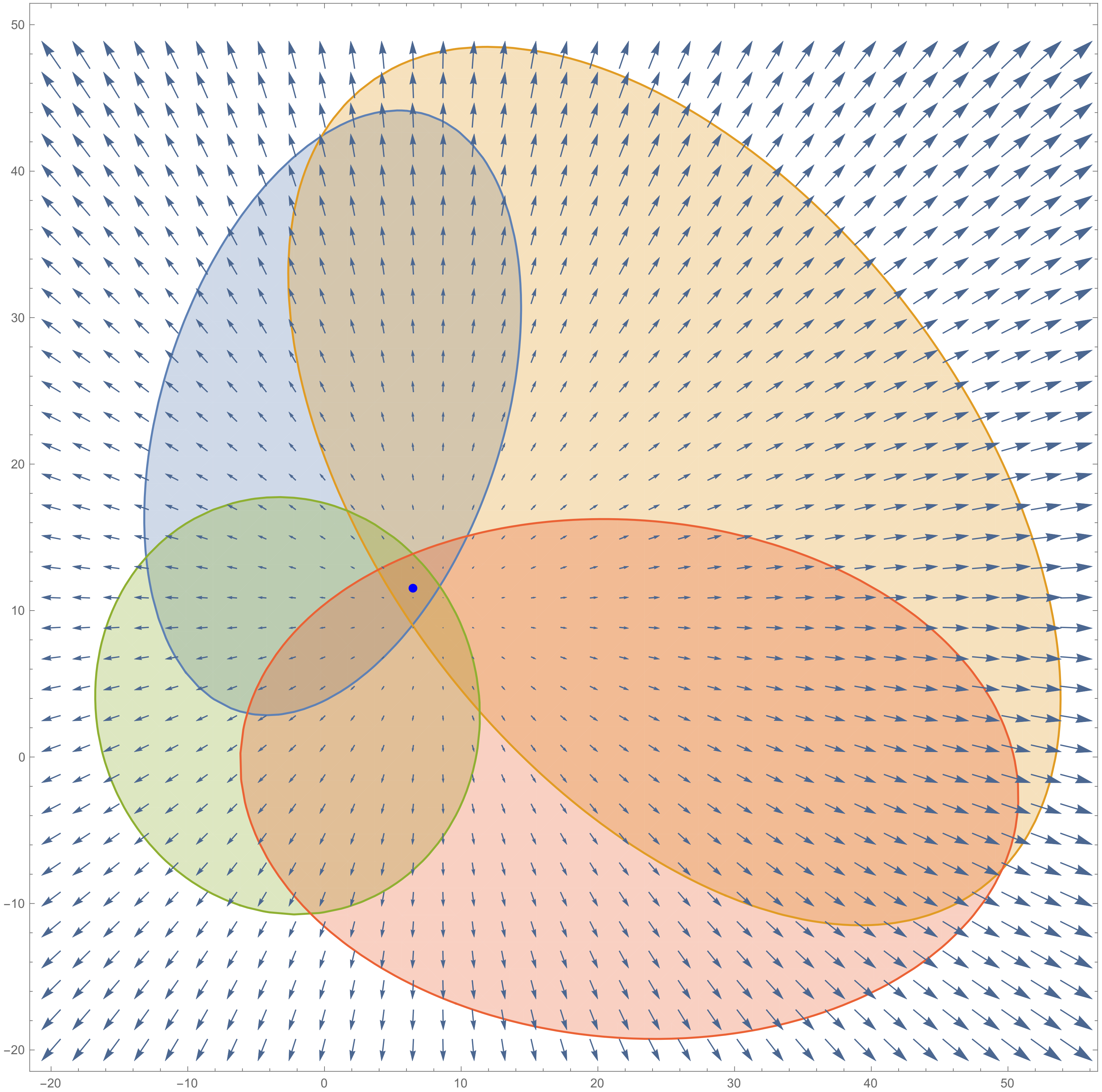
How to construct such a field? For example for this case:



[Capatti, VH, Kermanschah, Pelloni, Ruijl]
[arxiv:1906.06138]

DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

How to construct such a field? For example for this case:

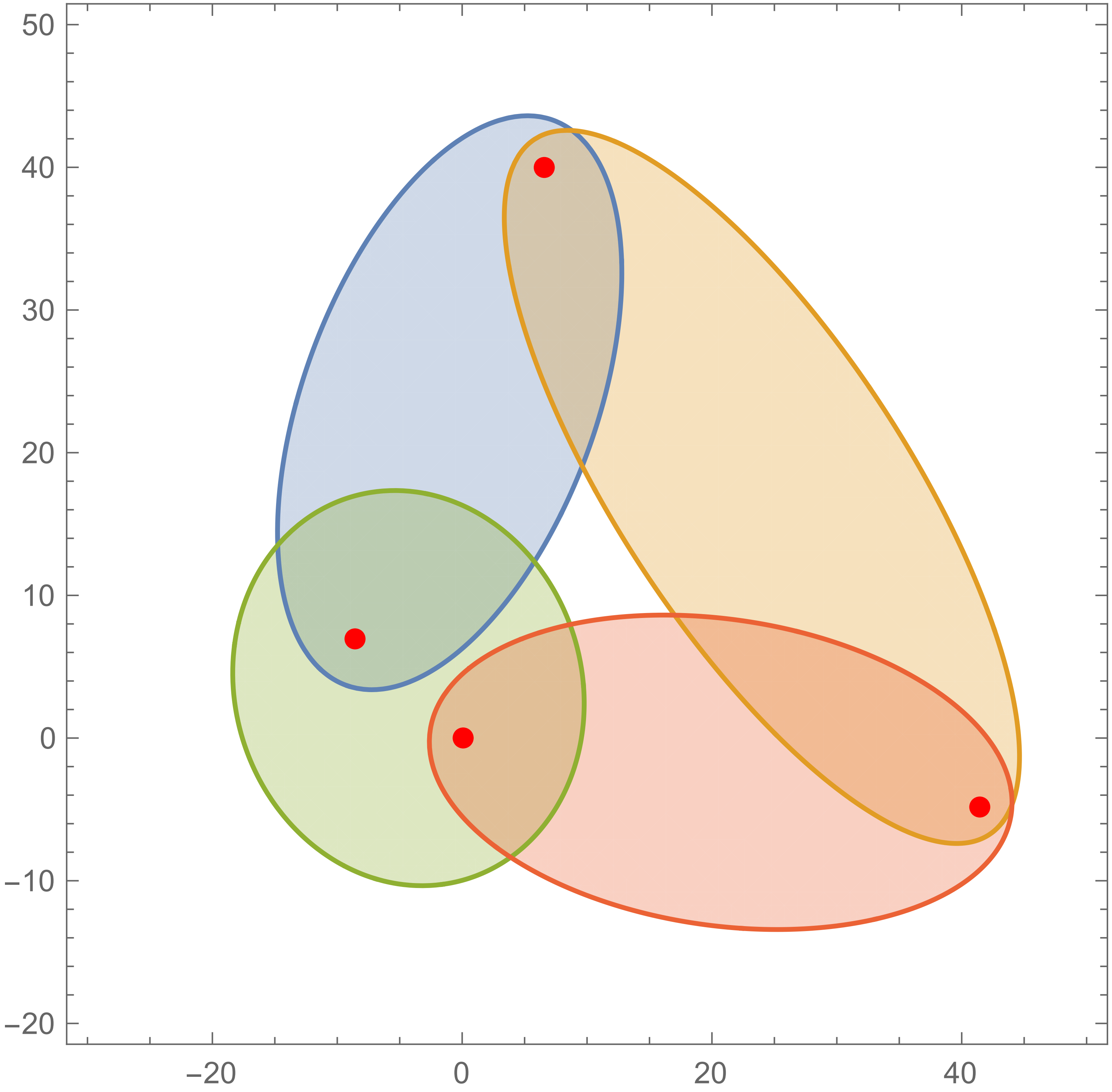


A **radial field** centered in the inside of all ellipsoids!

[Capatti, VH, Kermanschah, Pelloni, Ruijl]
[arxiv:1906.06138]

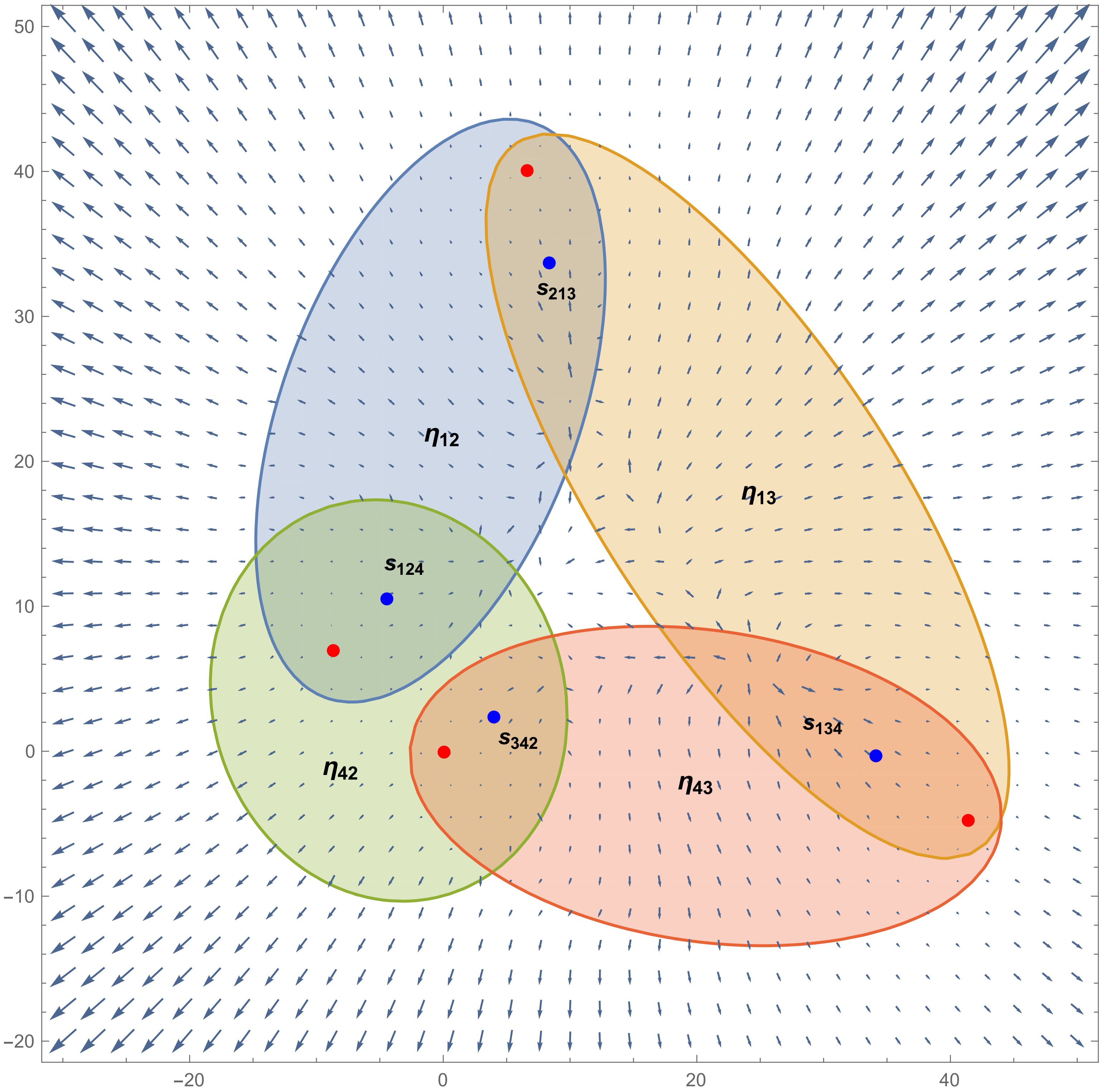
DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

But then what if there is no point in the inside of **all ellipsoids** (`Box4E` example) ?



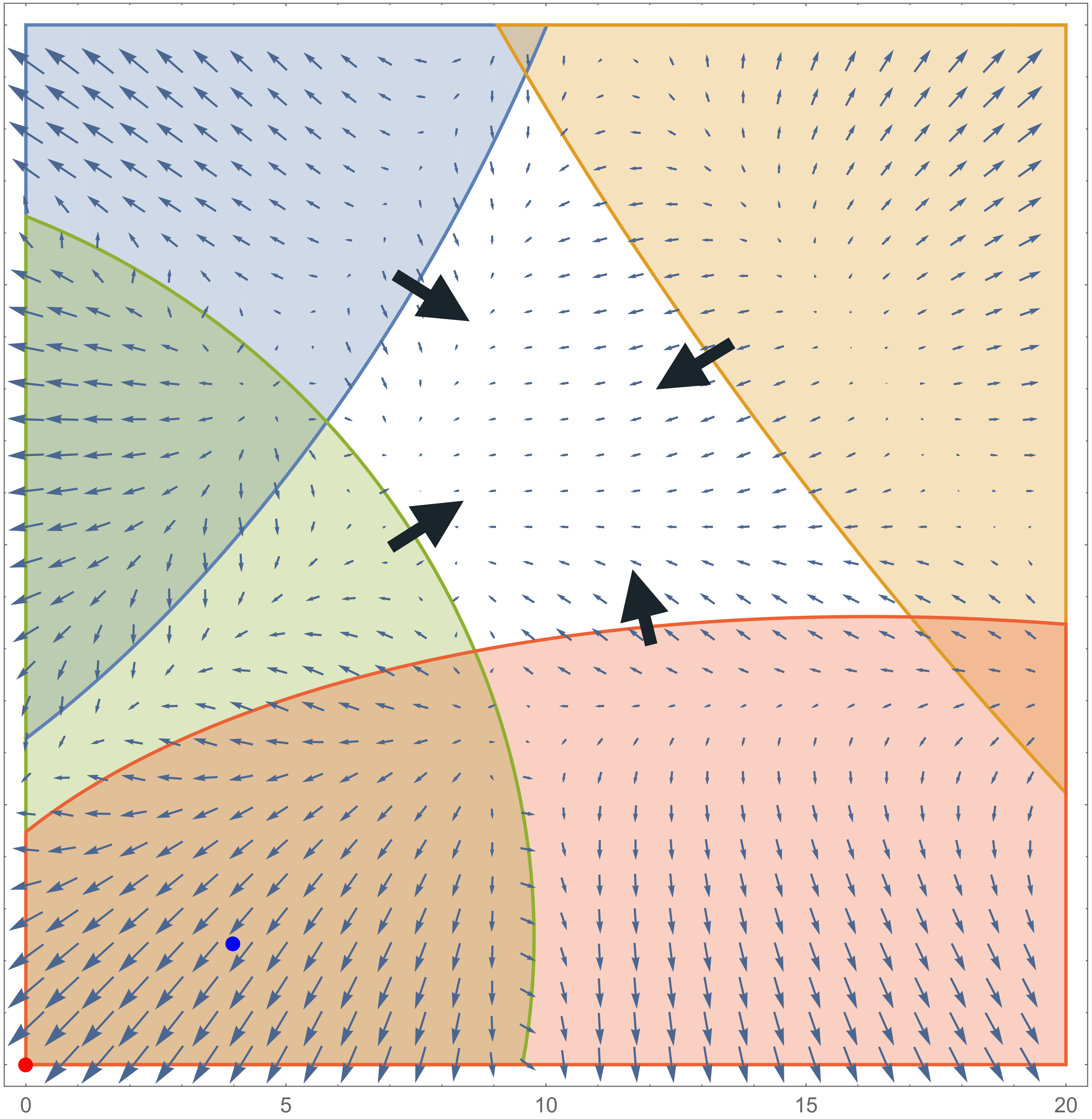
DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

But then what if there is no point in the inside of **all ellipsoids** (`Box4E` example) ?



DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

But then what if there is no point in the inside of **all ellipsoids** (`Box4E` example) ?



THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : 2110.06869]

$$\frac{1}{E_1+E_2-p_1^0} = \frac{1}{|\vec{k}|+|\vec{k}-\vec{p}_1|-p_1^0}$$

$$\overset{p_1^\mu=(2,\vec{0})}{=} \frac{1}{2|\vec{k}|-2} \propto \frac{1}{\sqrt{k_x^2+k_y^2}-1}$$

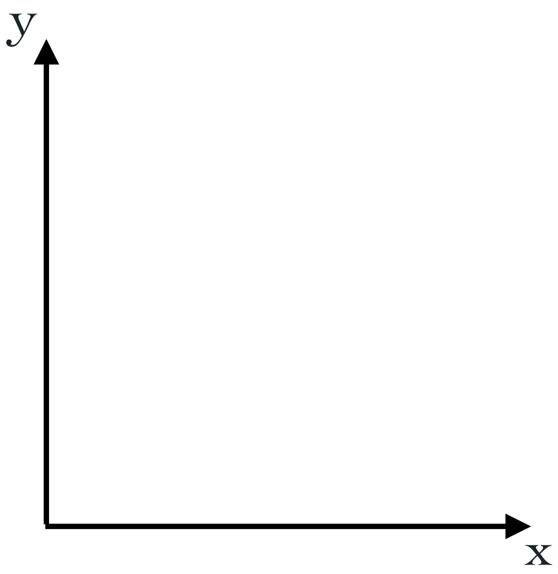
THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : 2110.06869]

$$\frac{1}{E_1+E_2-p_1^0} = \frac{1}{|\vec{k}|+|\vec{k}-\vec{p}_1|-p_1^0}$$

$$\overset{p_1^\mu=(2,\vec{0})}{=} \frac{1}{2|\vec{k}|-2} \propto \frac{1}{\sqrt{k_x^2+k_y^2}-1}$$

$$\lim_{\delta\to 0^+}\int_{-\infty}^{\infty} dxdy\,\frac{2}{\pi^2}\,\frac{1}{x^2+y^2+1}\,\frac{1}{\sqrt{x^2+y^2}-1\pm \mathrm{i}\delta} = 1\mp 2\,\mathrm{i}$$

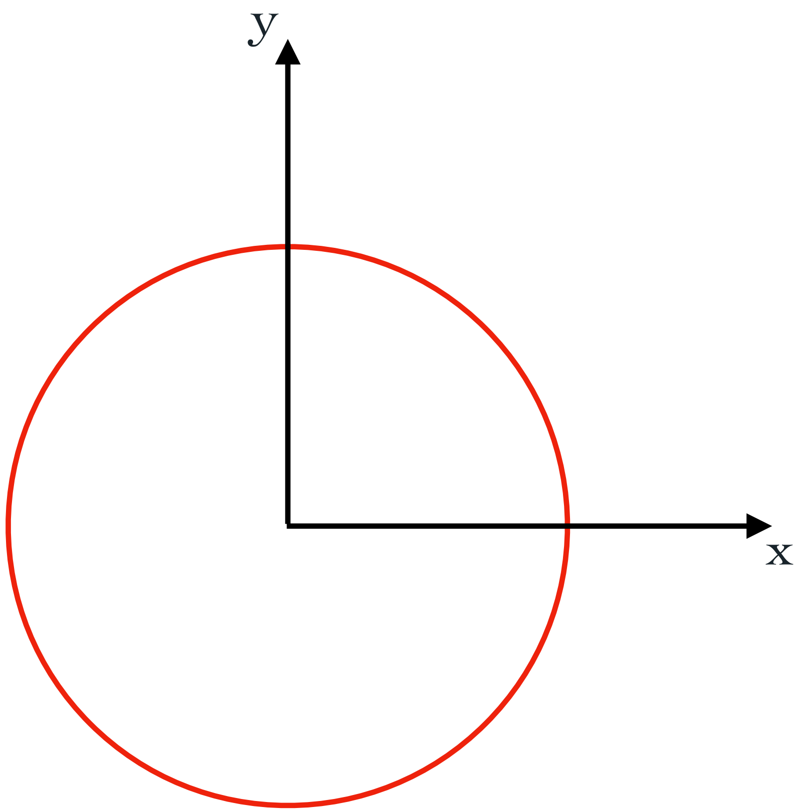


THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

$$\frac{1}{E_1 + E_2 - p_1^0} = \frac{1}{|\vec{k}| + |\vec{k} - \vec{p}_1| - p_1^0} \overset{p_1^\mu = (2,\vec{0})}{=} \frac{1}{2|\vec{k}| - 2} \propto \frac{1}{\sqrt{k_x^2 + k_y^2} - 1}$$

[D. Kermanschah, arXiv : 2110.06869]

$$\lim_{\delta \to 0^+} \int_{-\infty}^{\infty} dx dy \frac{2}{\pi^2} \frac{1}{x^2 + y^2 + 1} \frac{1}{\sqrt{x^2 + y^2} - 1 \pm \mathrm{i}\delta} = 1 \mp 2\,\mathrm{i}$$



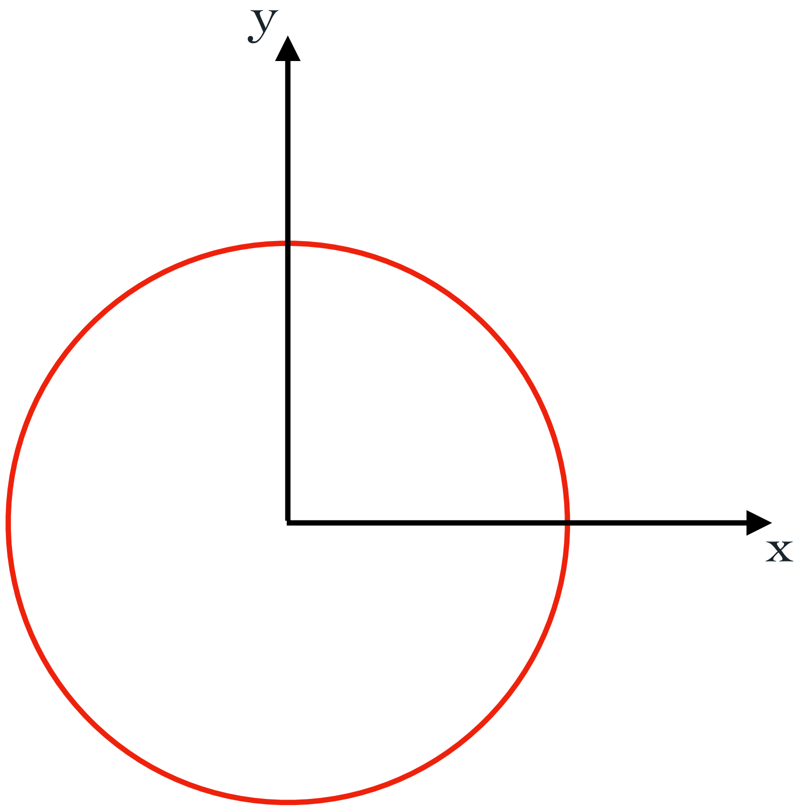
THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : 2110.06869]

$$\frac{1}{E_1+E_2-p_1^0} = \frac{1}{|\vec{k}|+|\vec{k}-\vec{p}_1|-p_1^0} \overset{p_1^\mu=(2,\vec{0})}{=} \frac{1}{2|\vec{k}|-2} \propto \frac{1}{\sqrt{k_x^2+k_y^2}-1}$$

$$\lim_{\delta\to 0^+}\int_{-\infty}^{\infty} dxdy\frac{2}{\pi^2}\frac{1}{x^2+y^2+1}\frac{1}{\sqrt{x^2+y^2}-1\pm \mathrm{i}\delta} = 1\mp 2\,\mathrm{i}$$

$$= \lim_{\delta\to 0^+}\frac{4}{\pi}\left[\int_0^\infty dr\left(\frac{1}{r^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta} \qquad\qquad\qquad\qquad \right]$$



THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : 2110.06869]

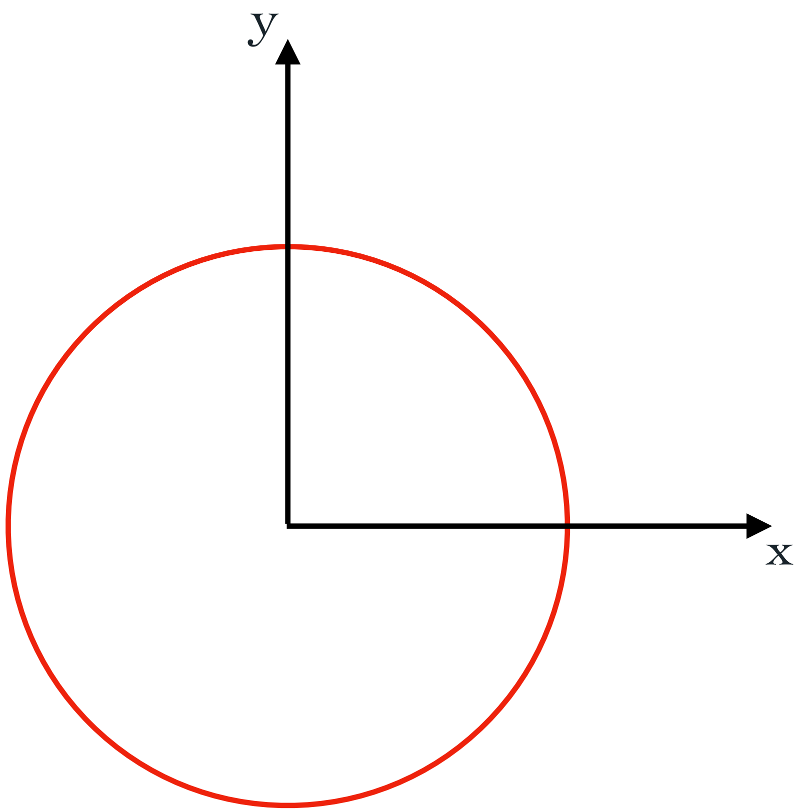
$$\frac{1}{E_1+E_2-p_1^0} = \frac{1}{|\vec{k}|+|\vec{k}-\vec{p}_1|-p_1^0}$$

$$\overset{p_1^\mu=(2,\vec{0})}{=} \frac{1}{2|\vec{k}|-2} \propto \frac{1}{\sqrt{k_x^2+k_y^2}-1}$$

$$\lim_{\delta\to 0^+} \int_{-\infty}^{\infty} dxdy \frac{2}{\pi^2} \frac{1}{x^2+y^2+1} \frac{1}{\sqrt{x^2+y^2}-1\pm \mathrm{i}\delta} = 1 \mp 2\,\mathrm{i}$$

Local threshold counterterm

$$= \lim_{\delta\to 0^+} \frac{4}{\pi}\left[\int_0^{\infty} dr \left(\frac{1}{r^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta} - \int_{1-\Delta}^{1+\Delta} dr \left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta}\right]$$



THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

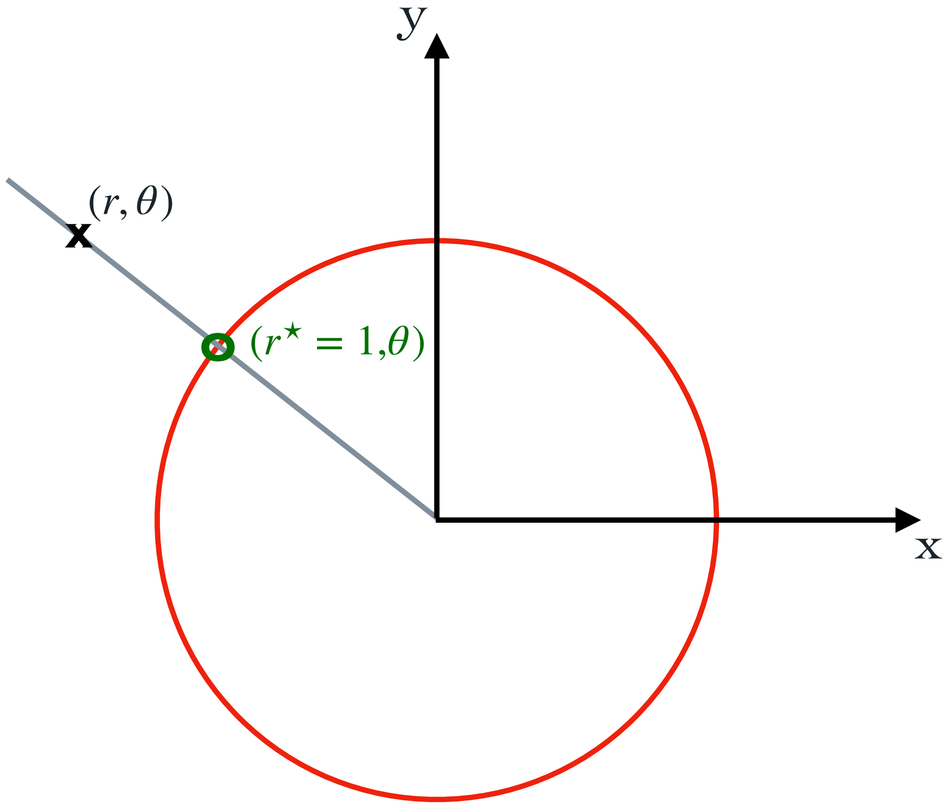
[D. Kermanschah, arXiv : 2110.06869]

$$\frac{1}{E_1+E_2-p_1^0} = \frac{1}{|\vec{k}|+|\vec{k}-\vec{p}_1|-p_1^0} \overset{p_1^\mu=(2,\vec{0})}{=} \frac{1}{2|\vec{k}|-2} \propto \frac{1}{\sqrt{k_x^2+k_y^2}-1}$$

$$\lim_{\delta\to 0^+}\int_{-\infty}^{\infty} dxdy \frac{2}{\pi^2}\frac{1}{x^2+y^2+1}\frac{1}{\sqrt{x^2+y^2}-1\pm \mathrm{i}\delta} = 1 \mp 2\,\mathrm{i}$$

Local threshold counterterm

$$= \lim_{\delta\to 0^+}\frac{4}{\pi}\left[\int_0^\infty dr\left(\frac{1}{r^2+1}\right)\frac{1}{1-r\pm\mathrm{i}\delta} - \int_{1-\Delta}^{1+\Delta} dr\left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm\mathrm{i}\delta}\right]$$



THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : 2110.06869]

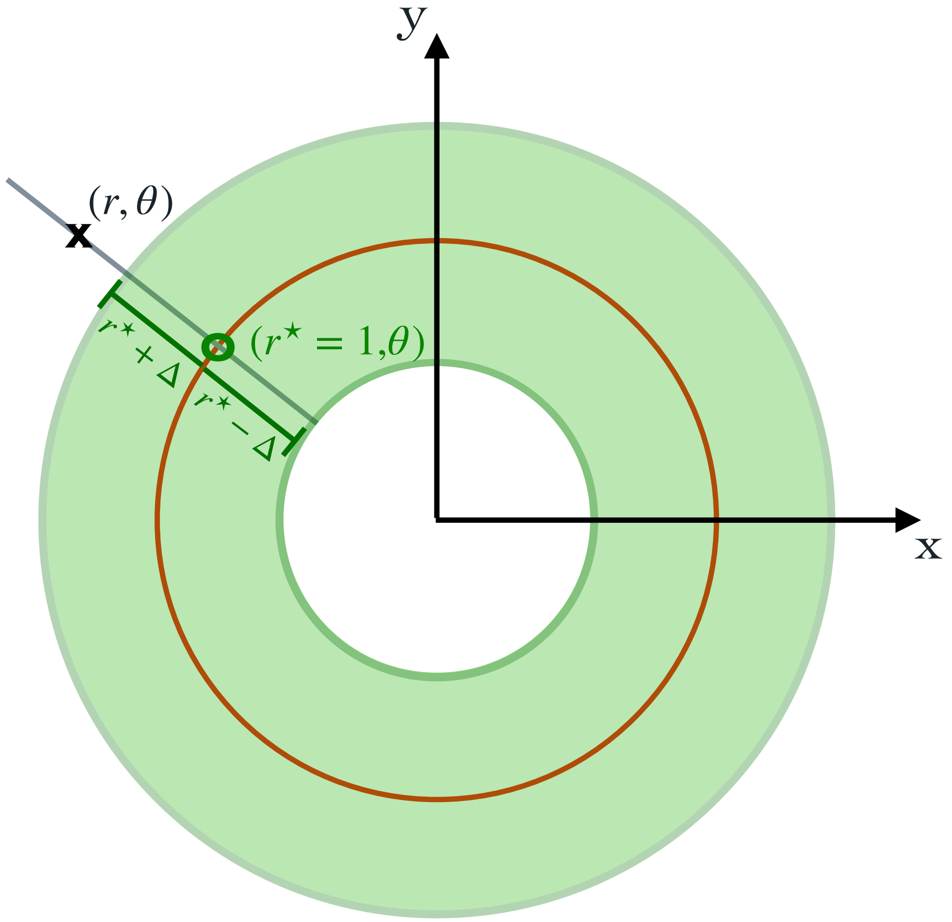
$$\frac{1}{E_1+E_2-p_1^0} = \frac{1}{|\vec{k}|+|\vec{k}-\vec{p}_1|-p_1^0}$$

$$\overset{p_1^\mu=(2,\vec{0})}{=} \frac{1}{2|\vec{k}|-2} \propto \frac{1}{\sqrt{k_x^2+k_y^2}-1}$$

$$\lim_{\delta\to 0^+}\int_{-\infty}^{\infty} dxdy\,\frac{2}{\pi^2}\,\frac{1}{x^2+y^2+1}\,\frac{1}{\sqrt{x^2+y^2}-1\pm \mathrm{i}\delta} = 1\mp 2\,\mathrm{i}$$

Local threshold counterterm

$$= \lim_{\delta\to 0^+}\frac{4}{\pi}\left[\int_0^\infty dr\left(\frac{1}{r^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta} - \int_{\boxed{1-\Delta}}^{\boxed{1+\Delta}} dr\left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta}\right]$$



THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : 2110.06869]

$$\frac{1}{E_1+E_2-p_1^0} = \frac{1}{|\vec{k}|+|\vec{k}-\vec{p}_1|-p_1^0} \overset{p_1^\mu=(2,\vec{0})}{=} \frac{1}{2|\vec{k}|-2} \propto \frac{1}{\sqrt{k_x^2+k_y^2}-1}$$

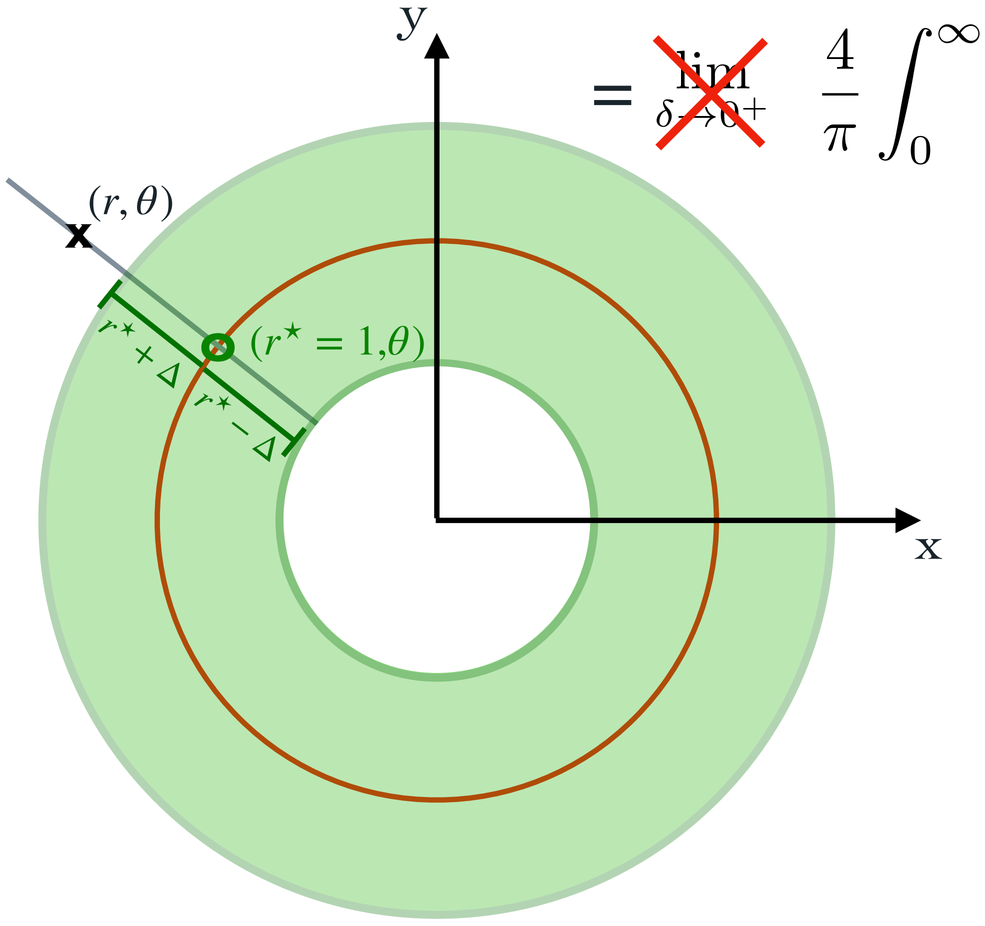
$$\lim_{\delta\to0^+}\int_{-\infty}^{\infty} dxdy\frac{2}{\pi^2}\frac{1}{x^2+y^2+1}\frac{1}{\sqrt{x^2+y^2}-1\pm\mathrm{i}\delta} = \boxed{1} \mp 2\,\mathrm{i}$$

Local threshold counterterm

$$= \lim_{\delta\to0^+}\frac{4}{\pi}\left[\int_0^\infty dr\left(\frac{1}{r^2+1}\right)\frac{1}{1-r\pm\mathrm{i}\delta} - \int_{\boxed{1-\Delta}}^{\boxed{1+\Delta}} dr\left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm\mathrm{i}\delta}\right]$$

$$= \xcancel{\lim_{\delta\to0^+}}\ \frac{4}{\pi}\int_0^\infty dr\left[\left(\frac{1}{r^2+1} - \frac{\Theta[\Delta+(r-1)]\Theta[\Delta-(r-1)]}{2}\right)\frac{1}{1-r}\right] = \boxed{1}$$

$\delta = 0$ can be taken safely !



THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : 2110.06869]

$$\frac{1}{E_1+E_2-p_1^0} = \frac{1}{|\vec{k}|+|\vec{k}-\vec{p}_1|-p_1^0}$$

$$\overset{p_1^\mu=(2,\vec{0})}{=} \frac{1}{2|\vec{k}|-2} \propto \frac{1}{\sqrt{k_x^2+k_y^2}-1}$$

$$\lim_{\delta\to0^+}\int_{-\infty}^{\infty} dxdy\,\frac{2}{\pi^2}\,\frac{1}{x^2+y^2+1}\,\frac{1}{\sqrt{x^2+y^2}-1\pm \mathrm{i}\delta} = \boxed{1} \mp 2\,\mathrm{i}$$

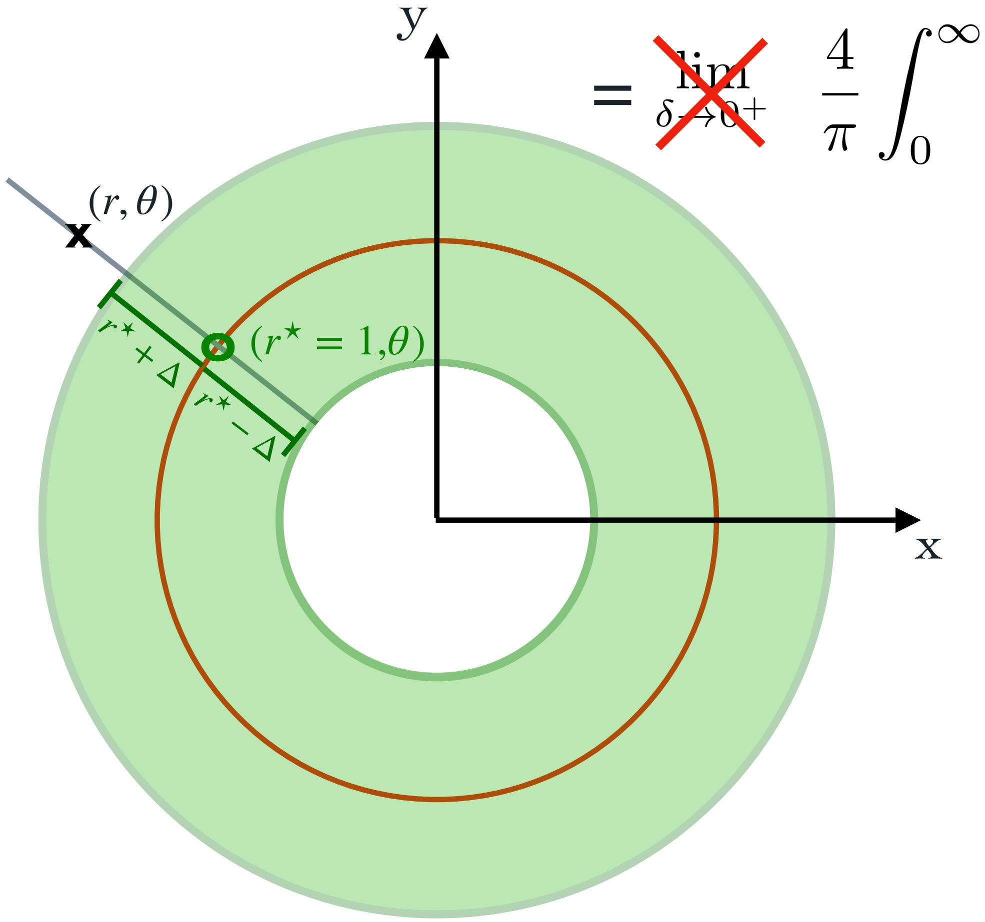
Local threshold counterterm

Integrated threshold counterterm

$$= \underline{\lim_{\delta\to0^+}\frac{4}{\pi}\left[\int_0^\infty dr\left(\frac{1}{r^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta} - \int_{\boxed{1-\Delta}}^{\boxed{1+\Delta}} dr \left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta}\right]} + \frac{4}{\pi}\int_{1-\Delta}^{1+\Delta} dr\left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta}$$

$$= \cancel{\lim_{\delta\to0^+}}\ \frac{4}{\pi}\int_0^\infty dr\left[\left(\frac{1}{r^2+1} - \frac{\Theta[\Delta+(r-1)]\Theta[\Delta-(r-1)]}{2}\right)\frac{1}{1-r}\right] = \boxed{1}$$

$\delta = 0$ can be taken safely !



THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

$$\frac{1}{E_1+E_2-p_1^0} = \frac{1}{|\vec{k}|+|\vec{k}-\vec{p}_1|-p_1^0} \overset{p_1^\mu=(2,\vec{0})}{=} \frac{1}{2|\vec{k}|-2} \propto \frac{1}{\sqrt{k_x^2+k_y^2}-1}$$

[D. Kermanschah, arXiv : 2110.06869]

$$\lim_{\delta\to 0^+}\int_{-\infty}^{\infty} dxdy\,\frac{2}{\pi^2}\frac{1}{x^2+y^2+1}\frac{1}{\sqrt{x^2+y^2}-1\pm \mathrm{i}\delta} = \boxed{1}\boxed{\mp 2\,\mathrm{i}}$$

Local threshold counterterm

Integrated threshold counterterm

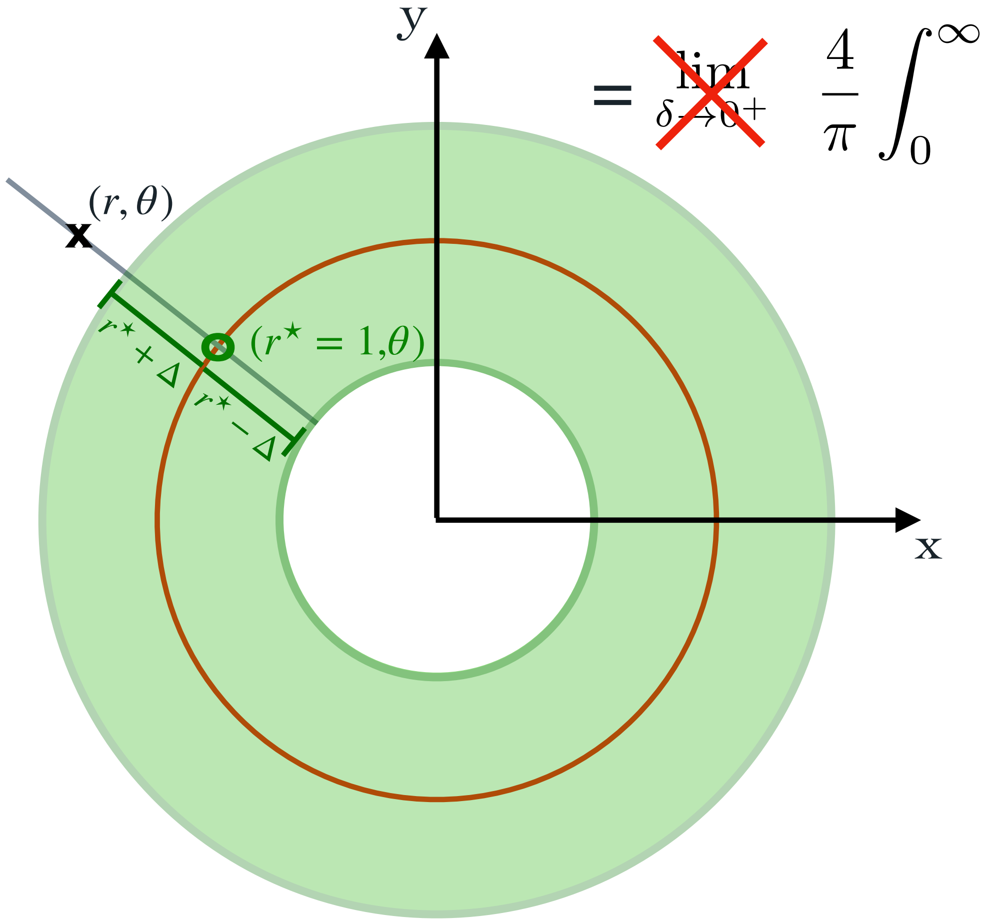
$$= \lim_{\delta\to 0^+}\frac{4}{\pi}\left[\int_0^{\infty} dr\left(\frac{1}{r^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta} - \int_{\boxed{1-\Delta}}^{\boxed{1+\Delta}} dr\left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta}\right] + \frac{4}{\pi}\int_{1-\Delta}^{1+\Delta} dr\left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm \mathrm{i}\delta}$$

$$= \cancel{\lim_{\delta\to 0^+}}\ \frac{4}{\pi}\int_0^{\infty} dr\left[\left(\frac{1}{r^2+1} - \frac{\Theta[\Delta+(r-1)]\Theta[\Delta-(r-1)]}{2}\right)\frac{1}{1-r}\right] = \boxed{1}$$

$\delta = 0$ can be taken safely !

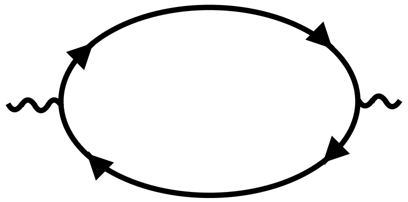
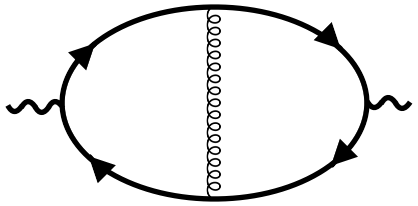
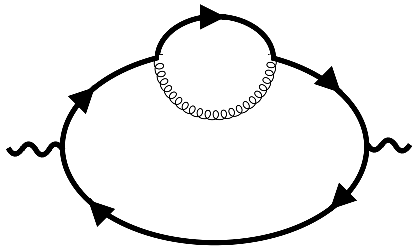
$$+ \lim_{\delta\to 0^+}\int_{1-\Delta}^{1+\Delta} dr\,\frac{4}{\pi}\frac{1}{2}\frac{1}{r-1\pm \mathrm{i}\delta} \overset{\Delta\leq 1}{=} \underbrace{\frac{2}{\pi}\mathrm{PV}\left[\frac{1}{r-1\pm \mathrm{i}\delta}\right]}_{0}\boxed{\mp 2\,\mathrm{i}}$$

Easy to compute since Principal Value is zero by construction !

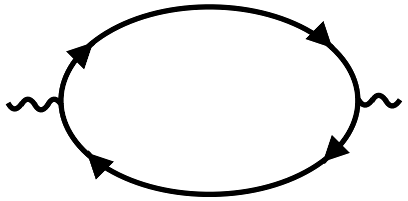
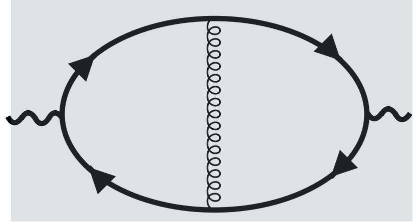
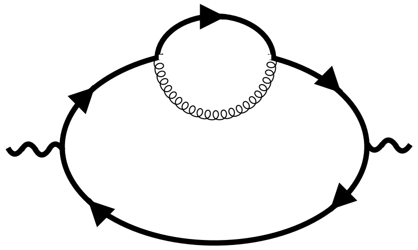


LOCAL UNITARITY: X-SEC RESULTS

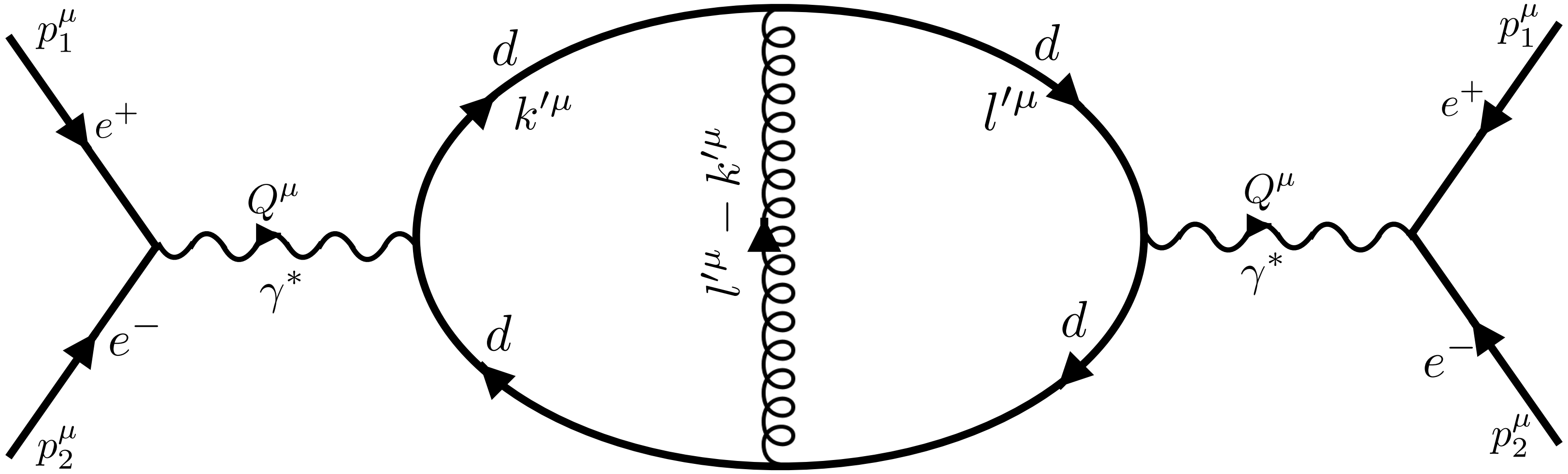
NUMERICAL RESULTS FOR $e^+e^- \to \gamma^\star \to d\bar{d}$ @ NLO

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $+$  $+\, 2\times$  $\Big]$

NUMERICAL RESULTS FOR $e^+e^- \to \gamma^\star \to d\bar{d}$ @ NLO

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\,\Big[$  $+$  $+\,2\times$  $\Big]$

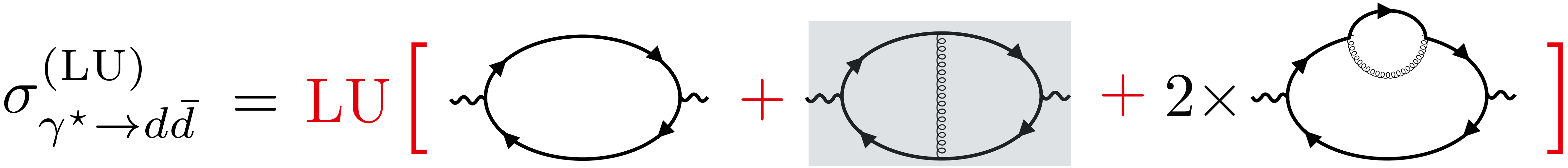
Visualisation of the **LU integrand** for the **Double-Triangle** supergraph and :



$$p_1^\mu = (1,0,0,1) \qquad 0.4 < p_{t,j_1} < 0.8$$

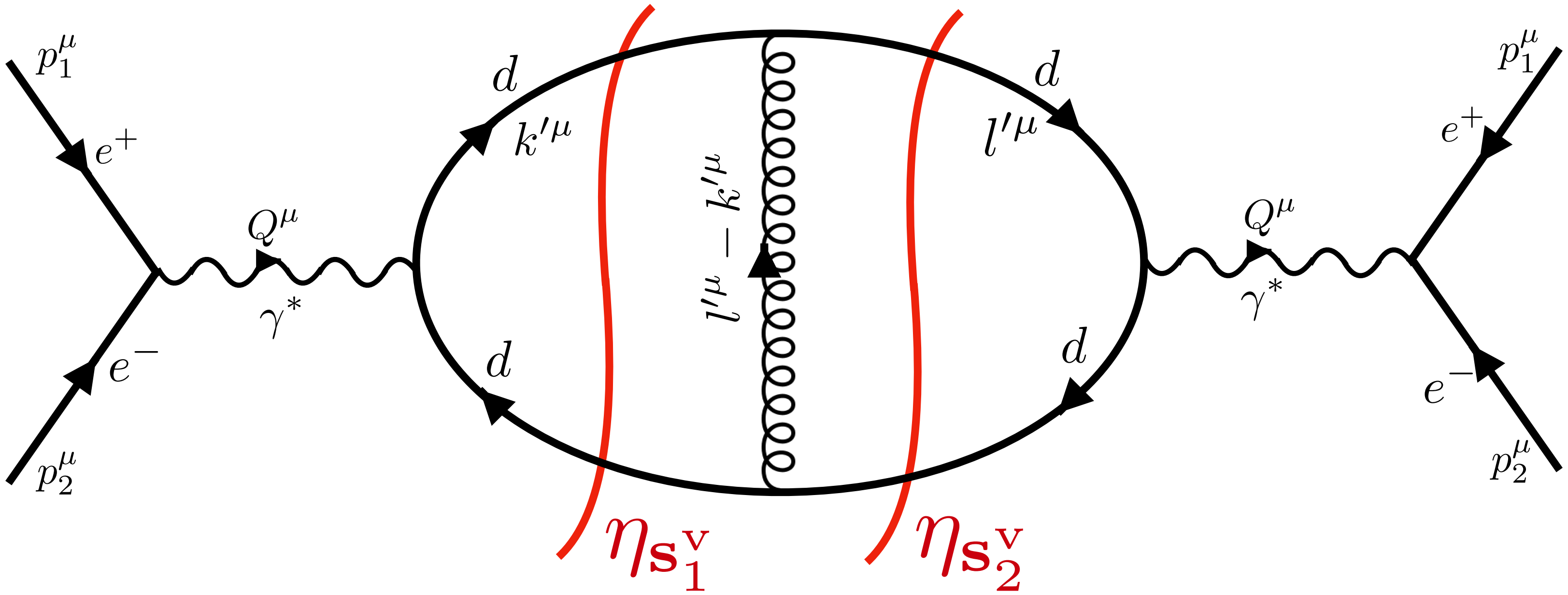
$$p_2^\mu = (1,0,0,-1) \qquad (\,\vec{k},\vec{l}\,) = (\;(0,k_y,\frac{1}{\sqrt{2}})\;,(0,\frac{1}{\sqrt{2}},l_z)\;)$$

NUMERICAL RESULTS FOR $e^+e^- \to \gamma^\star \to d\bar{d}$ @ NLO



$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[\quad + \quad + 2\times \quad \Big]$$

Visualisation of the **LU integrand** for the **Double-Triangle** supergraph and :

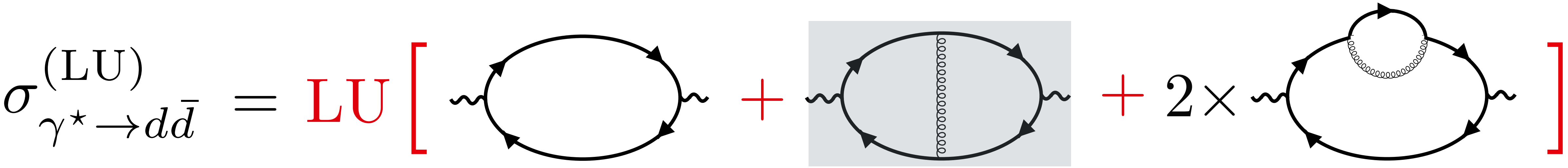


$$p_1^\mu = (1,0,0,1) \qquad 0.4 < p_{t,j_1} < 0.8$$

$$p_2^\mu = (1,0,0,-1) \qquad (\,\vec{k},\vec{l}\,) = (\,(0,k_y,\frac{1}{\sqrt{2}})\,,(0,\frac{1}{\sqrt{2}},l_z)\,)$$

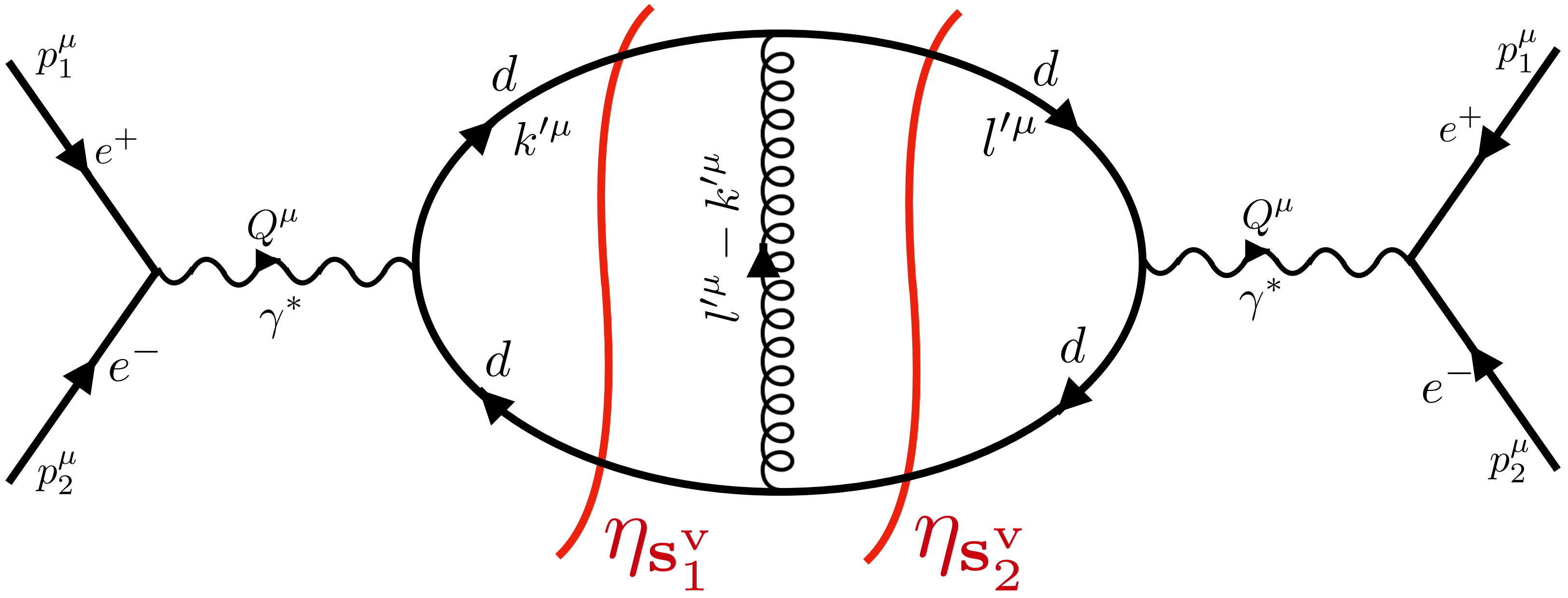
Cancellation of **non-pinched E-surfaces** for : $\eta_{\mathbf{s}_1^{\mathrm{v}}} = \eta_{\mathbf{s}_2^{\mathrm{v}}} \;\to\; k'_y = l'_z$

NUMERICAL RESULTS FOR $e^+e^- \to \gamma^\star \to d\bar{d}$ @ NLO



$$\sigma^{(\text{LU})}_{\gamma^\star \to d\bar{d}} = \text{LU}\Big[\quad + \quad + 2\times \quad \Big]$$

Visualisation of the **LU integrand** for the **Double-Triangle** supergraph and :



$p_1^\mu = (1, 0, 0, 1)$

$p_2^\mu = (1, 0, 0, -1)$

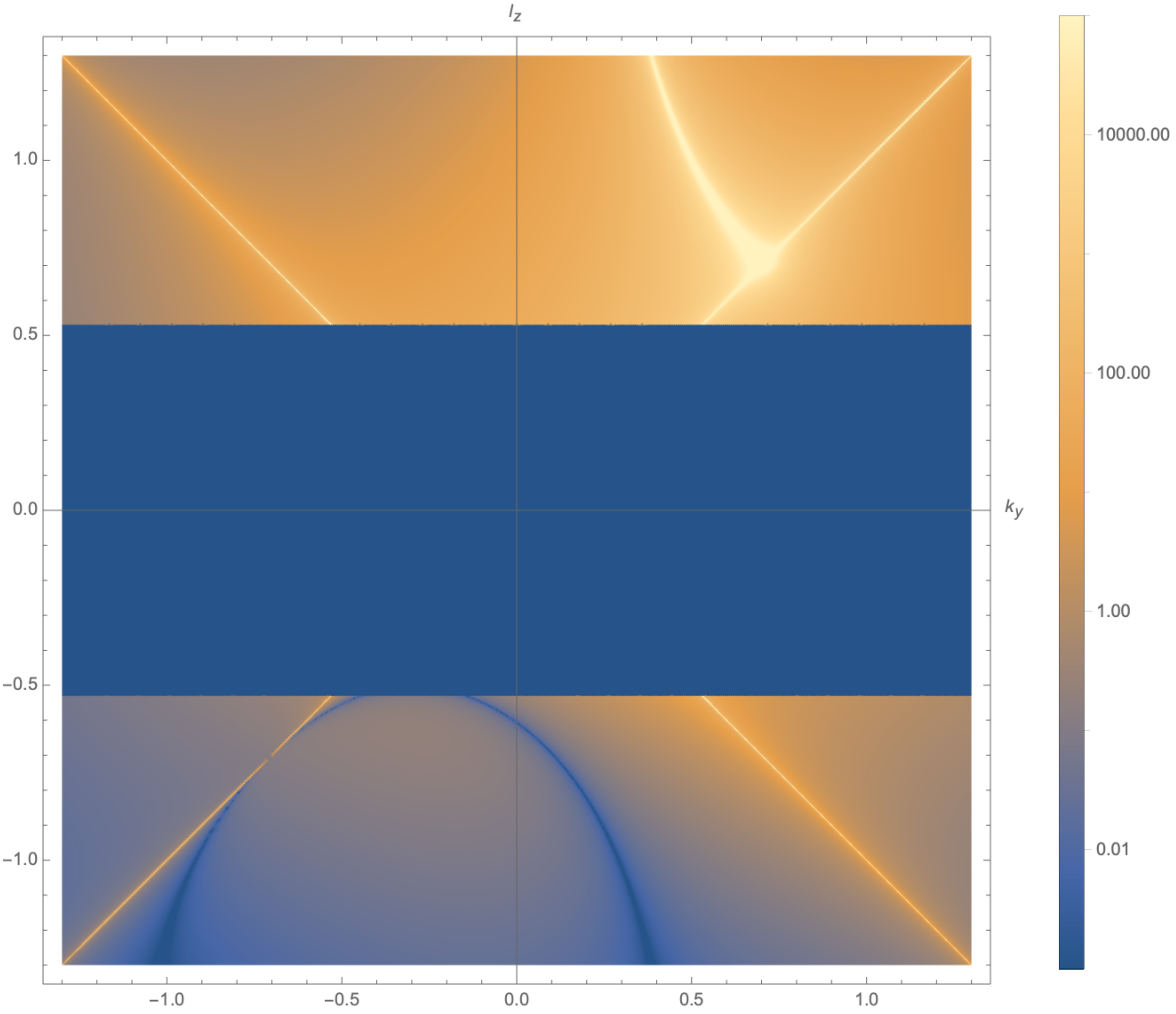
$0.4 < p_{t,j_1} < 0.8$

$(\, \vec{k}, \vec{l}\,) = (\, (0, k_y, \frac{1}{\sqrt{2}})\, , (0, \frac{1}{\sqrt{2}}, l_z)\,)$

Cancellation of **non-pinched E-surfaces** for : $\eta_{\mathbf{s}_1^{\mathrm{v}}} = \eta_{\mathbf{s}_2^{\mathrm{v}}} \to k'_y = l'_z$

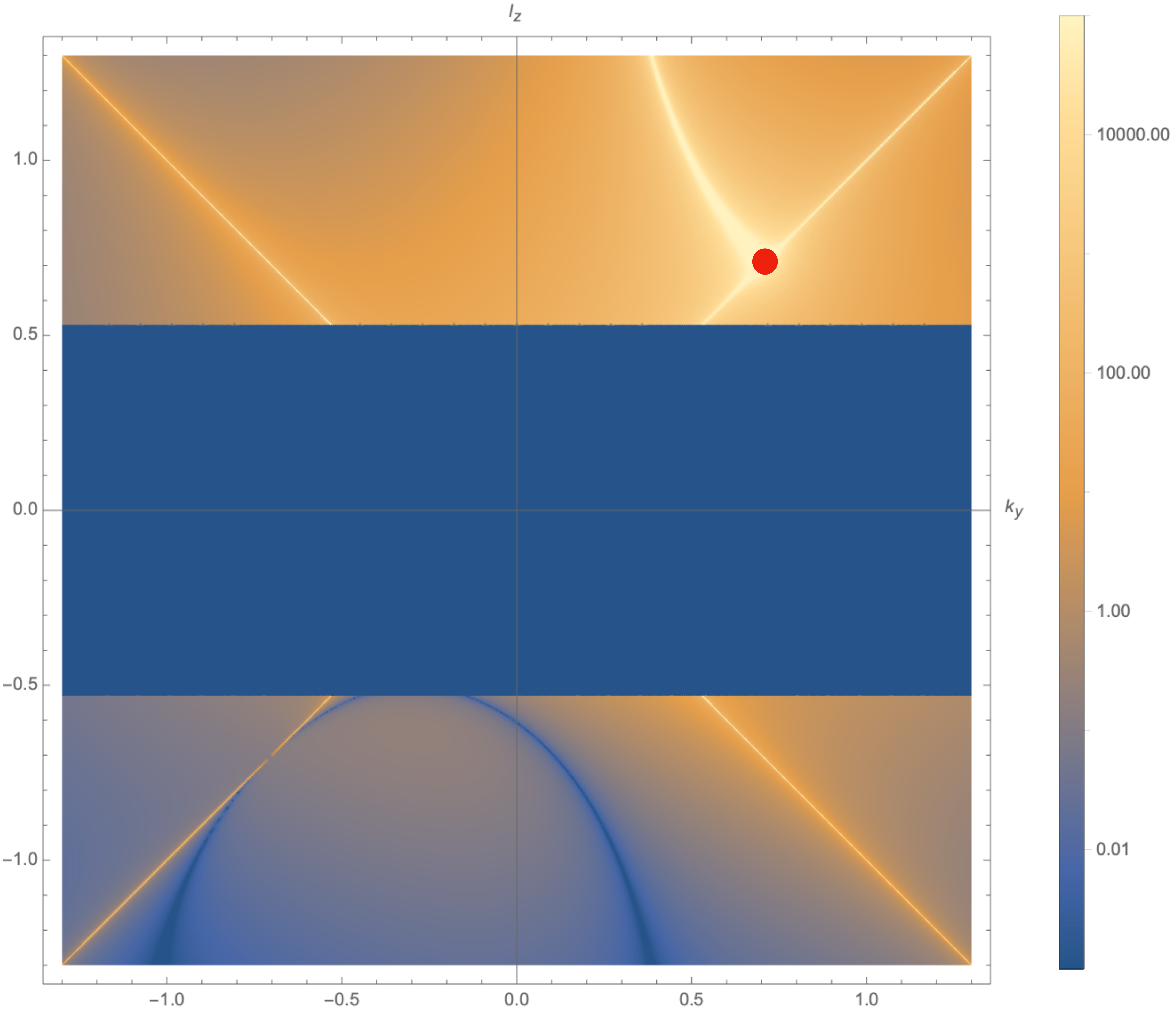
Soft configuration for : $|\vec{l}\,' - \vec{k}| = 0 \to k'_y = l'_z = \frac{1}{\sqrt{2}}$

$I_{\mathbf{s}_1^{\mathrm{v}}}$



$I_{\mathbf{s}_1^{\mathrm{v}}}$

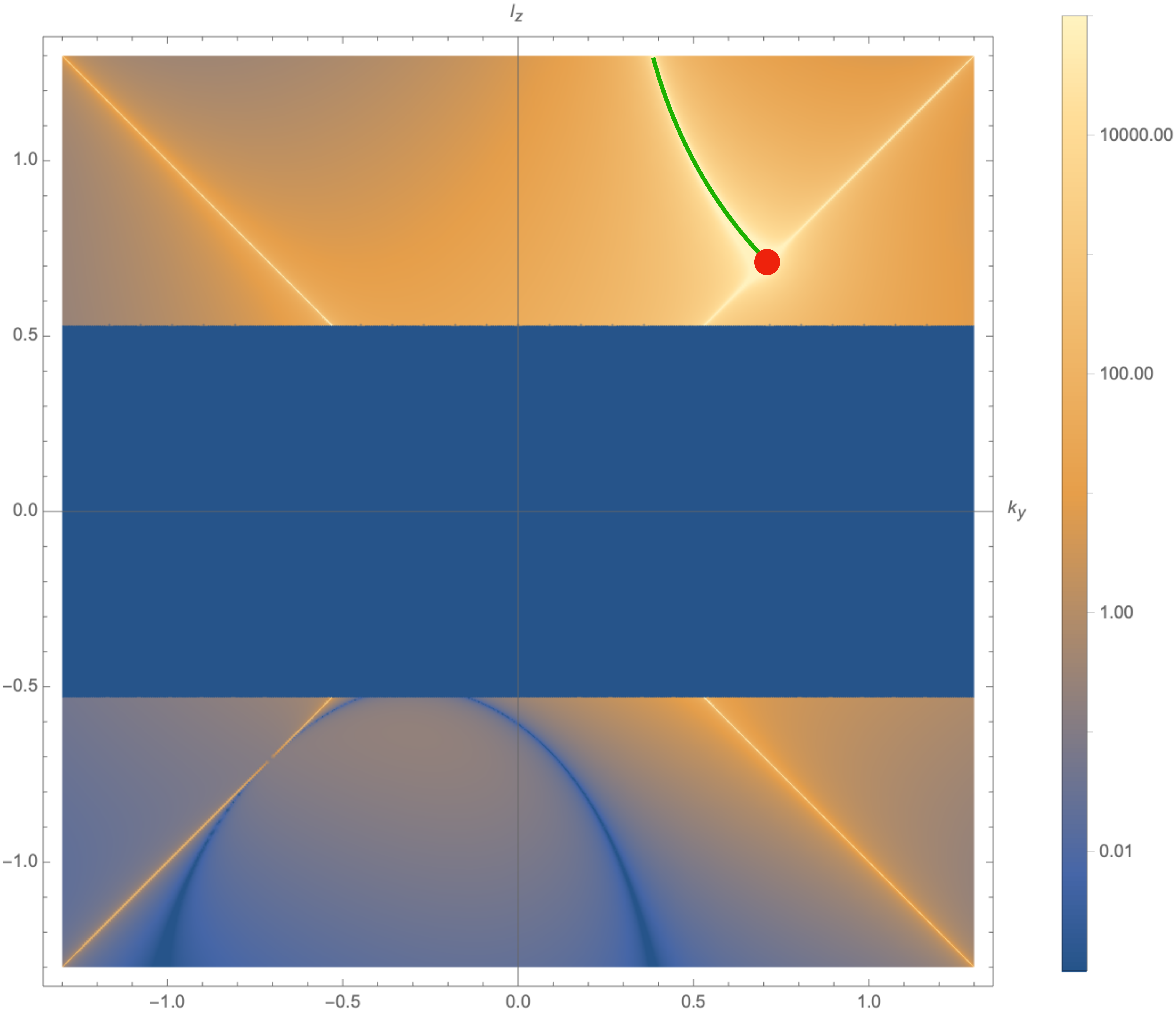
soft



$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

collinear

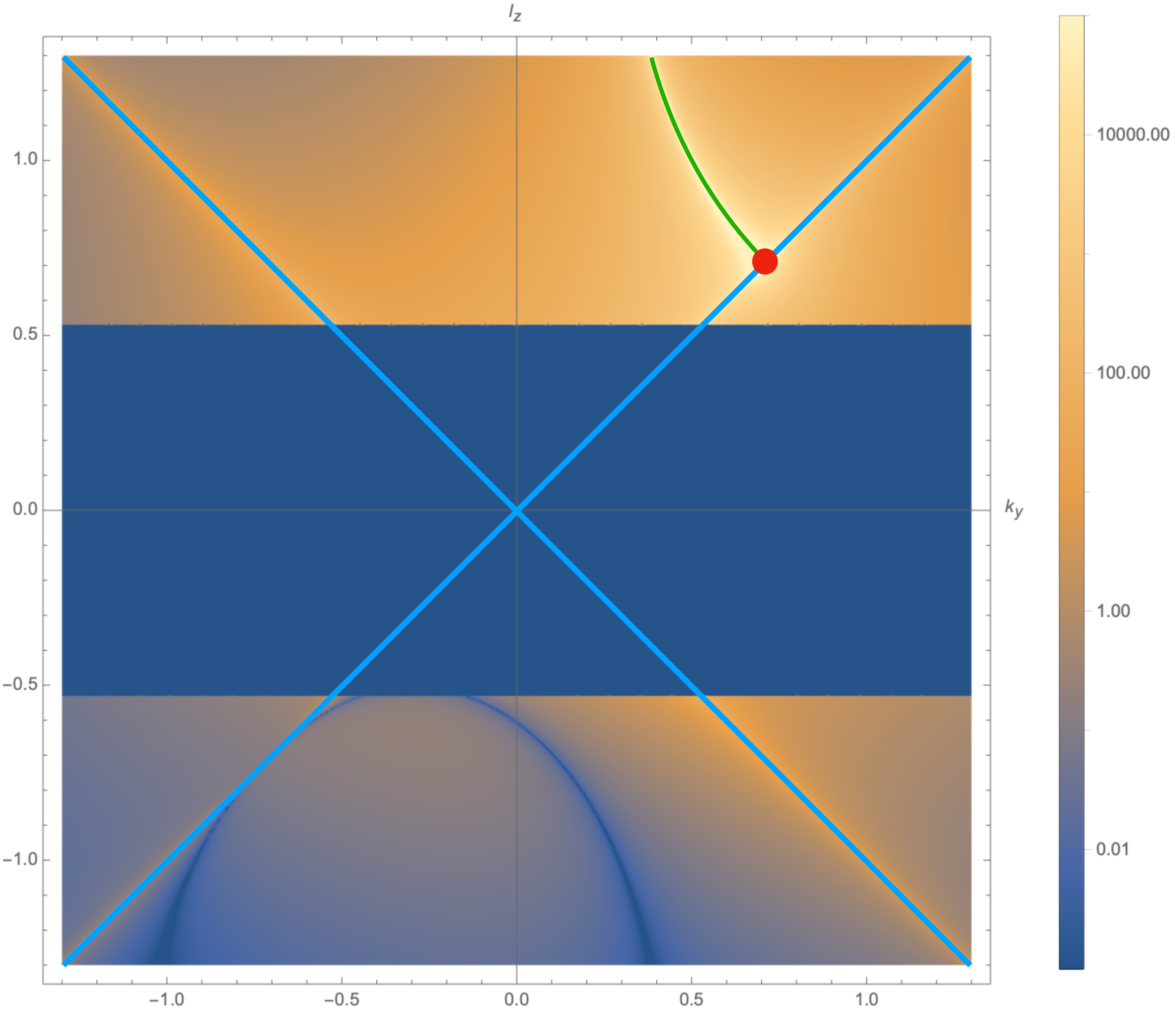


$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

collinear

non-pinch cancellation



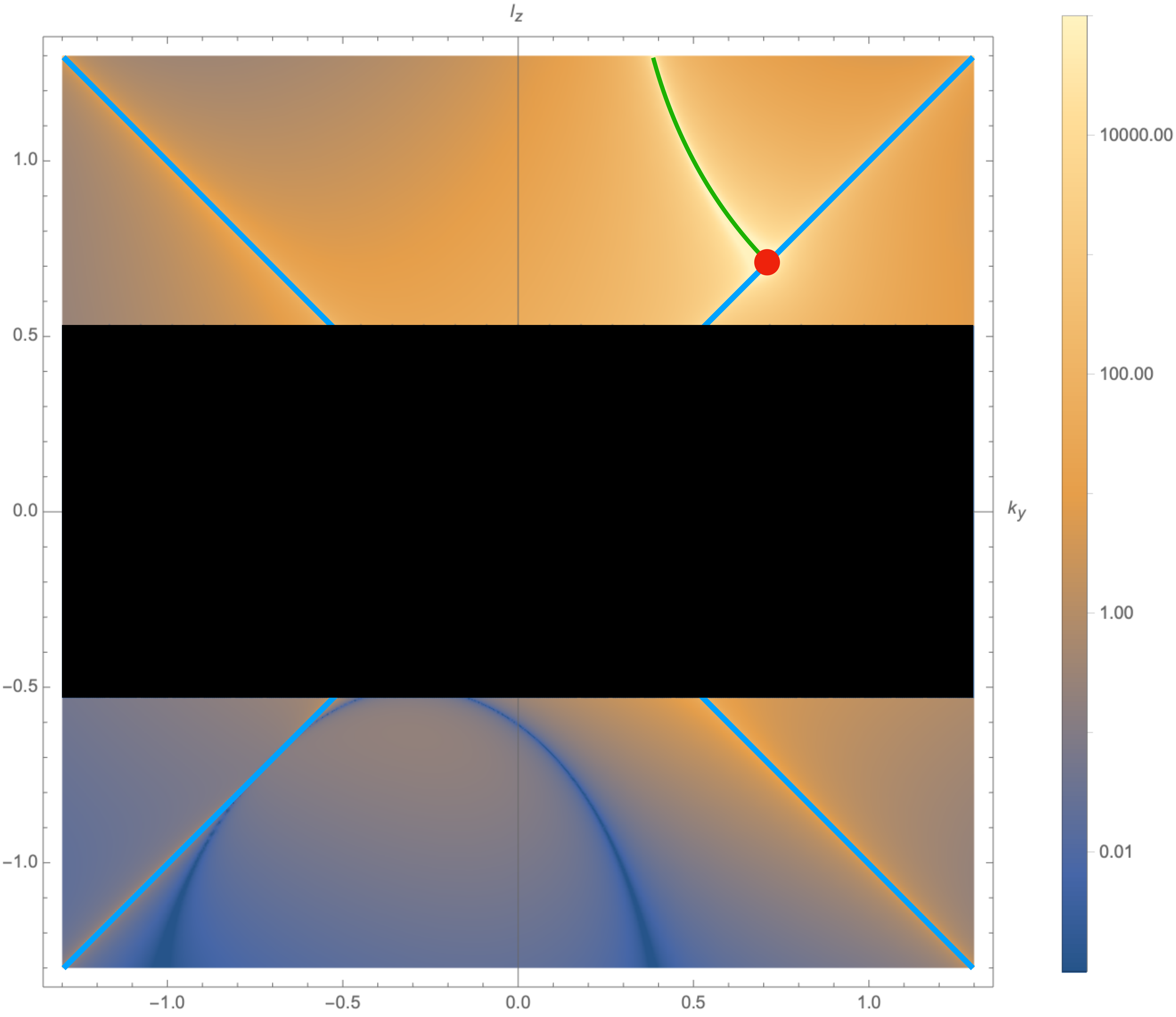
$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

collinear

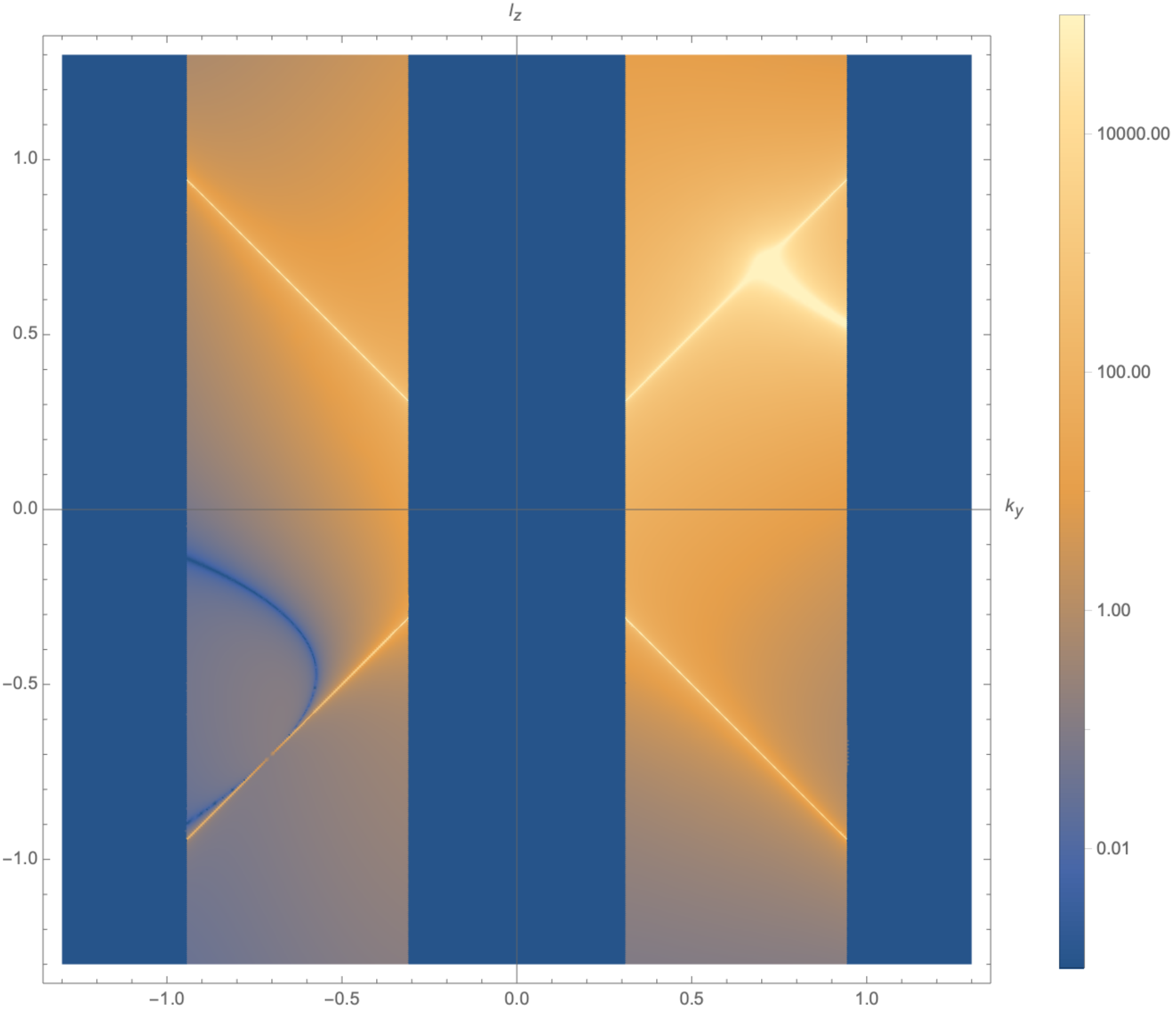
non-pinch cancellation

observable cuts

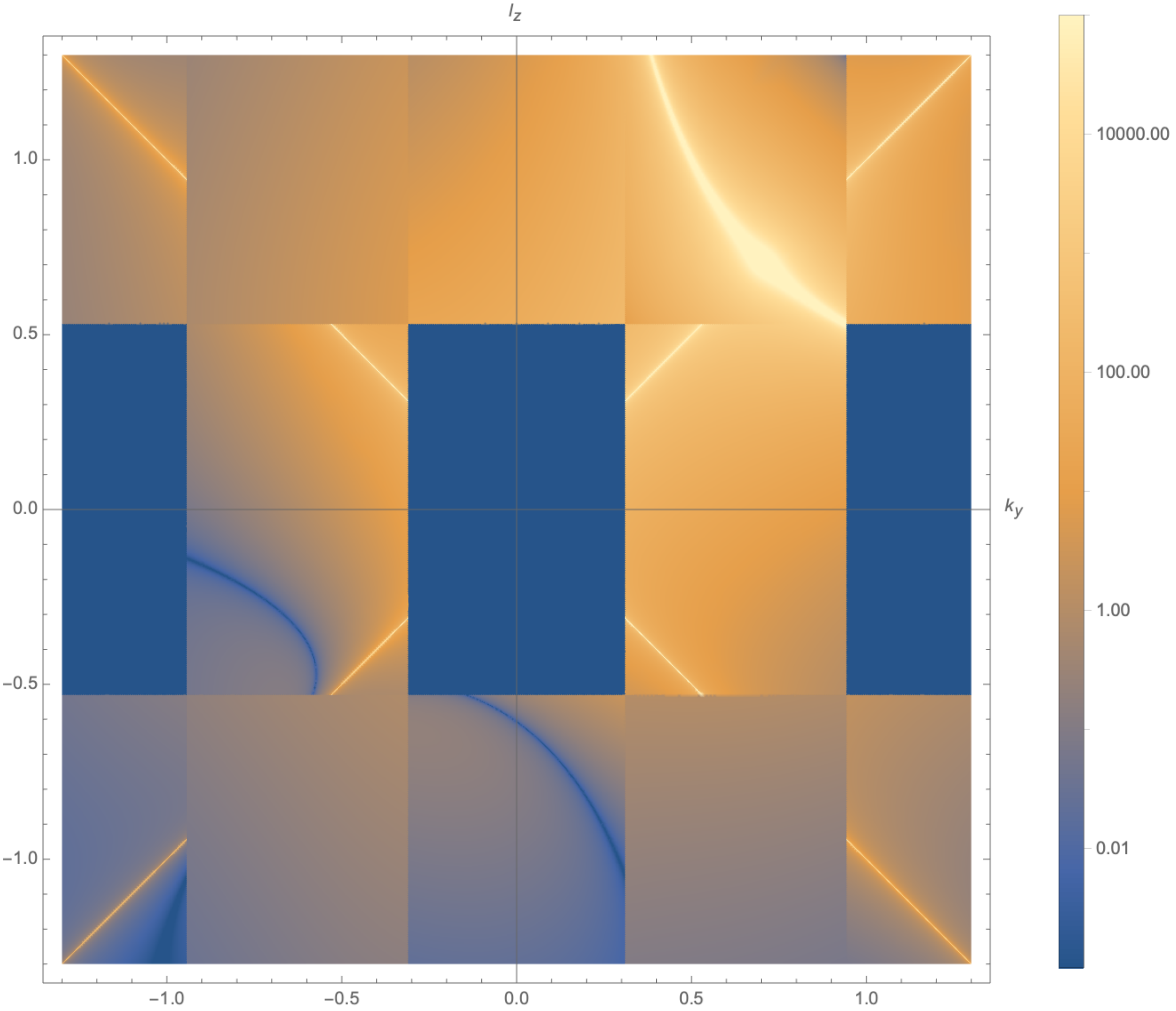


[redacted]

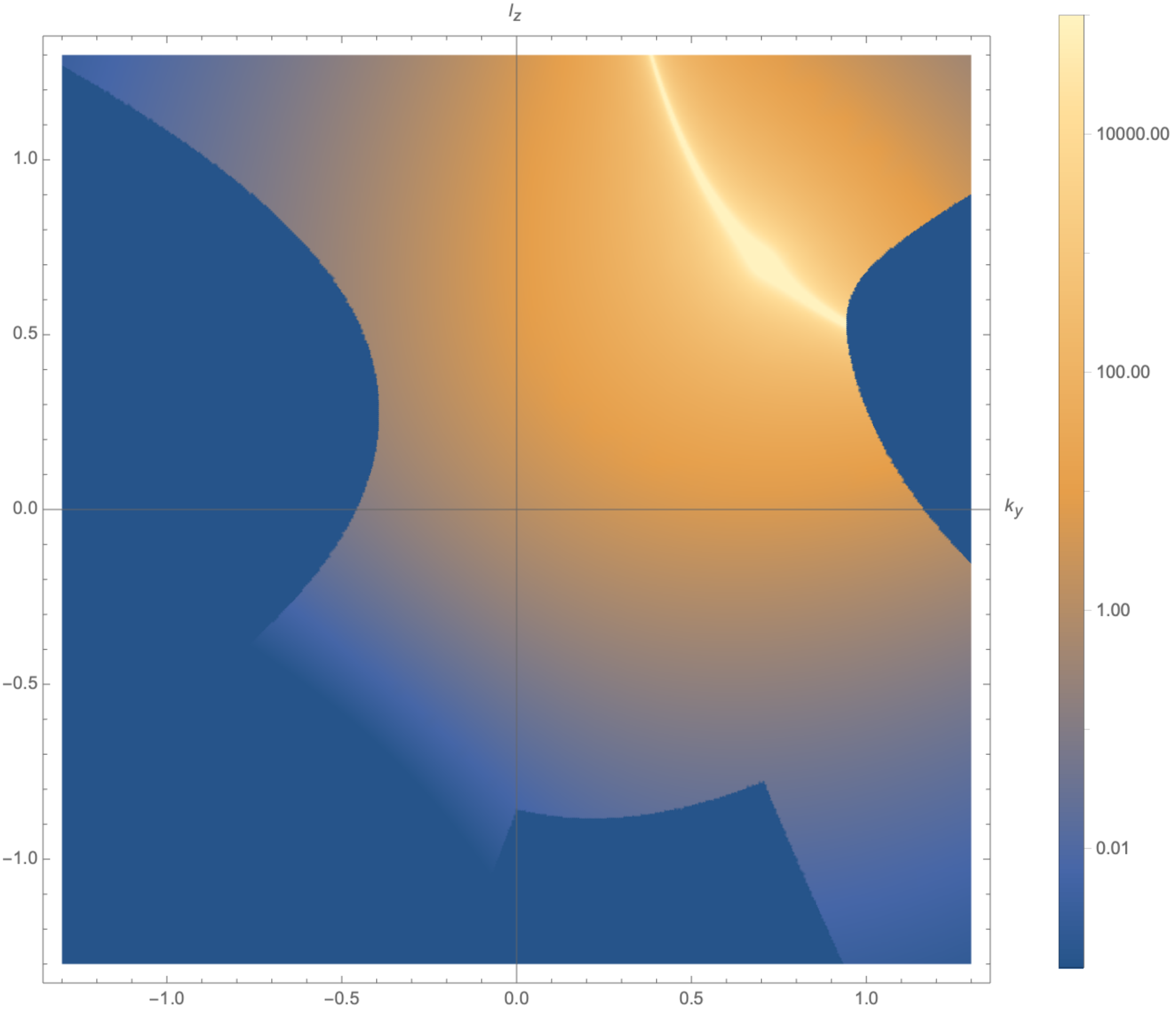
$I_{\mathbf{s}_2^{\mathrm{v}}}$



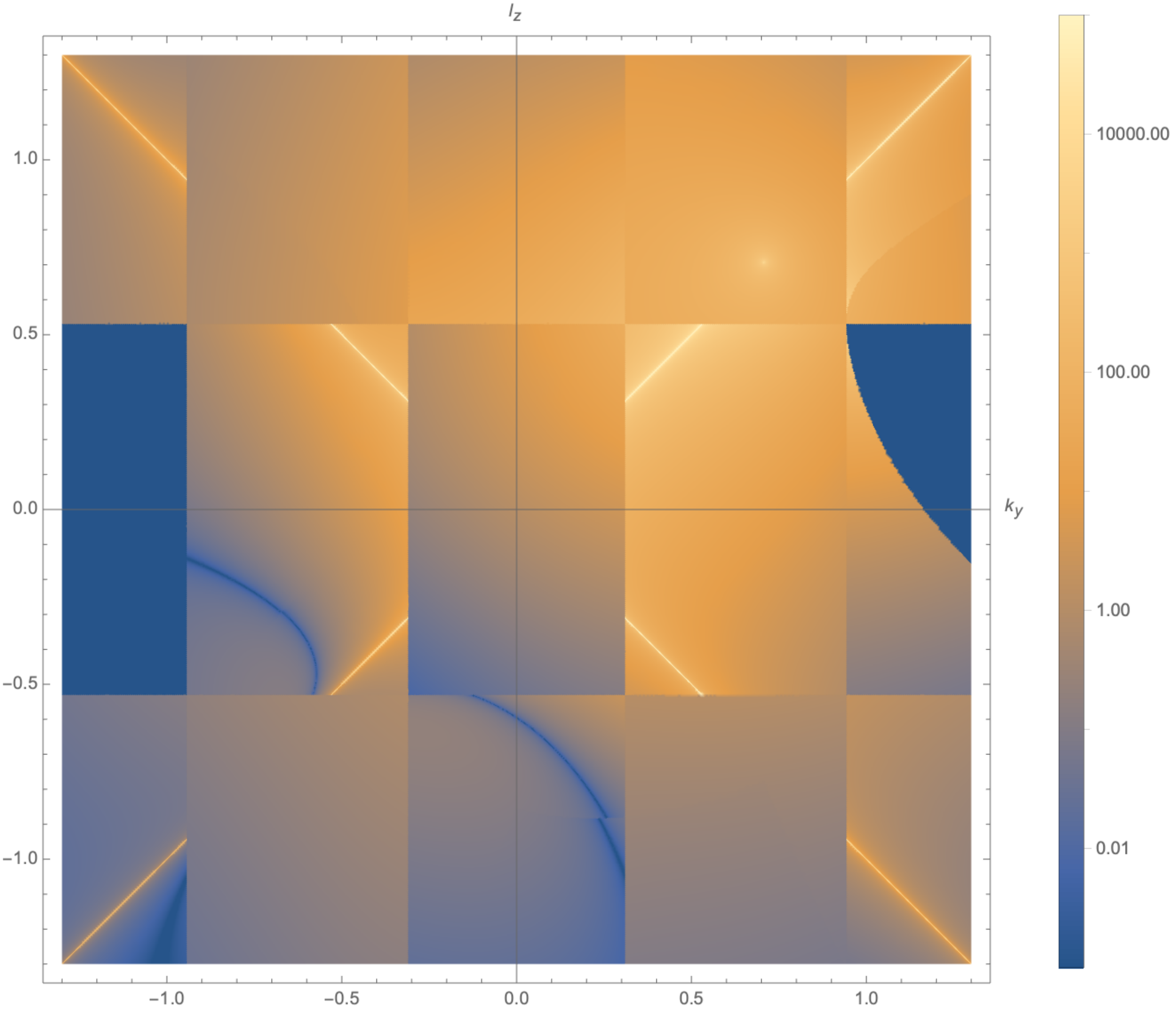
$$I_{\mathbf{s}_1^{\mathrm{v}}} + I_{\mathbf{s}_2^{\mathrm{v}}}$$



$I_{\mathbf{s}_1^{\mathrm{r}}} + I_{\mathbf{s}_2^{\mathrm{r}}}$

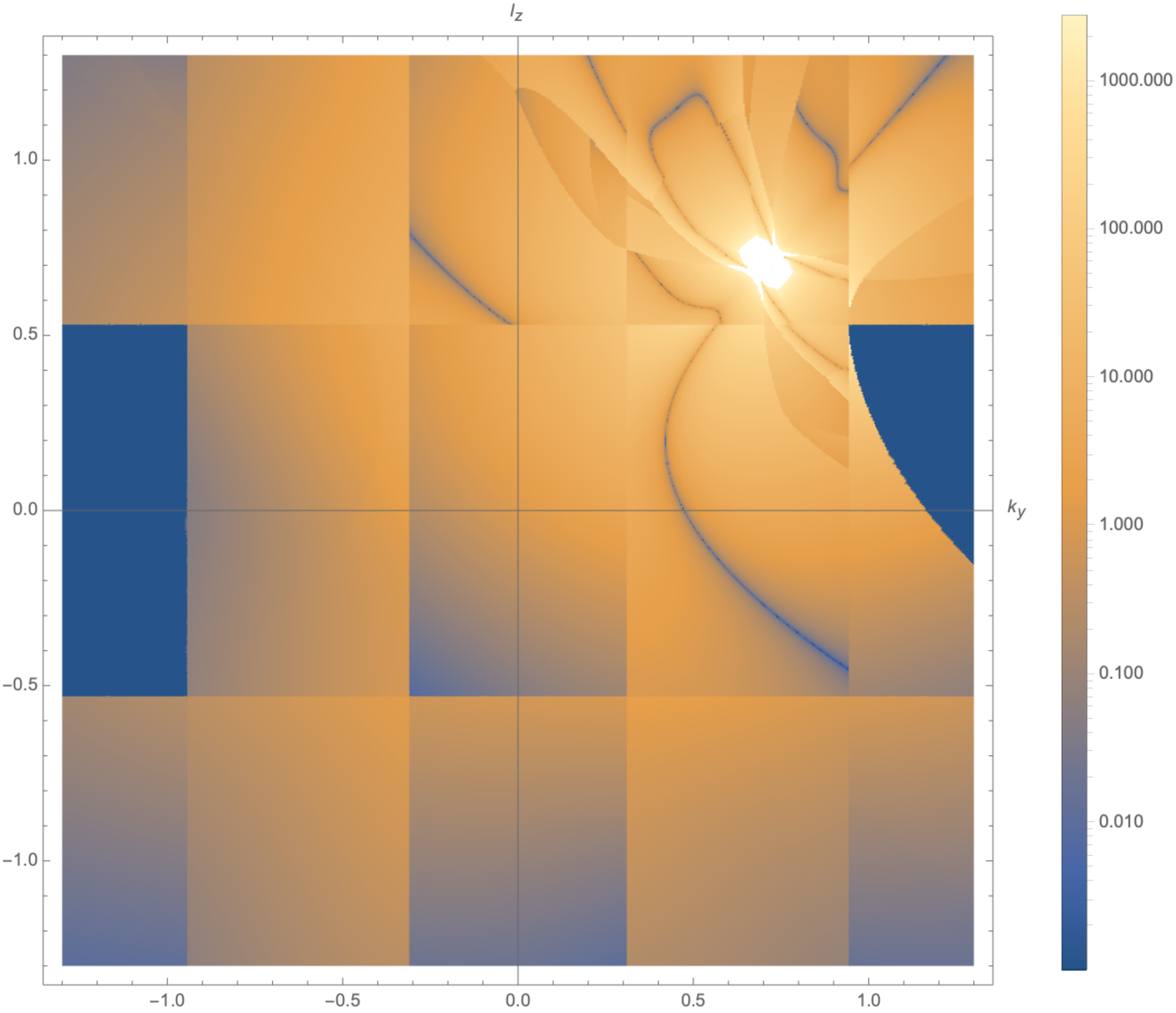


$I_{\sum}$

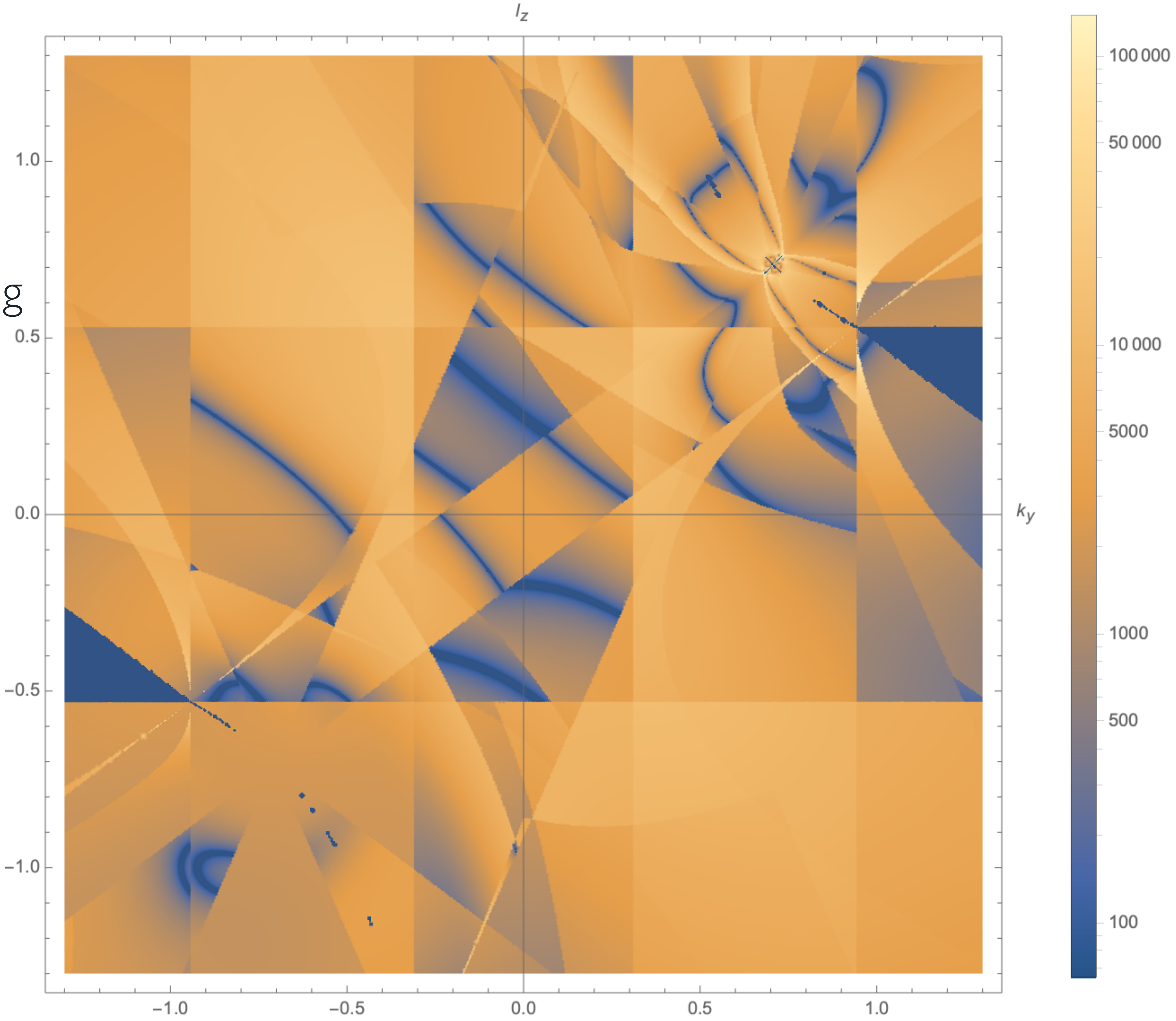


$\mathrm{Re}\left[I_{\sum}\right]$

with deformation



$\mathrm{Re}\left[I_{\sum}\right]$ with deformation and multichanneling



NUMERICAL RESULTS

We have computed the following fixed-order processes with **Local Unitarity**:

NLO

$e^+e^- \to \gamma \to jj$ $p_t(j_1)$ distribution

$e^+e^- \to \gamma \to jjj$ semi-inclusive

$e^+e^- \to \gamma \to t\bar{t}h$ (semi-)inclusive

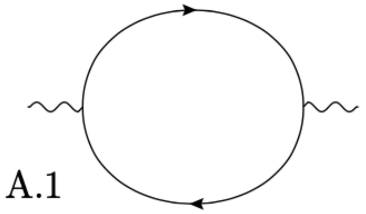
NNLO

$\gamma^\star \to jj$ inclusive

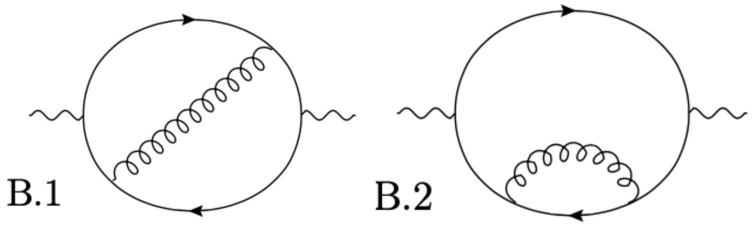
$\gamma^\star \to t\bar{t}$ inclusive

First NNLO cross-sections computed **fully numerically in momentum space**.

LO



NLO



NUMERICAL RESULTS

We have computed the following fixed-order processes with **Local Unitarity**:

NLO

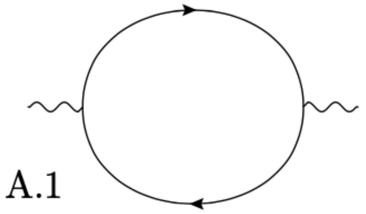
- $e^+e^- \to \gamma \to jj$ $p_t(j_1)$ distribution
- $e^+e^- \to \gamma \to jjj$ semi-inclusive
- $e^+e^- \to \gamma \to t\bar{t}h$ (semi-)inclusive

NNLO

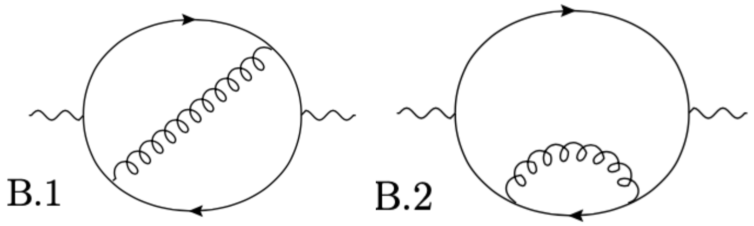
- $\gamma^\star \to jj$ inclusive
- $\gamma^\star \to t\bar{t}$ inclusive

First NNLO cross-sections computed **fully numerically in momentum space**.

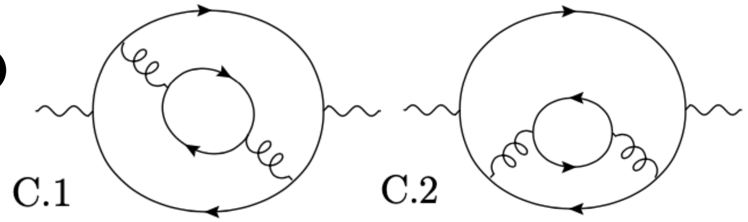
LO



NLO



NNLO
$\mathcal{O}(n_f)$



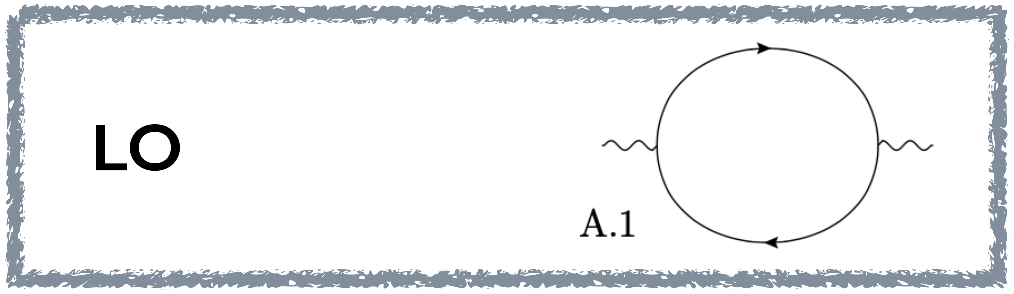
NUMERICAL RESULTS

We have computed the following fixed-order processes with **Local Unitarity**:

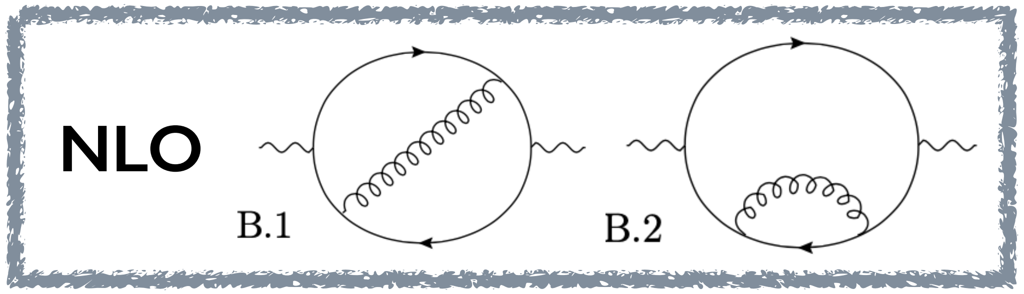
NLO
- $e^+e^- \to \gamma \to jj$ $p_t(j_1)$ distribution
- $e^+e^- \to \gamma \to jjj$ semi-inclusive
- $e^+e^- \to \gamma \to t\bar{t}h$ (semi-)inclusive

NNLO
- $\gamma^\star \to jj$ inclusive
- $\gamma^\star \to t\bar{t}$ inclusive

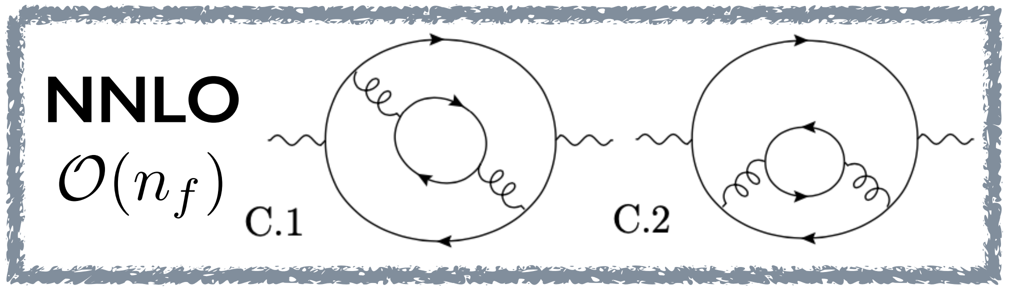
First NNLO cross-sections computed **fully numerically in momentum space**.



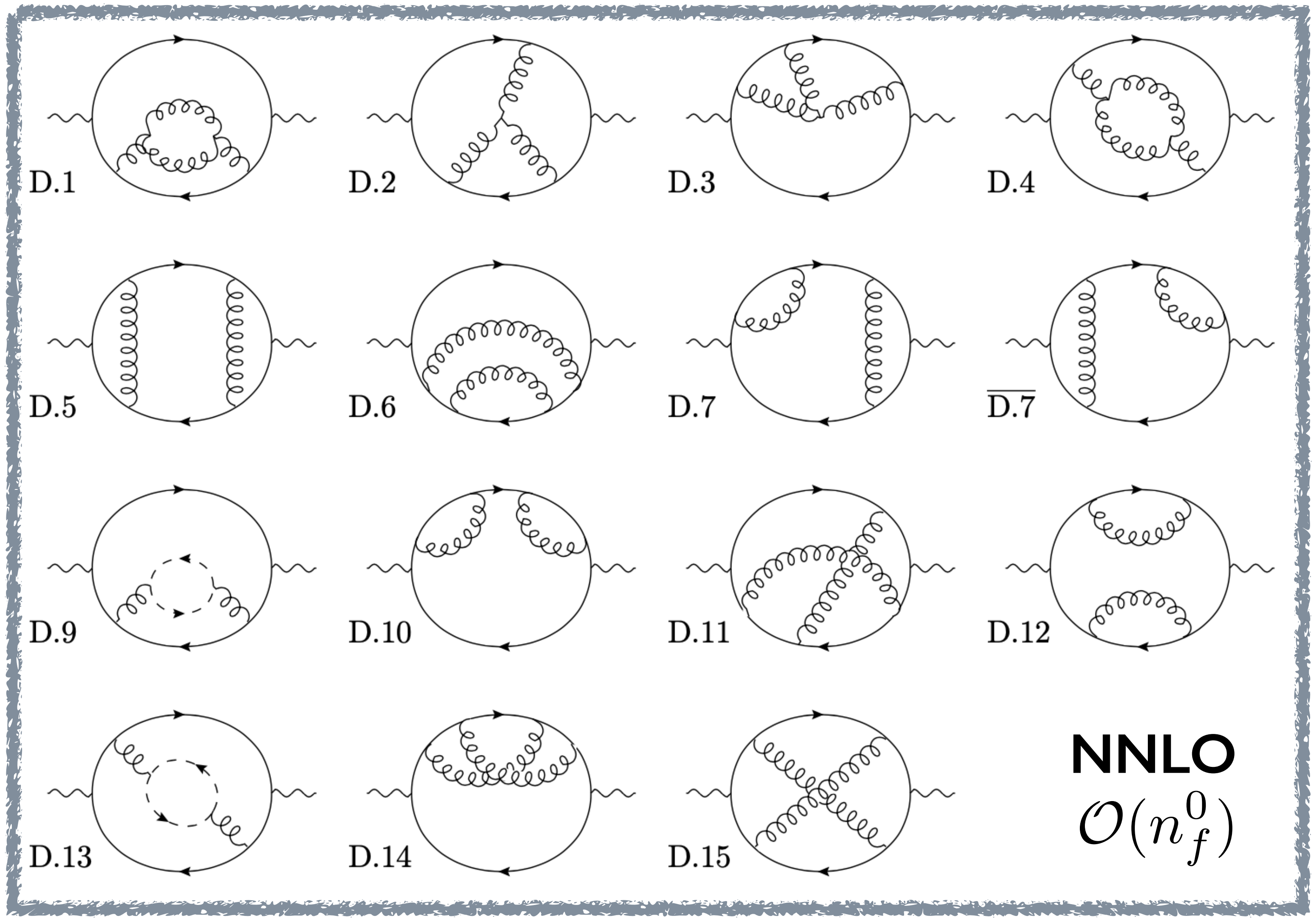
LO



NLO

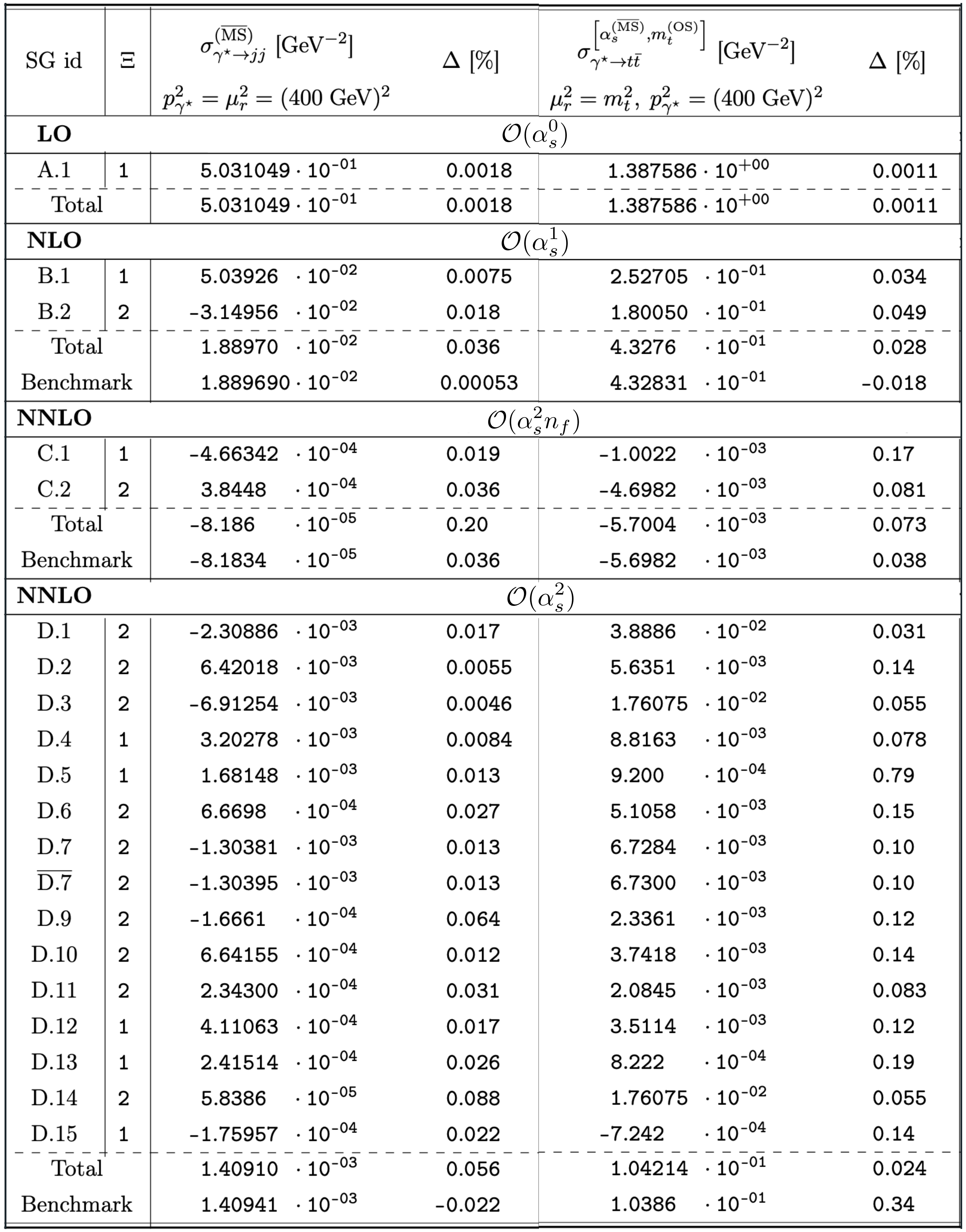


NNLO $\mathcal{O}(n_f)$



NNLO $\mathcal{O}(n_f^0)$

NUMERICAL RESULTS



SG id	Ξ	$\sigma^{(\overline{\text{MS}})}_{\gamma^\star\to jj}$ [GeV^{-2}] $p^2_{\gamma^\star}=\mu_r^2=(400\text{ GeV})^2$	Δ [%]	$\sigma^{\left[\alpha_s^{(\overline{\text{MS}})},m_t^{(\text{OS})}\right]}_{\gamma^\star\to t\bar{t}}$ [GeV^{-2}] $\mu_r^2=m_t^2,\ p^2_{\gamma^\star}=(400\text{ GeV})^2$	Δ [%]
LO		$\mathcal{O}(\alpha_s^0)$			
A.1	1	$5.031049\cdot 10^{-01}$	0.0018	$1.387586\cdot 10^{+00}$	0.0011
Total		$5.031049\cdot 10^{-01}$	0.0018	$1.387586\cdot 10^{+00}$	0.0011
NLO		$\mathcal{O}(\alpha_s^1)$			
B.1	1	$5.03926\cdot 10^{-02}$	0.0075	$2.52705\cdot 10^{-01}$	0.034
B.2	2	$-3.14956\cdot 10^{-02}$	0.018	$1.80050\cdot 10^{-01}$	0.049
Total		$1.88970\cdot 10^{-02}$	0.036	$4.3276\cdot 10^{-01}$	0.028
Benchmark		$1.889690\cdot 10^{-02}$	0.00053	$4.32831\cdot 10^{-01}$	-0.018
NNLO		$\mathcal{O}(\alpha_s^2 n_f)$			
C.1	1	$-4.66342\cdot 10^{-04}$	0.019	$-1.0022\cdot 10^{-03}$	0.17
C.2	2	$3.8448\cdot 10^{-04}$	0.036	$-4.6982\cdot 10^{-03}$	0.081
Total		$-8.186\cdot 10^{-05}$	0.20	$-5.7004\cdot 10^{-03}$	0.073
Benchmark		$-8.1834\cdot 10^{-05}$	0.036	$-5.6982\cdot 10^{-03}$	0.038
NNLO		$\mathcal{O}(\alpha_s^2)$			
D.1	2	$-2.30886\cdot 10^{-03}$	0.017	$3.8886\cdot 10^{-02}$	0.031
D.2	2	$6.42018\cdot 10^{-03}$	0.0055	$5.6351\cdot 10^{-03}$	0.14
D.3	2	$-6.91254\cdot 10^{-03}$	0.0046	$1.76075\cdot 10^{-02}$	0.055
D.4	1	$3.20278\cdot 10^{-03}$	0.0084	$8.8163\cdot 10^{-03}$	0.078
D.5	1	$1.68148\cdot 10^{-03}$	0.013	$9.200\cdot 10^{-04}$	0.79
D.6	2	$6.6698\cdot 10^{-04}$	0.027	$5.1058\cdot 10^{-03}$	0.15
D.7	2	$-1.30381\cdot 10^{-03}$	0.013	$6.7284\cdot 10^{-03}$	0.10
$\overline{\text{D}}$.7	2	$-1.30395\cdot 10^{-03}$	0.013	$6.7300\cdot 10^{-03}$	0.10
D.9	2	$-1.6661\cdot 10^{-04}$	0.064	$2.3361\cdot 10^{-03}$	0.12
D.10	2	$6.64155\cdot 10^{-04}$	0.012	$3.7418\cdot 10^{-03}$	0.14
D.11	2	$2.34300\cdot 10^{-04}$	0.031	$2.0845\cdot 10^{-03}$	0.083
D.12	1	$4.11063\cdot 10^{-04}$	0.017	$3.5114\cdot 10^{-03}$	0.12
D.13	1	$2.41514\cdot 10^{-04}$	0.026	$8.222\cdot 10^{-04}$	0.19
D.14	2	$5.8386\cdot 10^{-05}$	0.088	$1.76075\cdot 10^{-02}$	0.055
D.15	1	$-1.75957\cdot 10^{-04}$	0.022	$-7.242\cdot 10^{-04}$	0.14
Total		$1.40910\cdot 10^{-03}$	0.056	$1.04214\cdot 10^{-01}$	0.024
Benchmark		$1.40941\cdot 10^{-03}$	-0.022	$1.0386\cdot 10^{-01}$	0.34

Analytic benchmarks :

$$\sigma^{\overline{\text{MS}}}_{\gamma^\star\to X}(p^2,\mu_r^2=p^2)\supset K_X n_f\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^{(\text{LO})}_{\gamma^\star\to X}(p^2)$$

NUMERICAL RESULTS

SG id	Ξ	$\sigma^{(\overline{\mathrm{MS}})}_{\gamma^\star\to jj}$ [GeV^{-2}] $p^2_{\gamma^\star}=\mu_r^2=(400\text{ GeV})^2$	Δ [%]	$\sigma^{\left[\alpha_s^{(\overline{\mathrm{MS}})},m_t^{(\mathrm{OS})}\right]}_{\gamma^\star\to t\bar{t}}$ [GeV^{-2}] $\mu_r^2=m_t^2,\ p^2_{\gamma^\star}=(400\text{ GeV})^2$	Δ [%]
LO			$\mathcal{O}(\alpha_s^0)$		
A.1	1	$5.031049\cdot10^{-01}$	0.0018	$1.387586\cdot10^{+00}$	0.0011
Total		$5.031049\cdot10^{-01}$	0.0018	$1.387586\cdot10^{+00}$	0.0011
NLO			$\mathcal{O}(\alpha_s^1)$		
B.1	1	$5.03926\cdot10^{-02}$	0.0075	$2.52705\cdot10^{-01}$	0.034
B.2	2	$-3.14956\cdot10^{-02}$	0.018	$1.80050\cdot10^{-01}$	0.049
Total		$1.88970\cdot10^{-02}$	0.036	$4.3276\cdot10^{-01}$	0.028
Benchmark		$1.889690\cdot10^{-02}$	0.00053	$4.32831\cdot10^{-01}$	-0.018
NNLO			$\mathcal{O}(\alpha_s^2 n_f)$		
C.1	1	$-4.66342\cdot10^{-04}$	0.019	$-1.0022\cdot10^{-03}$	0.17
C.2	2	$3.8448\cdot10^{-04}$	0.036	$-4.6982\cdot10^{-03}$	0.081
Total		$-8.186\cdot10^{-05}$	0.20	$-5.7004\cdot10^{-03}$	0.073
Benchmark		$-8.1834\cdot10^{-05}$	0.036	$-5.6982\cdot10^{-03}$	0.038
NNLO			$\mathcal{O}(\alpha_s^2)$		
D.1	2	$-2.30886\cdot10^{-03}$	0.017	$3.8886\cdot10^{-02}$	0.031
D.2	2	$6.42018\cdot10^{-03}$	0.0055	$5.6351\cdot10^{-03}$	0.14
D.3	2	$-6.91254\cdot10^{-03}$	0.0046	$1.76075\cdot10^{-02}$	0.055
D.4	1	$3.20278\cdot10^{-03}$	0.0084	$8.8163\cdot10^{-03}$	0.078
D.5	1	$1.68148\cdot10^{-03}$	0.013	$9.200\cdot10^{-04}$	0.79
D.6	2	$6.6698\cdot10^{-04}$	0.027	$5.1058\cdot10^{-03}$	0.15
D.7	2	$-1.30381\cdot10^{-03}$	0.013	$6.7284\cdot10^{-03}$	0.10
$\overline{\mathrm{D.7}}$	2	$-1.30395\cdot10^{-03}$	0.013	$6.7300\cdot10^{-03}$	0.10
D.9	2	$-1.6661\cdot10^{-04}$	0.064	$2.3361\cdot10^{-03}$	0.12
D.10	2	$6.64155\cdot10^{-04}$	0.012	$3.7418\cdot10^{-03}$	0.14
D.11	2	$2.34300\cdot10^{-04}$	0.031	$2.0845\cdot10^{-03}$	0.083
D.12	1	$4.11063\cdot10^{-04}$	0.017	$3.5114\cdot10^{-03}$	0.12
D.13	1	$2.41514\cdot10^{-04}$	0.026	$8.222\cdot10^{-04}$	0.19
D.14	2	$5.8386\cdot10^{-05}$	0.088	$1.76075\cdot10^{-02}$	0.055
D.15	1	$-1.75957\cdot10^{-04}$	0.022	$-7.242\cdot10^{-04}$	0.14
Total		$1.40910\cdot10^{-03}$	0.056	$1.04214\cdot10^{-01}$	0.024
Benchmark		$1.40941\cdot10^{-03}$	-0.022	$1.0386\cdot10^{-01}$	0.34

Analytic benchmarks :

$$\sigma^{\overline{\mathrm{MS}}}_{\gamma^\star\to X}(p^2,\mu_r^2=p^2)\supset K_X n_f\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^{(\mathrm{LO})}_{\gamma^\star\to X}(p^2)$$

$$\gamma^\star\to jj$$

$$\boxed{K_{jj}}=-C_F\left[11-8\zeta_3\right]$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

NUMERICAL RESULTS

SG id	Ξ	$\sigma^{(\overline{\text{MS}})}_{\gamma^\star\to jj}$ [GeV^{-2}] $p^2_{\gamma^\star}=\mu_r^2=(400\text{ GeV})^2$	Δ [%]	$\sigma^{\left[\alpha_s^{(\overline{\text{MS}})},m_t^{(\text{OS})}\right]}_{\gamma^\star\to t\bar{t}}$ [GeV^{-2}] $\mu_r^2=m_t^2,\ p^2_{\gamma^\star}=(400\text{ GeV})^2$	Δ [%]
LO		$\mathcal{O}(\alpha_s^0)$			
A.1	1	$5.031049\cdot10^{-01}$	0.0018	$1.387586\cdot10^{+00}$	0.0011
Total		$5.031049\cdot10^{-01}$	0.0018	$1.387586\cdot10^{+00}$	0.0011
NLO		$\mathcal{O}(\alpha_s^1)$			
B.1	1	$5.03926\cdot10^{-02}$	0.0075	$2.52705\cdot10^{-01}$	0.034
B.2	2	$-3.14956\cdot10^{-02}$	0.018	$1.80050\cdot10^{-01}$	0.049
Total		$1.88970\cdot10^{-02}$	0.036	$4.3276\cdot10^{-01}$	0.028
Benchmark		$1.889690\cdot10^{-02}$	0.00053	$4.32831\cdot10^{-01}$	-0.018
NNLO		$\mathcal{O}(\alpha_s^2 n_f)$			
C.1	1	$-4.66342\cdot10^{-04}$	0.019	$-1.0022\cdot10^{-03}$	0.17
C.2	2	$3.8448\cdot10^{-04}$	0.036	$-4.6982\cdot10^{-03}$	0.081
Total		$-8.186\cdot10^{-05}$	0.20	$-5.7004\cdot10^{-03}$	0.073
Benchmark		$-8.1834\cdot10^{-05}$	0.036	$-5.6982\cdot10^{-03}$	0.038
NNLO		$\mathcal{O}(\alpha_s^2)$			
D.1	2	$-2.30886\cdot10^{-03}$	0.017	$3.8886\cdot10^{-02}$	0.031
D.2	2	$6.42018\cdot10^{-03}$	0.0055	$5.6351\cdot10^{-03}$	0.14
D.3	2	$-6.91254\cdot10^{-03}$	0.0046	$1.76075\cdot10^{-02}$	0.055
D.4	1	$3.20278\cdot10^{-03}$	0.0084	$8.8163\cdot10^{-03}$	0.078
D.5	1	$1.68148\cdot10^{-03}$	0.013	$9.200\cdot10^{-04}$	0.79
D.6	2	$6.6698\cdot10^{-04}$	0.027	$5.1058\cdot10^{-03}$	0.15
D.7	2	$-1.30381\cdot10^{-03}$	0.013	$6.7284\cdot10^{-03}$	0.10
$\overline{\text{D}}$.7	2	$-1.30395\cdot10^{-03}$	0.013	$6.7300\cdot10^{-03}$	0.10
D.9	2	$-1.6661\cdot10^{-04}$	0.064	$2.3361\cdot10^{-03}$	0.12
D.10	2	$6.64155\cdot10^{-04}$	0.012	$3.7418\cdot10^{-03}$	0.14
D.11	2	$2.34300\cdot10^{-04}$	0.031	$2.0845\cdot10^{-03}$	0.083
D.12	1	$4.11063\cdot10^{-04}$	0.017	$3.5114\cdot10^{-03}$	0.12
D.13	1	$2.41514\cdot10^{-04}$	0.026	$8.222\cdot10^{-04}$	0.19
D.14	2	$5.8386\cdot10^{-05}$	0.088	$1.76075\cdot10^{-02}$	0.055
D.15	1	$-1.75957\cdot10^{-04}$	0.022	$-7.242\cdot10^{-04}$	0.14
Total		$1.40910\cdot10^{-03}$	0.056	$1.04214\cdot10^{-01}$	0.024
Benchmark		$1.40941\cdot10^{-03}$	-0.022	$1.0386\cdot10^{-01}$	0.34

Analytic benchmarks :

$$\sigma^{\overline{\text{MS}}}_{\gamma^\star\to X}(p^2,\mu_r^2=p^2)\supset K_X n_f\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^{(\text{LO})}_{\gamma^\star\to X}(p^2)$$

$$\gamma^\star\to jj$$

$$\boxed{K_{jj}}=-C_F\left[11-8\zeta_3\right]$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

$$\gamma^\star\to t\bar{t}$$

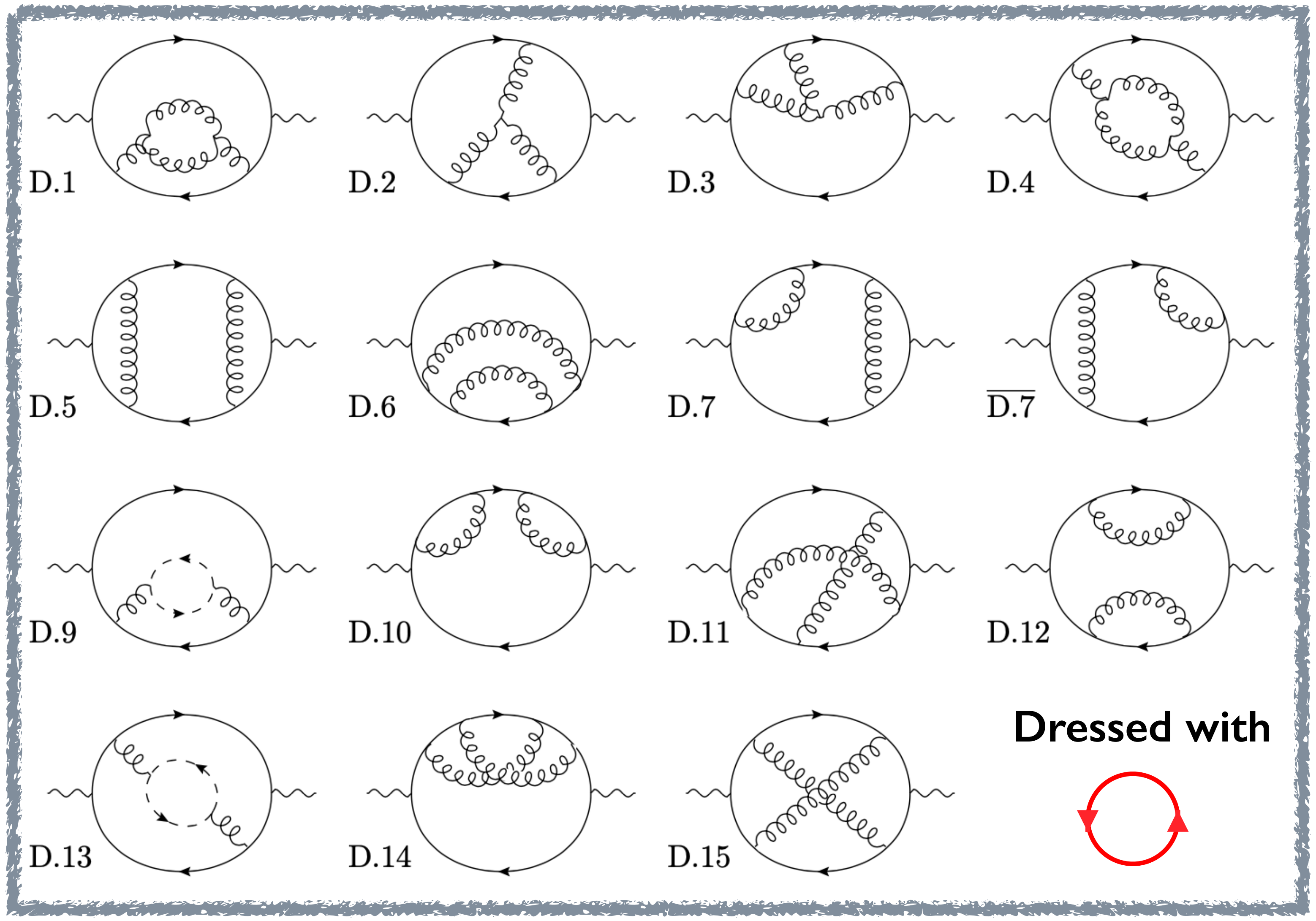
$$\boxed{K_{t\bar{t}}}=$$

$$\begin{aligned}\delta^{(2)}=&-\frac{(3-v^2)\,(1+v^2)}{6}\times\\&\Big\{\mathrm{Li}_3(p)-2\,\mathrm{Li}_3(1-p)-3\,\mathrm{Li}_3(p^2)-4\,\mathrm{Li}_3\Big(\frac{p}{1+p}\Big)-5\,\mathrm{Li}_3(1-p^2)+\frac{11}{2}\zeta(3)\\&+\mathrm{Li}_2(p)\,\ln\Big(\frac{4\,(1-v^2)}{v^4}\Big)+2\,\mathrm{Li}_2(p^2)\,\ln\Big(\frac{1-v^2}{2\,v^2}\Big)+2\,\zeta(2)\Big[\ln p-\ln\Big(\frac{1-v^2}{4\,v}\Big)\Big]\\&-\frac{1}{6}\,\ln\Big(\frac{1+v}{2}\Big)\Big[36\,\ln 2\,\ln p-44\,\ln^2 p+49\,\ln p\,\ln\Big(\frac{1-v^2}{4}\Big)+\ln^2\Big(\frac{1-v^2}{4}\Big)\Big]\\&-\frac{1}{2}\,\ln p\,\ln v\Big[36\,\ln 2+21\,\ln p+16\,\ln v-22\,\ln(1-v^2)\Big]\Big\}\\&+\frac{1}{24}\Big\{(15-6\,v^2-v^4)\Big(\mathrm{Li}_2(p)+\mathrm{Li}_2(p^2)\Big)+3\,(7-22\,v^2+7\,v^4)\,\mathrm{Li}_2(p)\\&-(1-v)\,(51-45\,v-27\,v^2+5\,v^3)\,\zeta(2)\\&+\frac{(1+v)\,(-9+33\,v-9\,v^2-15\,v^3+4\,v^4)}{v}\,\ln^2 p\\&+\Big[(33+22\,v^2-7\,v^4)\,\ln 2-10\,(3-v^2)\,(1+v^2)\,\ln v\\&\qquad-(15-22\,v^2+3\,v^4)\,\ln\Big(\frac{1-v^2}{4\,v^2}\Big)\Big]\,\ln p\\&+2\,v\,(3-v^2)\,\ln\Big(\frac{4\,(1-v^2)}{v^4}\Big)\Big[\ln v-3\,\ln\Big(\frac{1-v^2}{4\,v}\Big)\Big]\\&+\frac{237-96\,v+62\,v^2+32\,v^3-59\,v^4}{4}\,\ln p-16\,v\,(3-v^2)\,\ln\Big(\frac{1+v}{4}\Big)\\&-2\,v\,(39-17\,v^2)\,\ln\Big(\frac{1-v^2}{2\,v^2}\Big)-\frac{v\,(75-29\,v^2)}{2}\Big\}.\,\ldots\end{aligned}\qquad\text{(B.3)}$$

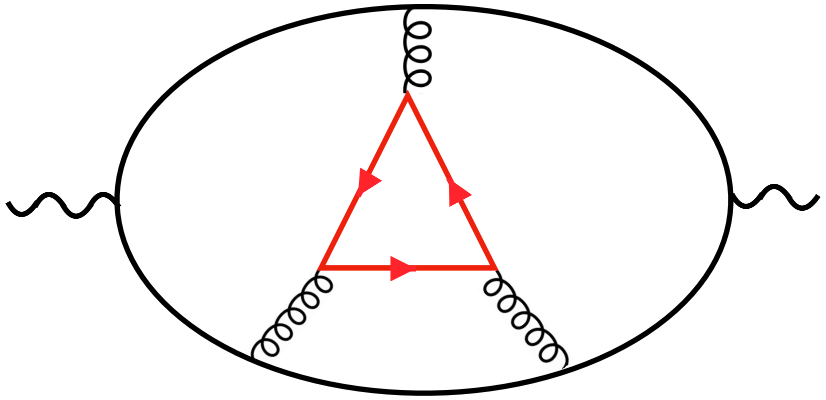
[Chetykrin, Kuehn, Steinhauser, arxiv : 9606230]

PRELIMINARY N3LO RESULTS

n_f contributions :



+ new topologies, such as:

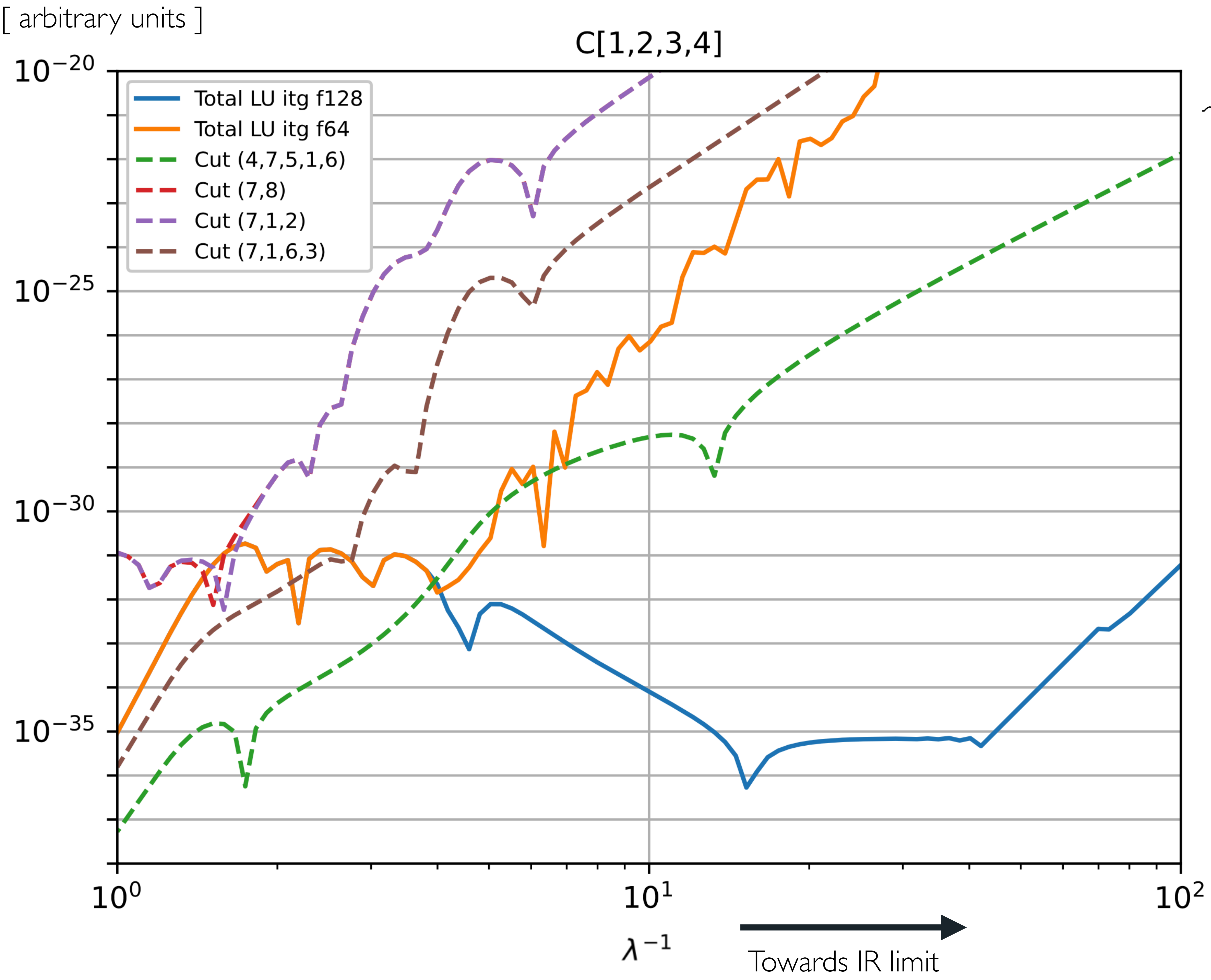


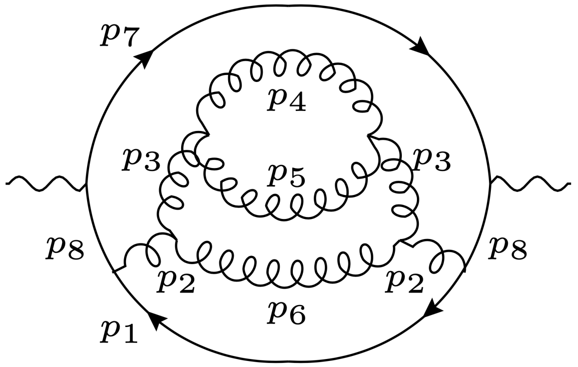
$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f),(\text{MC LU})} = -77.1(1.7)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f)} = -C_F^2\left(\frac{29}{2} - 152\zeta_3 + 160\zeta_5\right) - C_F C_A\left(\frac{15520}{27} - \frac{88}{3}\zeta_2 - \frac{3584}{9}\zeta_3 - \frac{80}{3}\zeta_5\right) = -76.8086$$

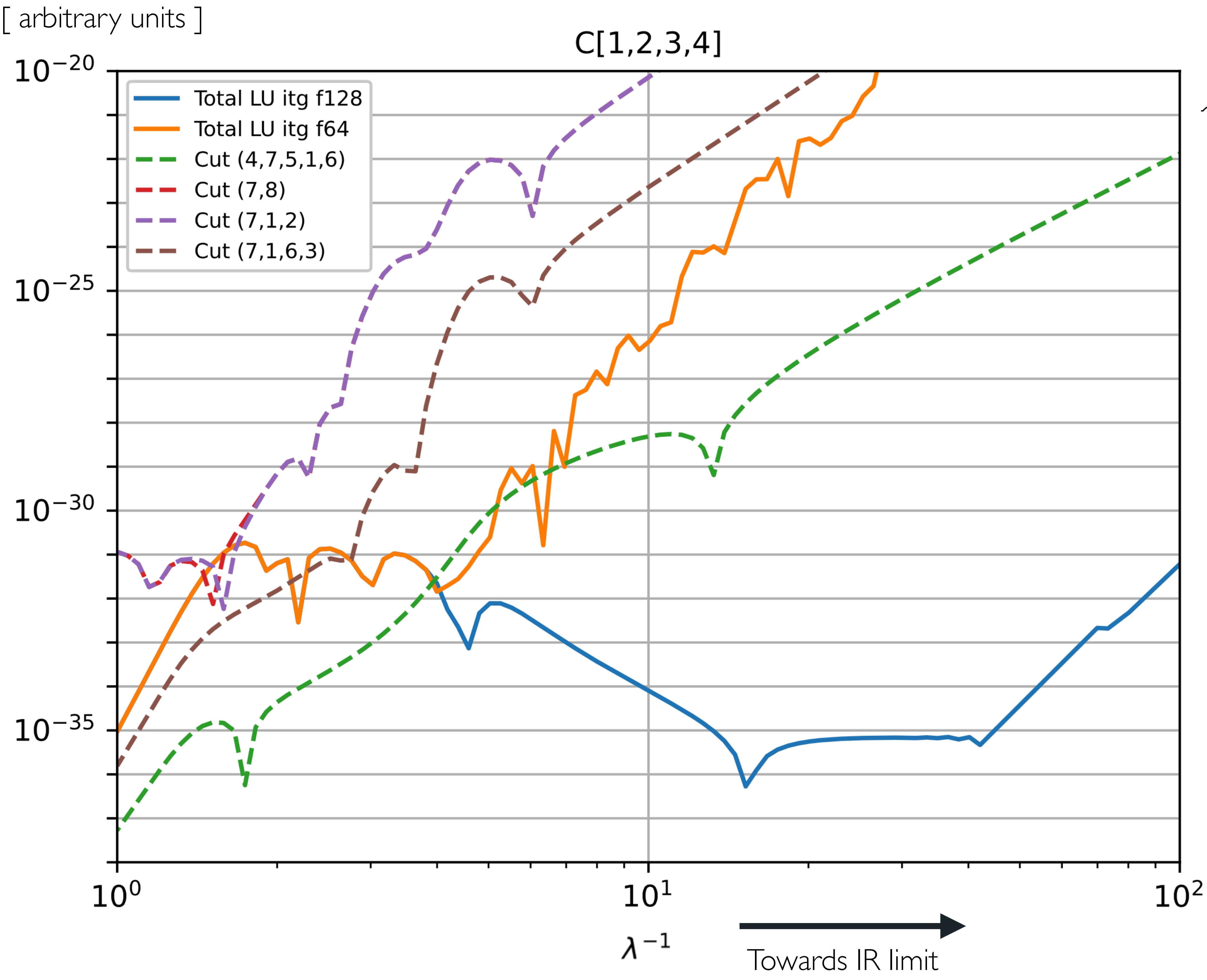
[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

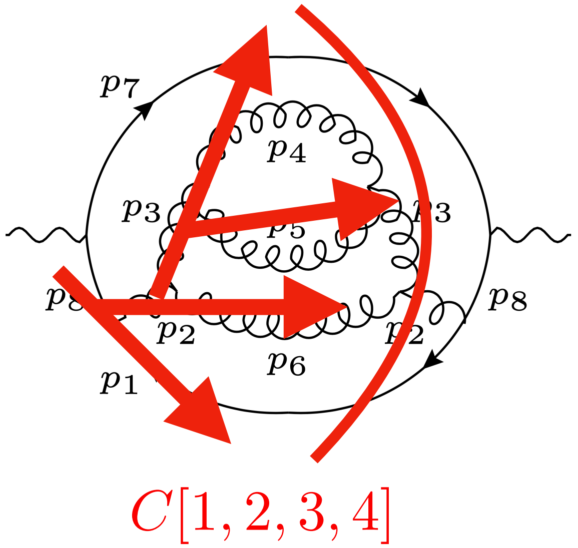
TESTING IR QUADRUPLE COLLINEAR LIMITS



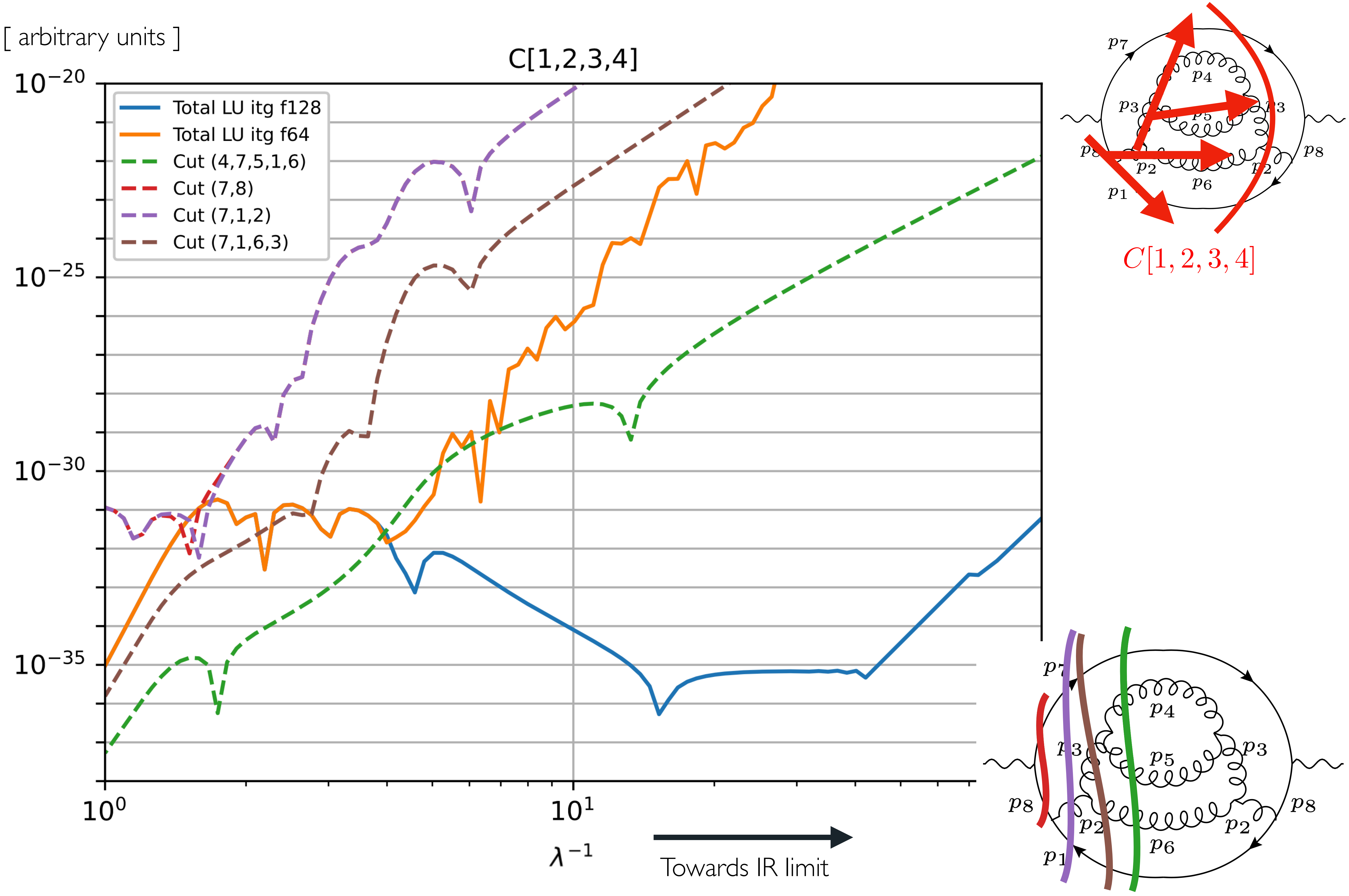


TESTING IR QUADRUPLE COLLINEAR LIMITS





TESTING IR QUADRUPLE COLLINEAR LIMITS

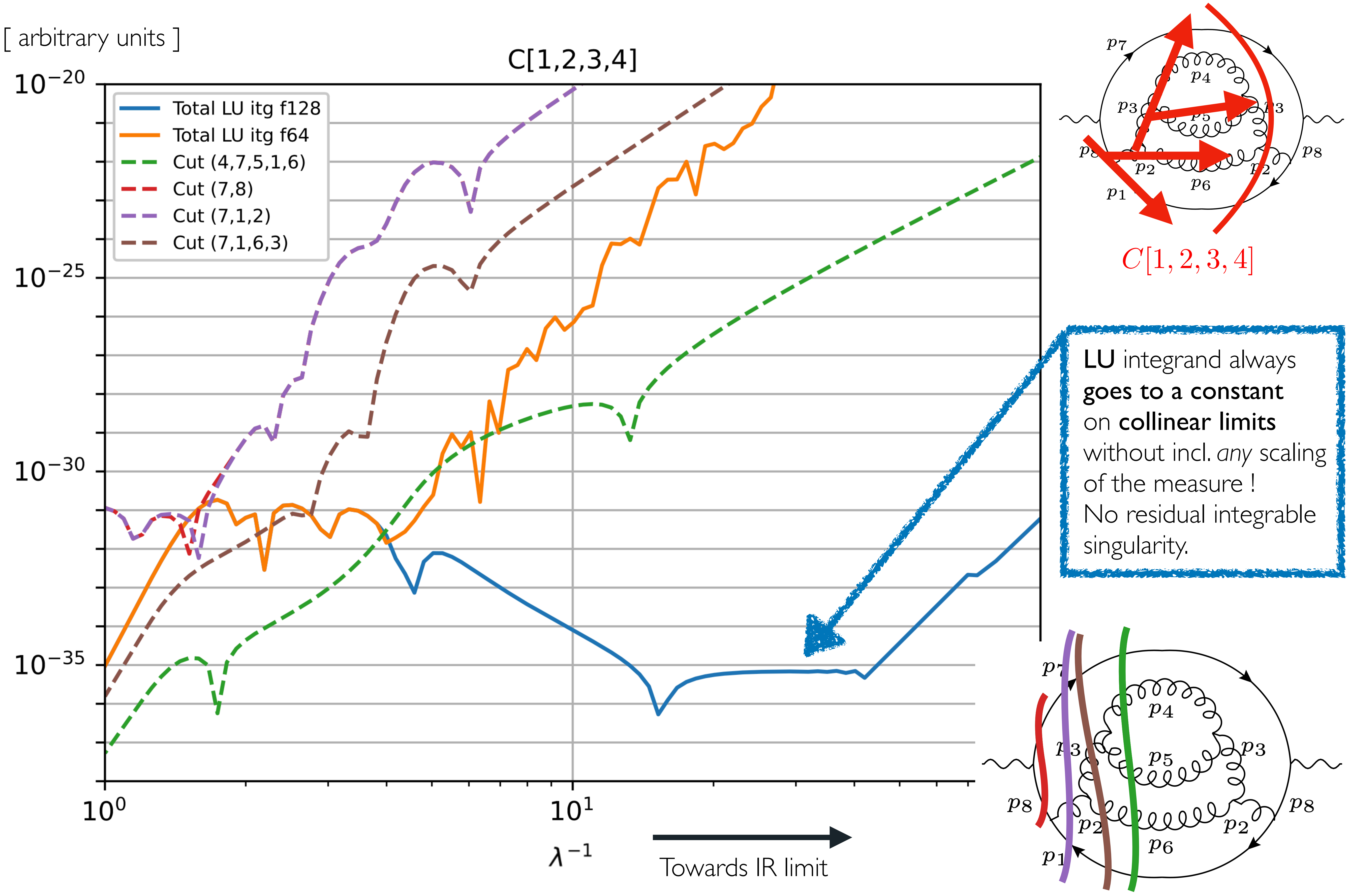


[arbitrary units]

$C[1,2,3,4]$

Towards IR limit

TESTING IR QUADRUPLE COLLINEAR LIMITS

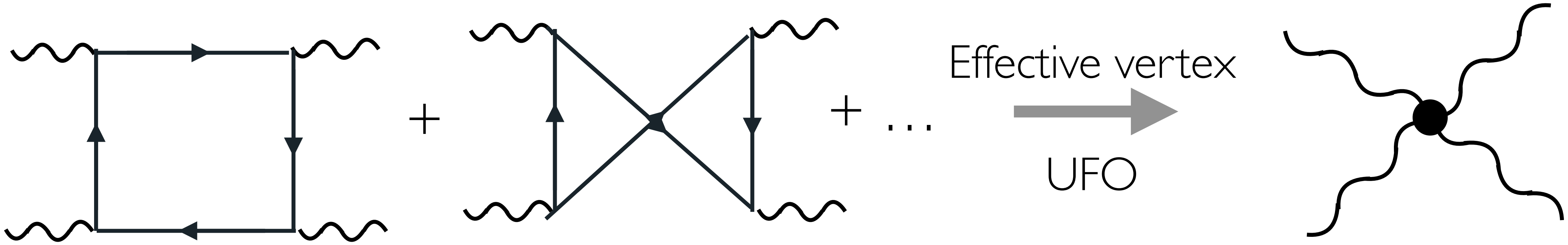


LU integrand always **goes to a constant** on **collinear limits** without incl. *any* scaling of the measure !
No residual integrable singularity.

PHOTON-PHOTON SCATTERING IN HI UPC

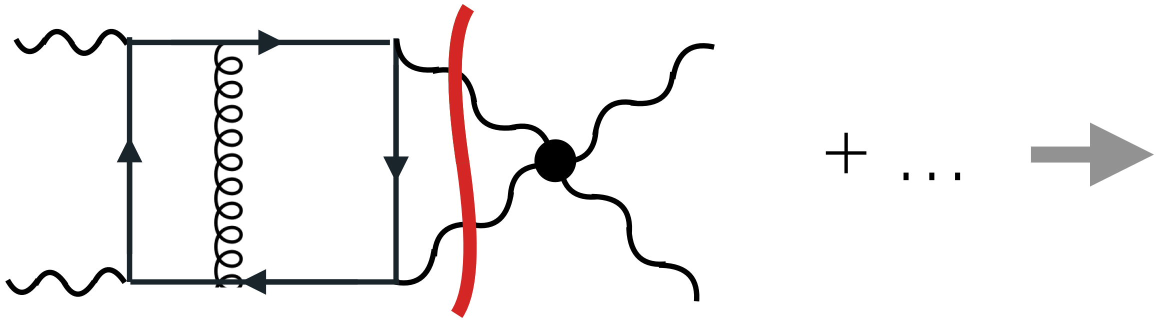
[In collaboration with H.S. -Shao, M. Fraaije, E. Chaubet]

- Too early to present / show much, but alphaLoop delivered a first NLO unknown x-sec!



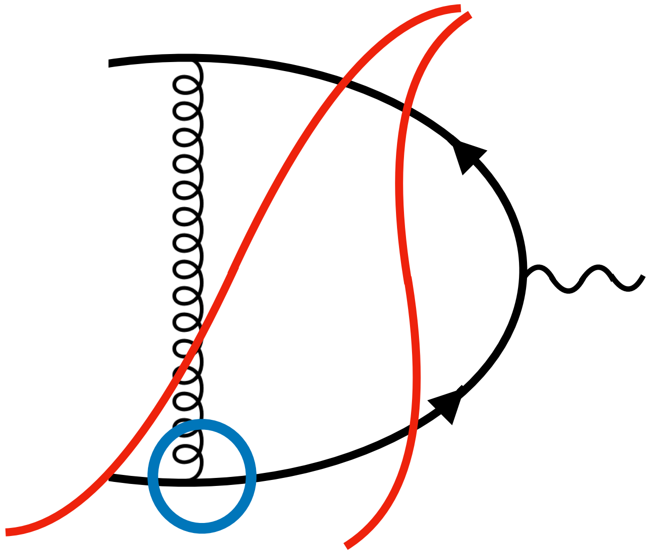
$$\sim \sum_{i=1}^{3} C_i(s,t,u,m_f^2) T_i^{\mu_1\mu_2\mu_3\mu_4}(\{p_i\})$$

- In alphaloop we then do



Yielding our first "prediction" for a piece of a yet unknown cross-section: NLO photon scattering. Successful validation vs analytics!

INITIAL-STATE SINGULARITIES

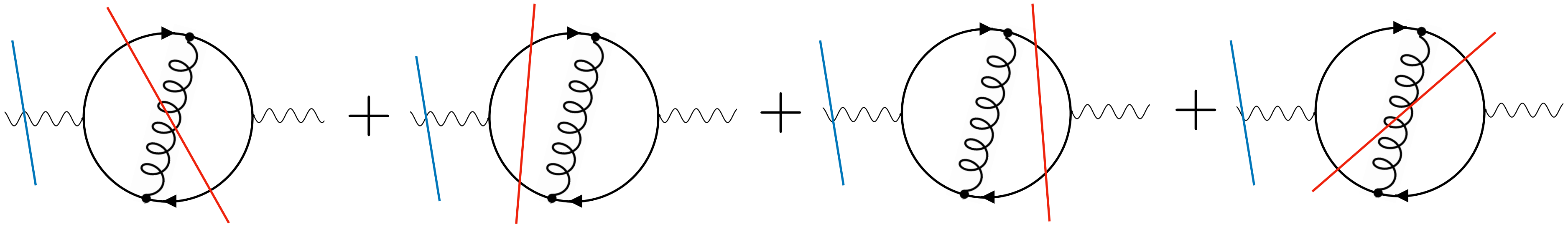


KLN CAN WORK FOR INITIAL-STATE !

INITIAL-STATE SINGULARITIES: IDEA

INITIAL-STATE SINGULARITIES: IDEA

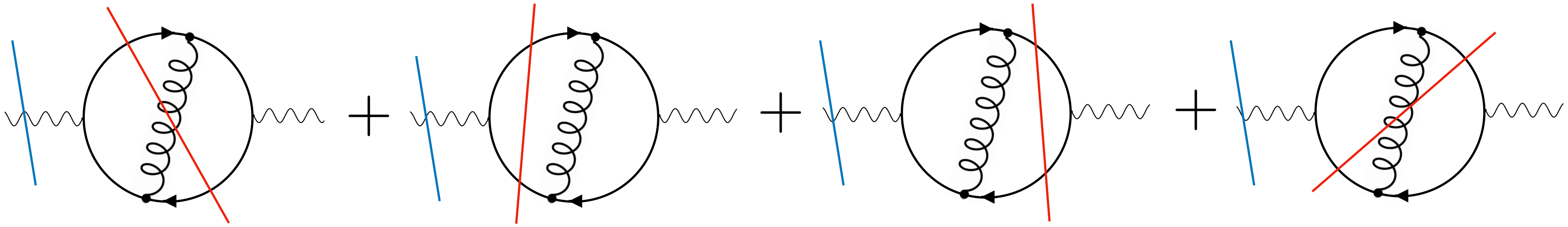
For $e^+e^- \rightarrow 2j$ @ NLO we used



(Including all degenerate configurations, higher final-state multiplicities)

INITIAL-STATE SINGULARITIES: IDEA

For $e^+e^- \to 2j$ @ NLO we used

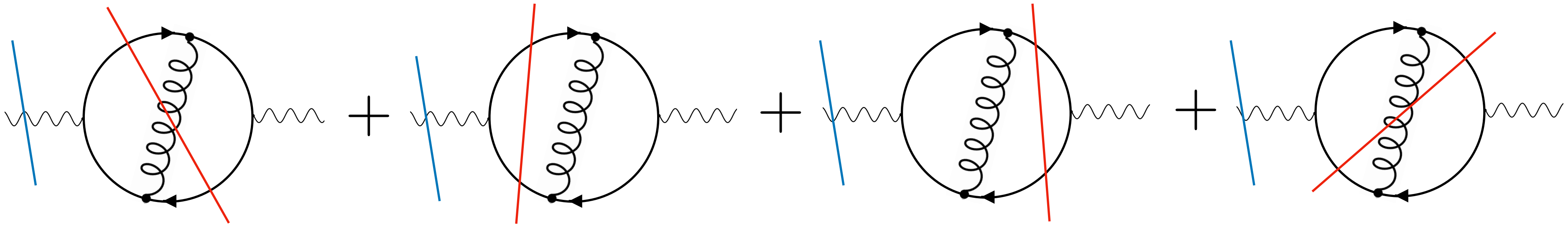


(Including all degenerate configurations, higher final-state multiplicities)

Flip it, and obtain the answer for Drell-Yan, $2\,\mathrm{j} \to e^+e^-$

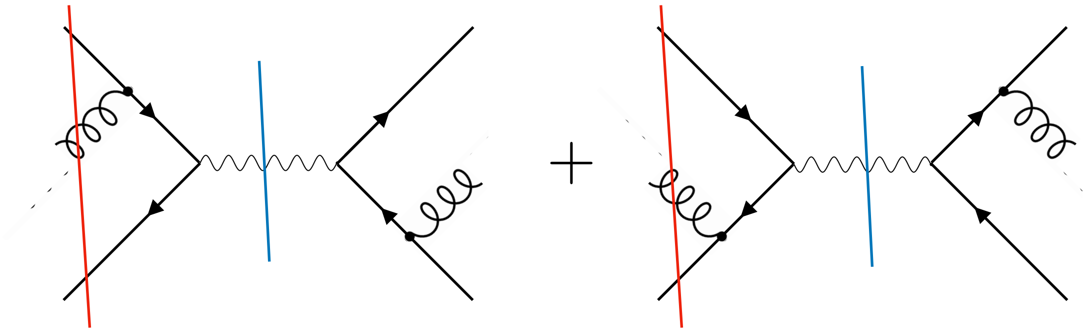
INITIAL-STATE SINGULARITIES: IDEA

For $e^+e^- \to 2j$ @ NLO we used



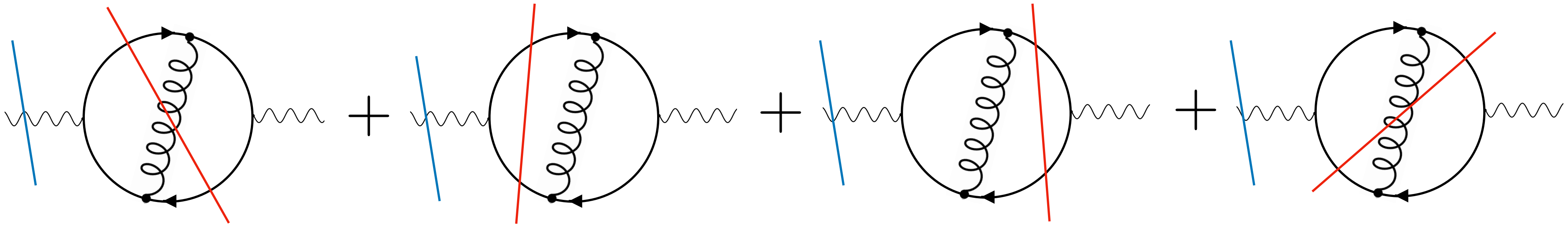
(Including all degenerate configurations, higher final-state multiplicities)

Flip it, and obtain the answer for Drell-Yan, $2\,\mathrm{j} \to e^+e^-$



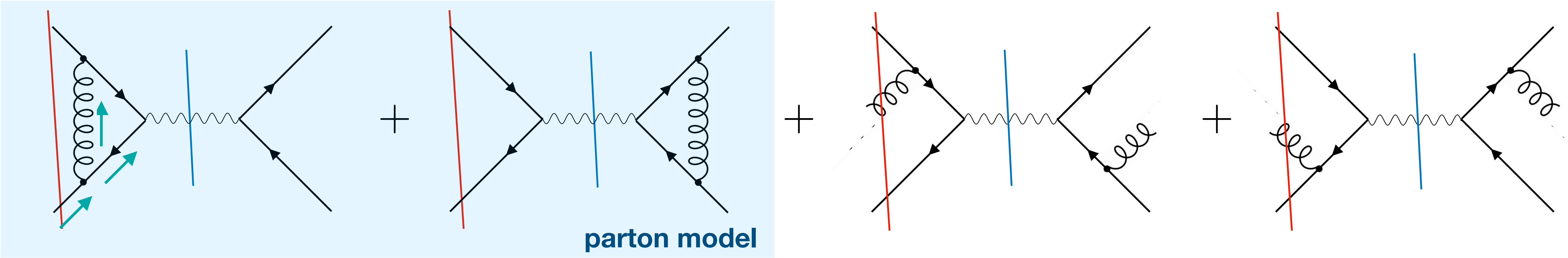
INITIAL-STATE SINGULARITIES: IDEA

For $e^+e^- \rightarrow 2j$ @ NLO we used



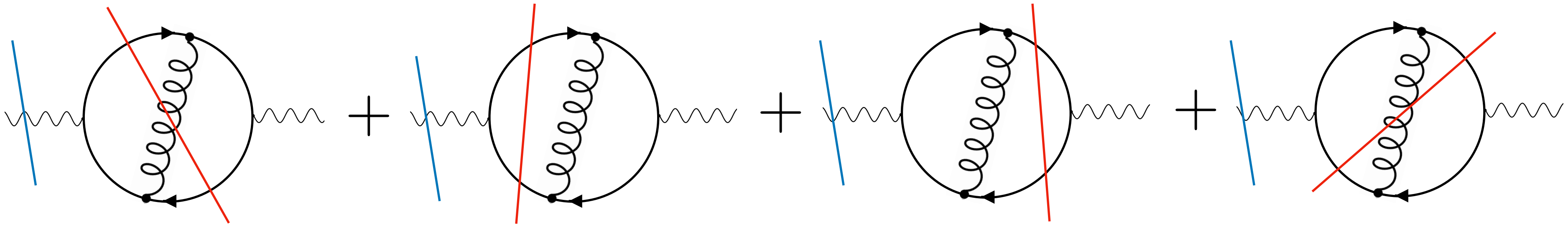
(Including all degenerate configurations, higher final-state multiplicities)

Flip it, and obtain the answer for Drell-Yan, $2\,\mathrm{j} \rightarrow e^+e^-$



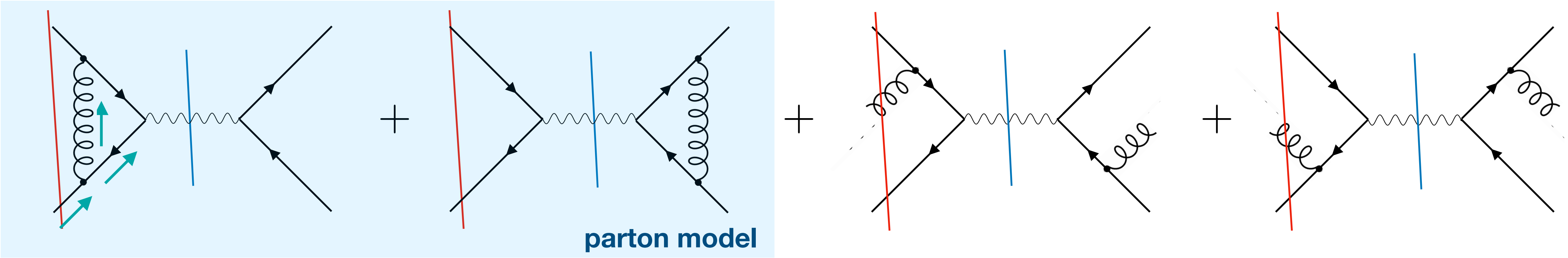
INITIAL-STATE SINGULARITIES: IDEA

For $e^+e^- \to 2j$ @ NLO we used



(Including all degenerate configurations, higher final-state multiplicities)

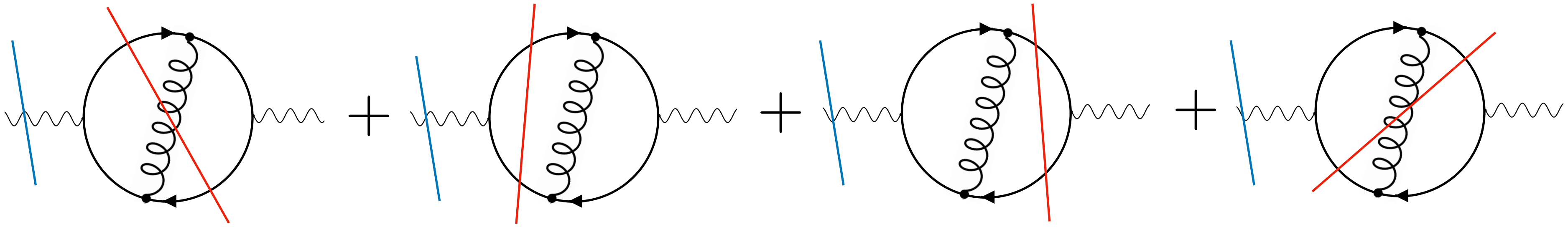
Flip it, and obtain the answer for Drell-Yan, $2\,\mathrm{j} \to e^+e^-$



The initial state singularity is now absent!

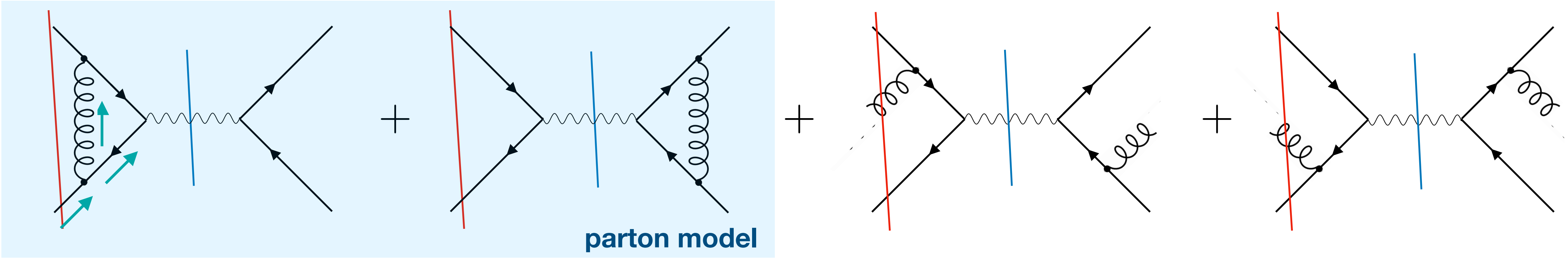
INITIAL-STATE SINGULARITIES: IDEA

For $e^+e^- \to 2j$ @ NLO we used



(Including all degenerate configurations, higher final-state multiplicities)

Flip it, and obtain the answer for Drell-Yan, $2\,\mathrm{j} \to e^+e^-$



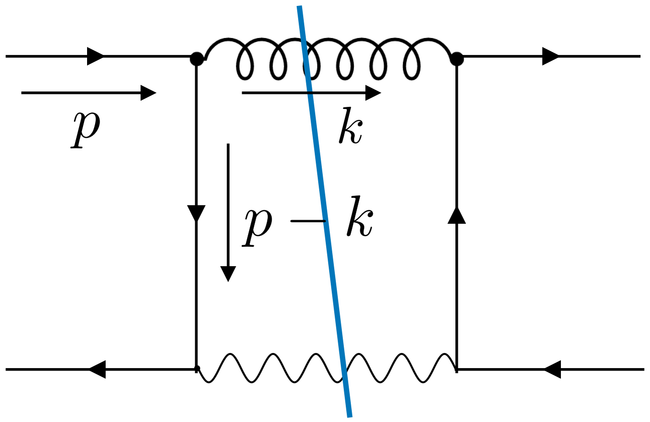
The initial state singularity is now absent!

Include degenerate initial states $\longrightarrow$ **Higher multiplicity initial states**

INITIAL-STATE SINGULARITIES: IDEA

INITIAL-STATE SINGULARITIES: IDEA

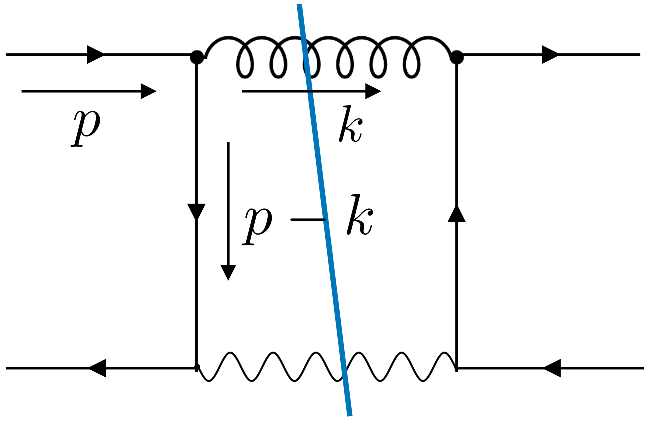
What about
this diagram?



Also has collinear
singularity at $k = xp$

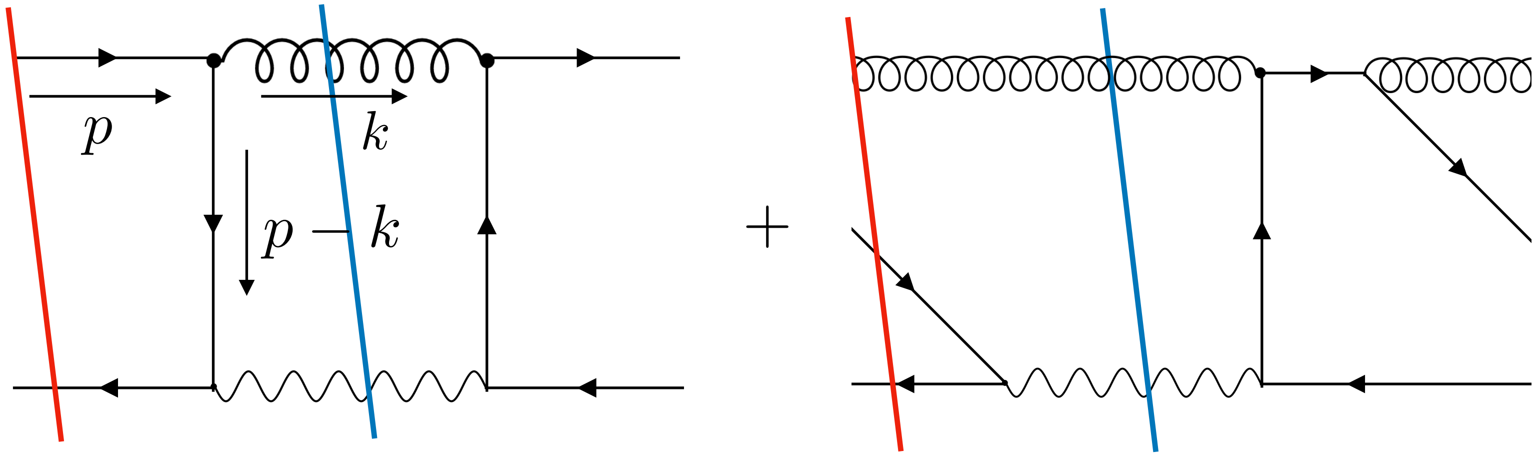
INITIAL-STATE SINGULARITIES: IDEA

What about
this diagram?



Also has collinear
singularity at $k = xp$

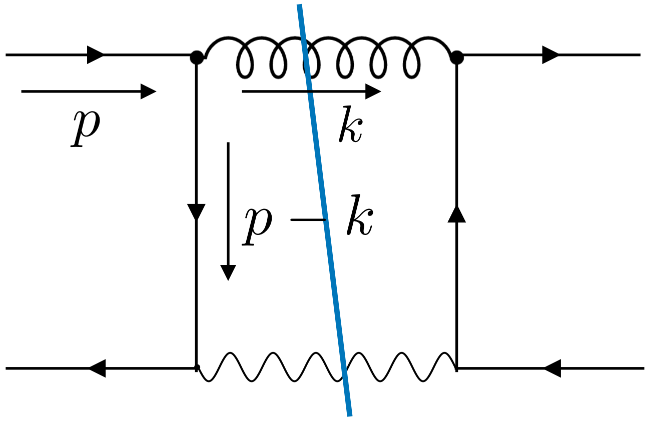
In this case, the cancelling partner is



Higher multiplicity initial states, but also **disconnected**! **Free travelling gluon!**

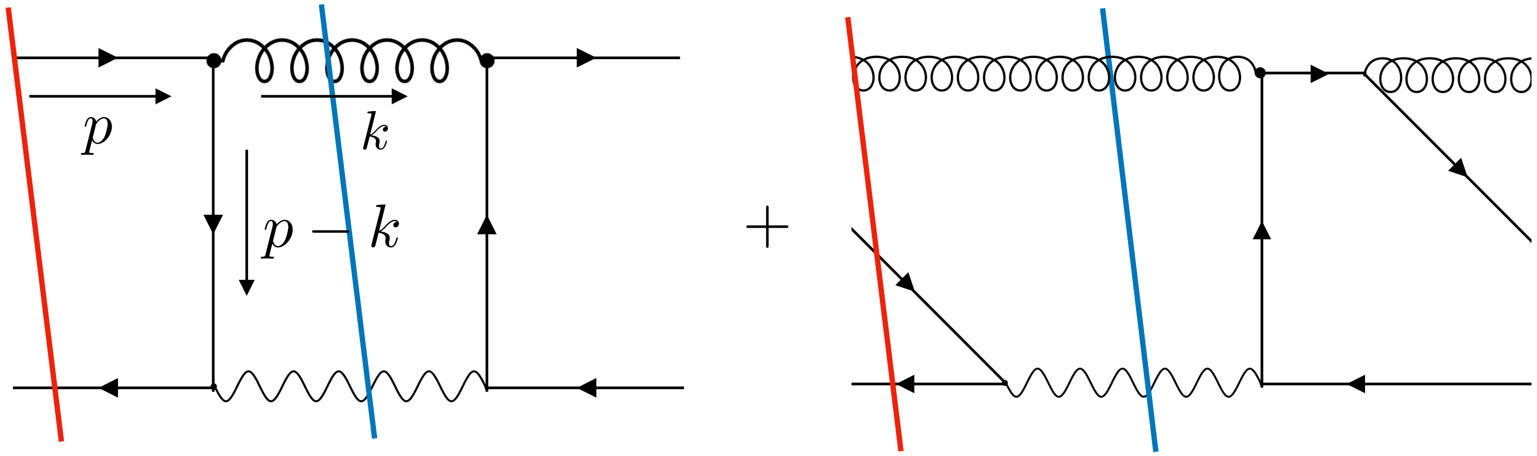
INITIAL-STATE SINGULARITIES: IDEA

What about
this diagram?



Also has collinear
singularity at $k = xp$

In this case, the cancelling partner is

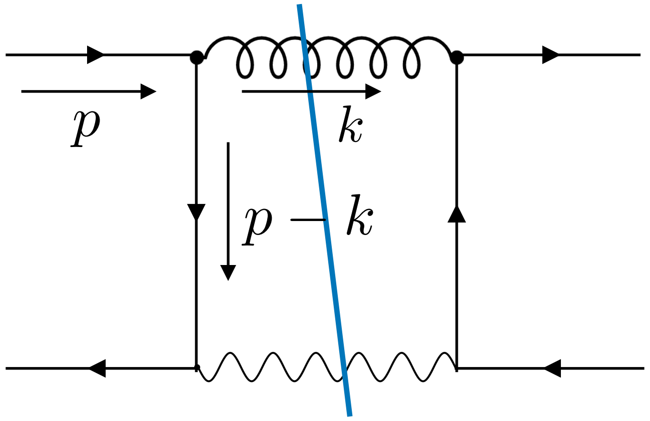


Higher multiplicity initial states, but also **disconnected**! **Free travelling gluon!**

The sum of these two diagrams is finite everywhere in phase space

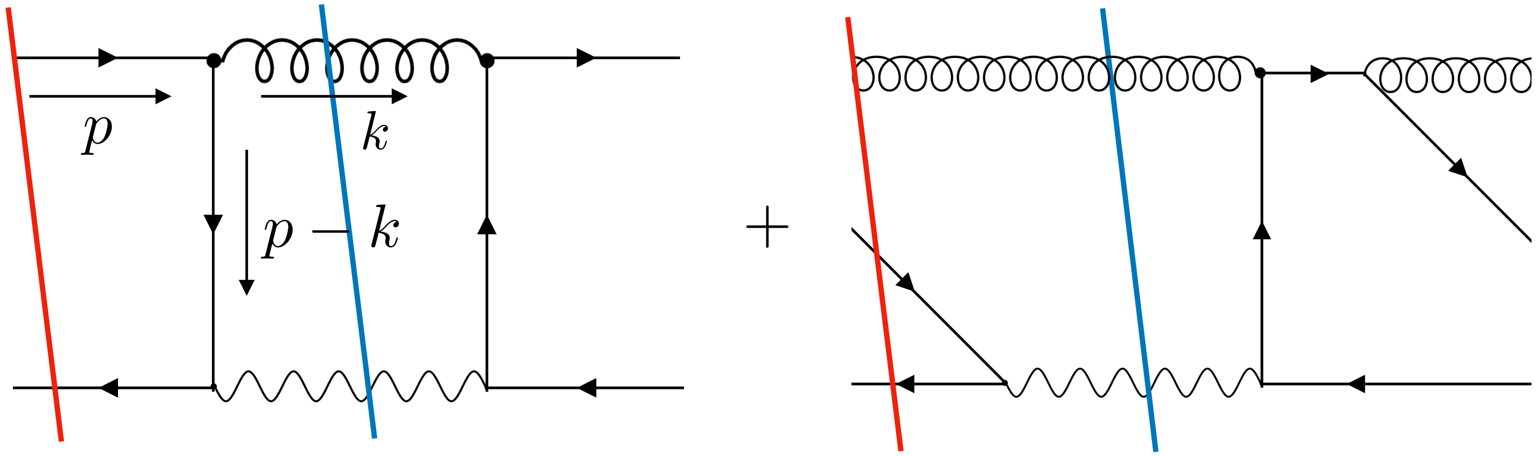
INITIAL-STATE SINGULARITIES: IDEA

What about
this diagram?



Also has collinear
singularity at $k = xp$

In this case, the cancelling partner is



Higher multiplicity initial states, but also **disconnected**! **Free travelling gluon!**

The sum of these two diagrams is finite everywhere in phase space

Looks radical, but actually not a new idea, e.g. :

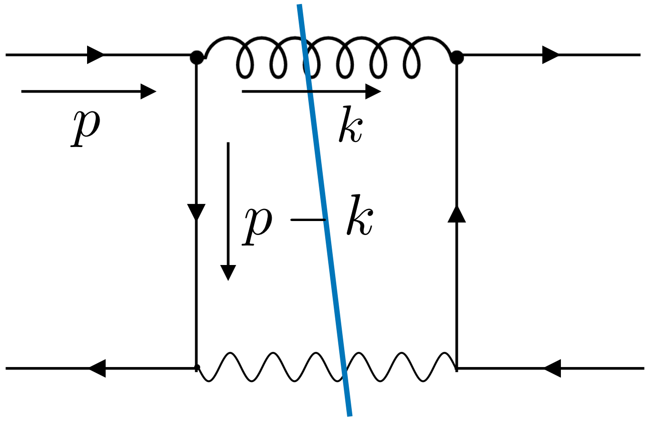
T. Kinoshita
“Mass singularities of
Feynman amplitudes”
(1962)

G. Sterman, S. Weinberg,
“Jets from Quantum Chromodynamics”
(1977)

T.D. Lee, M. Nauenberg
“Degenerate systems and
mass singularities”
(1963)

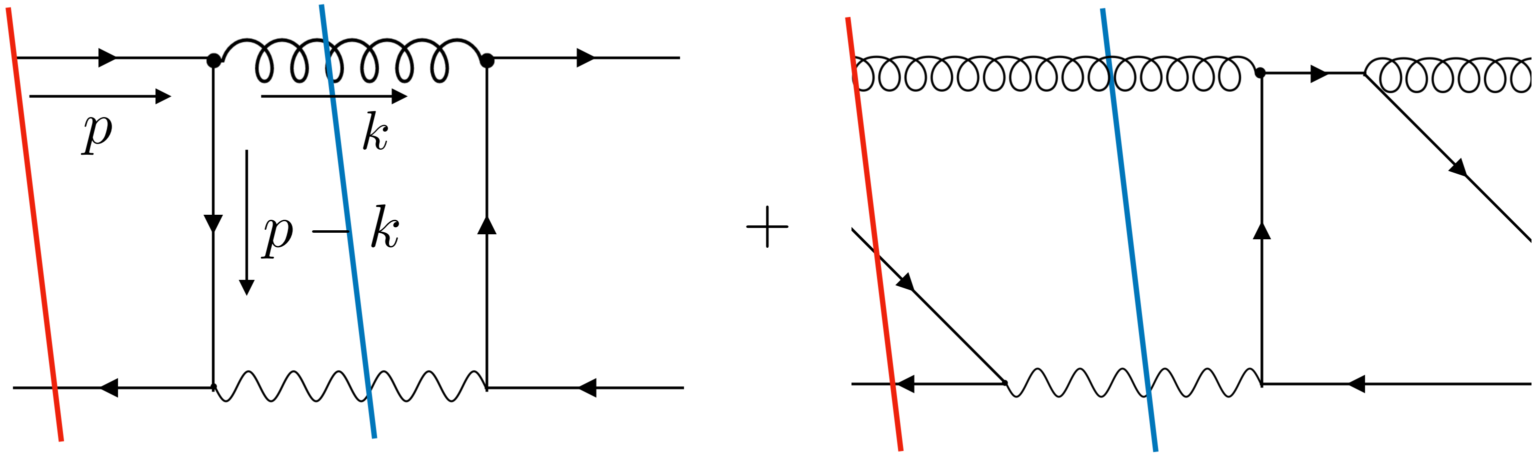
INITIAL-STATE SINGULARITIES: IDEA

What about
this diagram?



Also has collinear
singularity at $k = xp$

In this case, the cancelling partner is



Higher multiplicity initial states, but also **disconnected**! **Free travelling gluon!**

The sum of these two diagrams is finite everywhere in phase space

Looks radical, but actually not a new idea, e.g. :

T. Kinoshita
“Mass singularities of
Feynman amplitudes”
(1962)

G. Sterman, S. Weinberg,
“Jets from Quantum Chromodynamics”
(1977)

T.D. Lee, M. Nauenberg
“Degenerate systems and
mass singularities”
(1963)

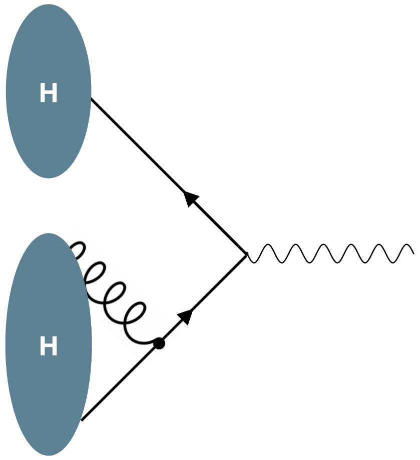
But also more recently, they were studied in:

Frye, Hannesdottir, Paul, Schwartz, Yan
arXiv:1810.10022 (2019)

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

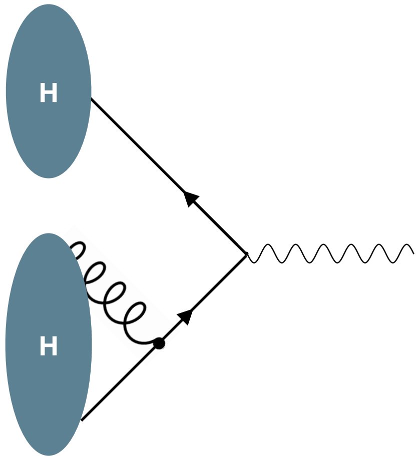
INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

This argument suggests that, in order to maintain IR-finiteness, one requires more than two initial state partons

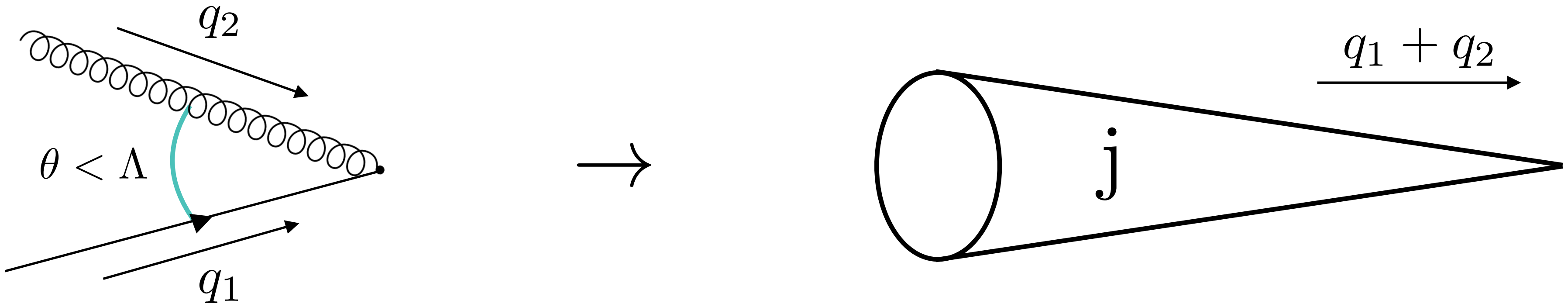


INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

This argument suggests that, in order to maintain IR-finiteness, one requires more than two initial state partons



and that the multiple partons should be clustered into **two jet-like objects** that resemble boosted hadrons

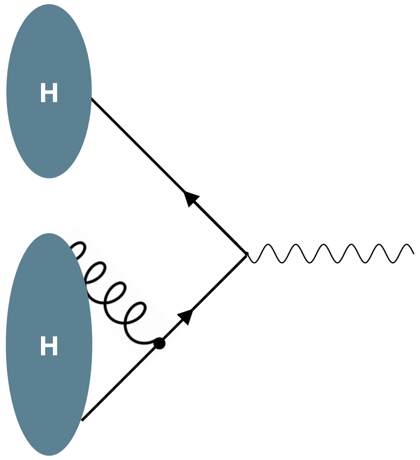


After clustering, we get two jets with momenta

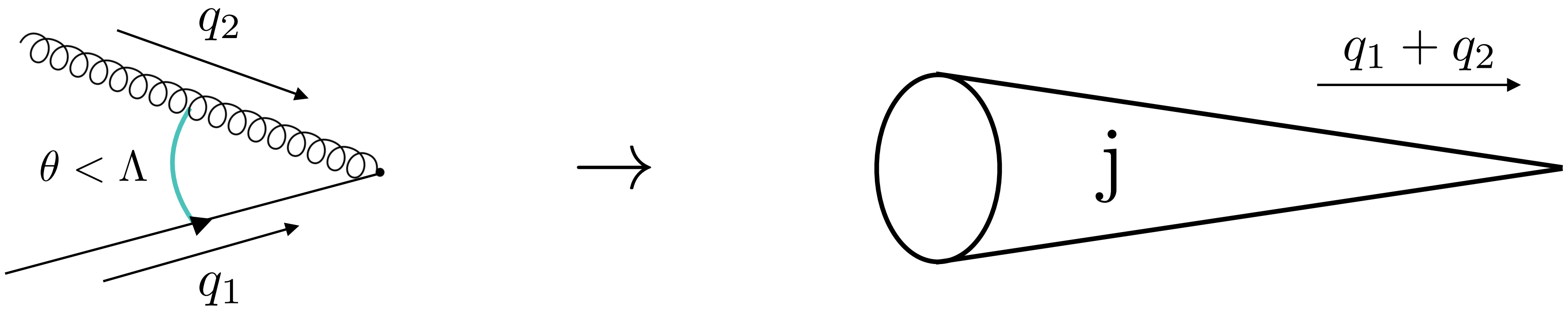
$$P_1^{\mathrm{j}} \qquad\qquad P_2^{\mathrm{j}}$$

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

This argument suggests that, in order to maintain IR-finiteness, one requires more than two initial state partons



and that the multiple partons should be clustered into **two jet-like objects** that resemble boosted hadrons



After clustering, we get two jets with momenta

$$P_1^{\text{j}} \qquad\qquad P_2^{\text{j}}$$

Cluster initial states analogously to final states: symmetry initial-final state

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

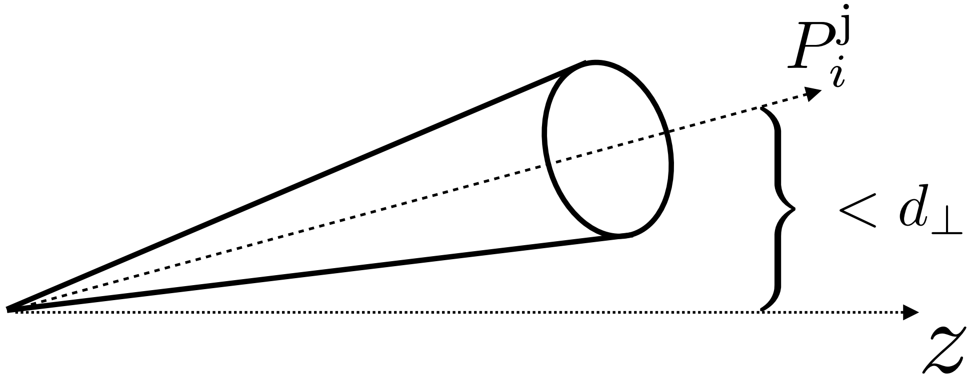
There are two relevant scales for the two initial state jets reconstructed:

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

There are two relevant scales for the two initial state jets reconstructed:

- One measuring the allowed phase space for the **total momentum** of the jet

$$(P_i^{\mathrm{j}})_\perp < d_\perp$$



If the scale is zero, the jet lies **exactly** on the z axis

If $d_\perp = 0$ the two jets are exactly back-to-back. This is equivalent to the parton's model

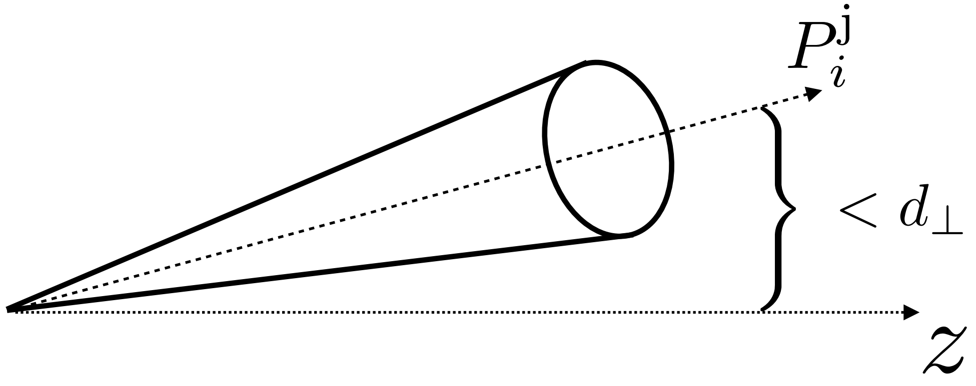
$$p_1 = (x_1\sqrt{s}, 0, 0, x_1\sqrt{s}), \quad p_2 = (x_2\sqrt{s}, 0, 0, -x_2\sqrt{s})$$

Initial-state singularities: preliminary tests

There are two relevant scales for the two initial state jets reconstructed:

- One measuring the allowed phase space for the **total momentum** of the jet

$$(P_i^{\mathrm{j}})_\perp < d_\perp$$



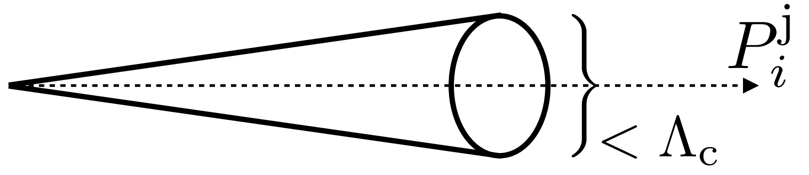
If the scale is zero, the jet lies **exactly** on the z axis

If $d_\perp = 0$ the two jets are exactly back-to-back. This is equivalent to the parton's model

$$p_1 = (x_1\sqrt{s}, 0, 0, x_1\sqrt{s}), \quad p_2 = (x_2\sqrt{s}, 0, 0, -x_2\sqrt{s})$$

- One measuring the maximum angular separation between two partons in a jet

$$(P_{ij})_\perp < \Lambda_\mathrm{c}$$



The smaller this scale, the more **collinear** the partons are

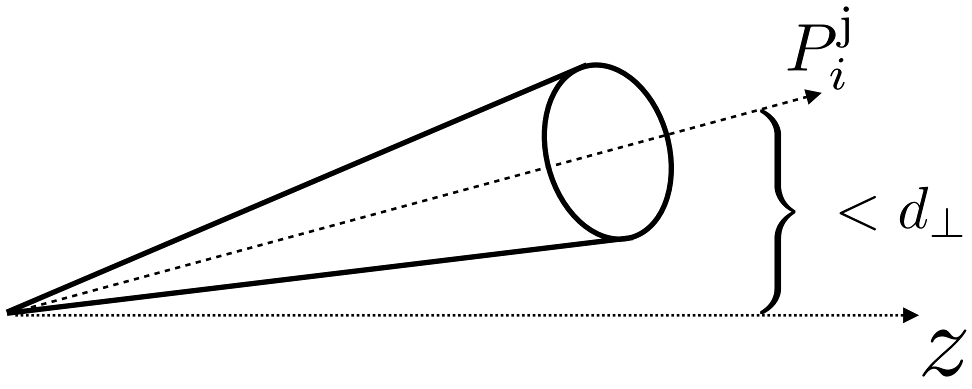
The more collinear the partons, the more divergent the observable

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

There are two relevant scales for the two initial state jets reconstructed:

- One measuring the allowed phase space for the **total momentum** of the jet

$$(P_i^{\mathrm{j}})_\perp < d_\perp$$



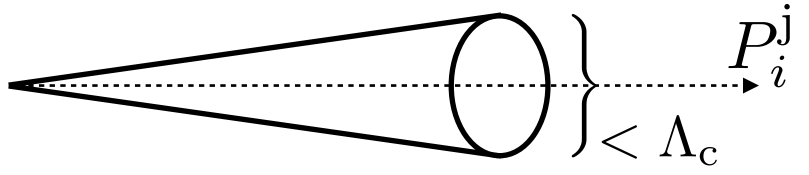
If the scale is zero, the jet lies **exactly** on the z axis

If $d_\perp = 0$ the two jets are exactly back-to-back. This is equivalent to the parton's model

$$p_1 = (x_1\sqrt{s}, 0, 0, x_1\sqrt{s}), \quad p_2 = (x_2\sqrt{s}, 0, 0, -x_2\sqrt{s})$$

- One measuring the maximum angular separation between two partons in a jet

$$(P_{ij})_\perp < \Lambda_{\mathrm{c}}$$



The smaller this scale, the more **collinear** the partons are

The more collinear the partons, the more divergent the observable

Λ_{c} **is the equivalent of the factorisation scale!** $\approx \log(\Lambda_{\mathrm{c}})$

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

- Take the limit $d_\perp \to 0$ **analytically** and obtain **exact back to back jets**

$$P_i^{\mathrm{j}} = ((P_i^{\mathrm{j}})^0, 0, 0, (P_i^{\mathrm{j}})^3)$$

This allows us to define **Bjorken variables**

$$x_1 = \frac{(P_1^{\mathrm{j}})^0 + (P_1^{\mathrm{j}})^3}{2} \qquad\qquad x_2 = \frac{(P_2^{\mathrm{j}})^0 - (P_2^{\mathrm{j}})^3}{2}$$

For a 2 to N diagram this reproduces the parton model

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

- Take the limit $d_\perp \to 0$ **analytically** and obtain **exact back to back jets**

$$P_i^{\mathrm{j}} = ((P_i^{\mathrm{j}})^0, 0, 0, (P_i^{\mathrm{j}})^3)$$

This allows us to define **Bjorken variables**

$$x_1 = \frac{(P_1^{\mathrm{j}})^0 + (P_1^{\mathrm{j}})^3}{2} \qquad x_2 = \frac{(P_2^{\mathrm{j}})^0 - (P_2^{\mathrm{j}})^3}{2}$$

For a 2 to N diagram this reproduces the parton model

- Bin the distribution in the Bjorken variables → **Fit PDFs!**

(not in MS bar)

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

- Take the limit $d_\perp \to 0$ **analytically** and obtain **exact back to back jets**

$$P_i^{\mathrm{j}} = ((P_i^{\mathrm{j}})^0, 0, 0, (P_i^{\mathrm{j}})^3)$$

This allows us to define **Bjorken variables**

$$x_1 = \frac{(P_1^{\mathrm{j}})^0 + (P_1^{\mathrm{j}})^3}{2} \qquad x_2 = \frac{(P_2^{\mathrm{j}})^0 - (P_2^{\mathrm{j}})^3}{2}$$

For a 2 to N diagram this reproduces the parton model

- Bin the distribution in the Bjorken variables $\longrightarrow$ **Fit PDFs!**

(not in MS bar)

- Vary the factorisation scale Λ_c and interpolate the dependence on the factorisation scale **Numerical resummation?**

Banfi, Salam, Zanderighi, arXiv:0407286 (2004)

INITIAL-STATE SINGULARITIES: "PDFS"

INITIAL-STATE SINGULARITIES: "PDFS"

We started with a very generic formalism for scattering

$$\sigma(HH \to X + n\,\mathrm{j}) = \sum_{m} \int \left[\prod_{i=1}^{m} \mathrm{d}^3\vec{p}_i\right] f(p_1,...,p_m) \;\; \frac{\mathrm{d}^m\sigma}{\mathrm{d}p_1...\mathrm{d}p_m}(p_1,...,p_m \to X + n\mathrm{j})$$

Sum over number of initial state partons

Integration over initial state partons momenta

Weight

Cross-sections for m initial state partons

INITIAL-STATE SINGULARITIES: “PDFS”

We started with a very generic formalism for scattering

$$\sigma(HH \to X + n\,\mathrm{j}) = \sum_m \int \left[\prod_{i=1}^{m} \mathrm{d}^3\vec{p}_i\right] f(p_1,...,p_m)\ \frac{\mathrm{d}^m\sigma}{\mathrm{d}p_1...\mathrm{d}p_m}(p_1,...,p_m \to X + n\mathrm{j})$$

Sum over number of initial state partons

Integration over initial state partons momenta

Weight

Cross-sections for m initial state partons

And we “forced” the initial-state observable to reproduce the usual factorised structure:

$$\sigma(HH \to X + n\,\mathrm{j}) = \int \mathrm{d}x_1\mathrm{d}x_2\ f(x_1,\Lambda_\mathrm{c})\ f(x_2,\Lambda_\mathrm{c})\ \frac{\mathrm{d}^2\sigma_p}{\mathrm{d}x_1\mathrm{d}x_2}(2\,\mathrm{j} \to X + n\,\mathrm{j}, \Lambda_\mathrm{c})$$

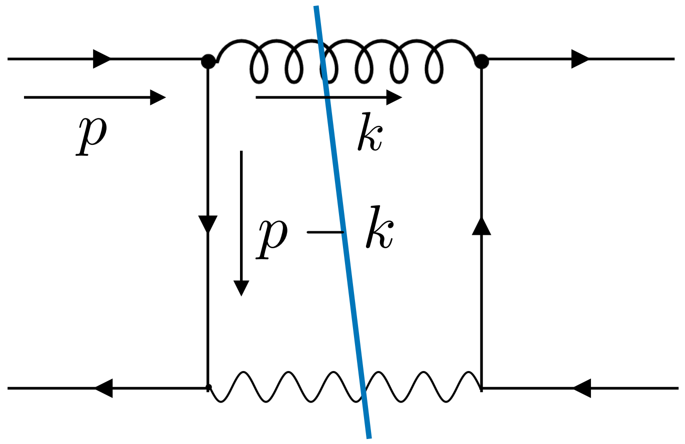
But we did not need to start from this factorised ansatz!

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

Numerical example result for this finite sum of two interference diagrams:

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

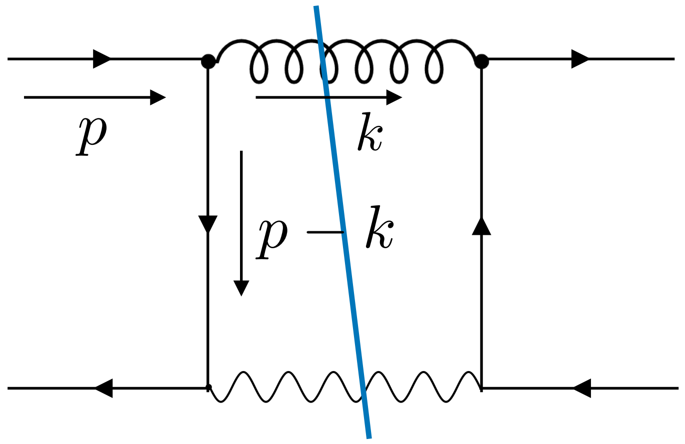
Numerical example result for this finite sum of two interference diagrams:



Always included!
This is the usual contribution.

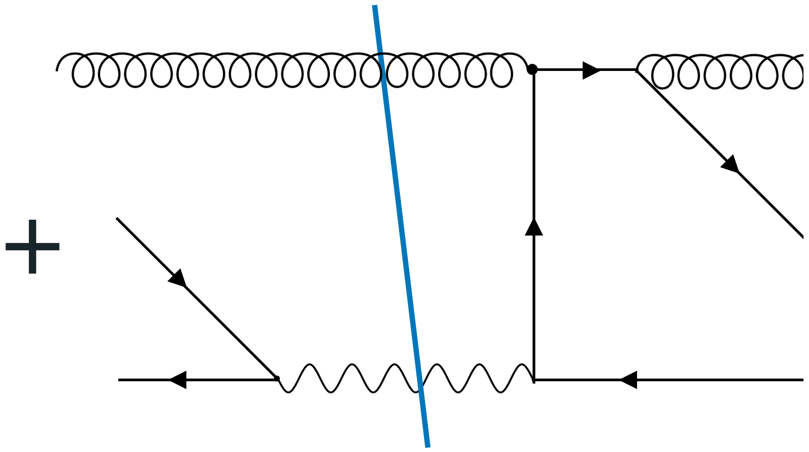
INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

Numerical example result for this finite sum of two interference diagrams:



Always included!
This is the usual contribution.

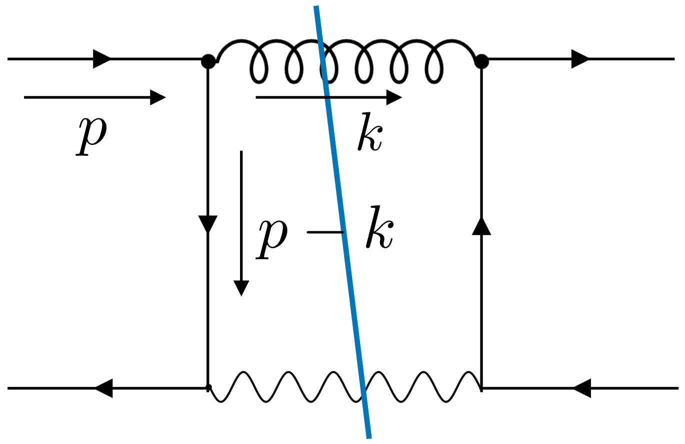
+



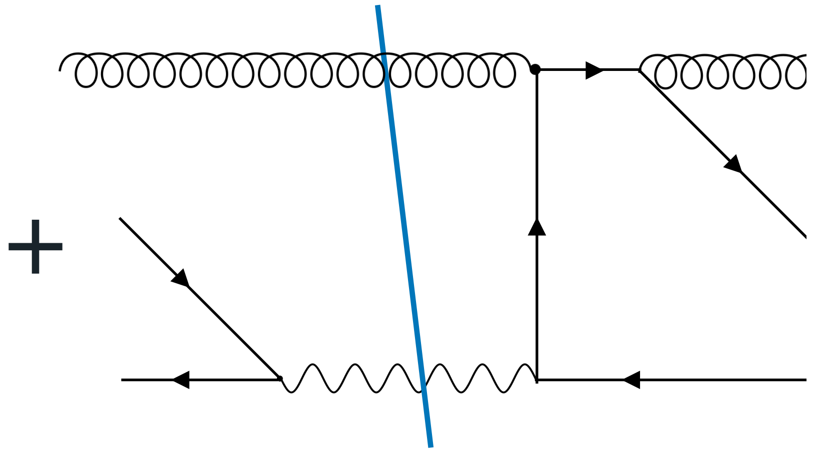
Included only when two initial-state
quarks with collinearity < Λ_c

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

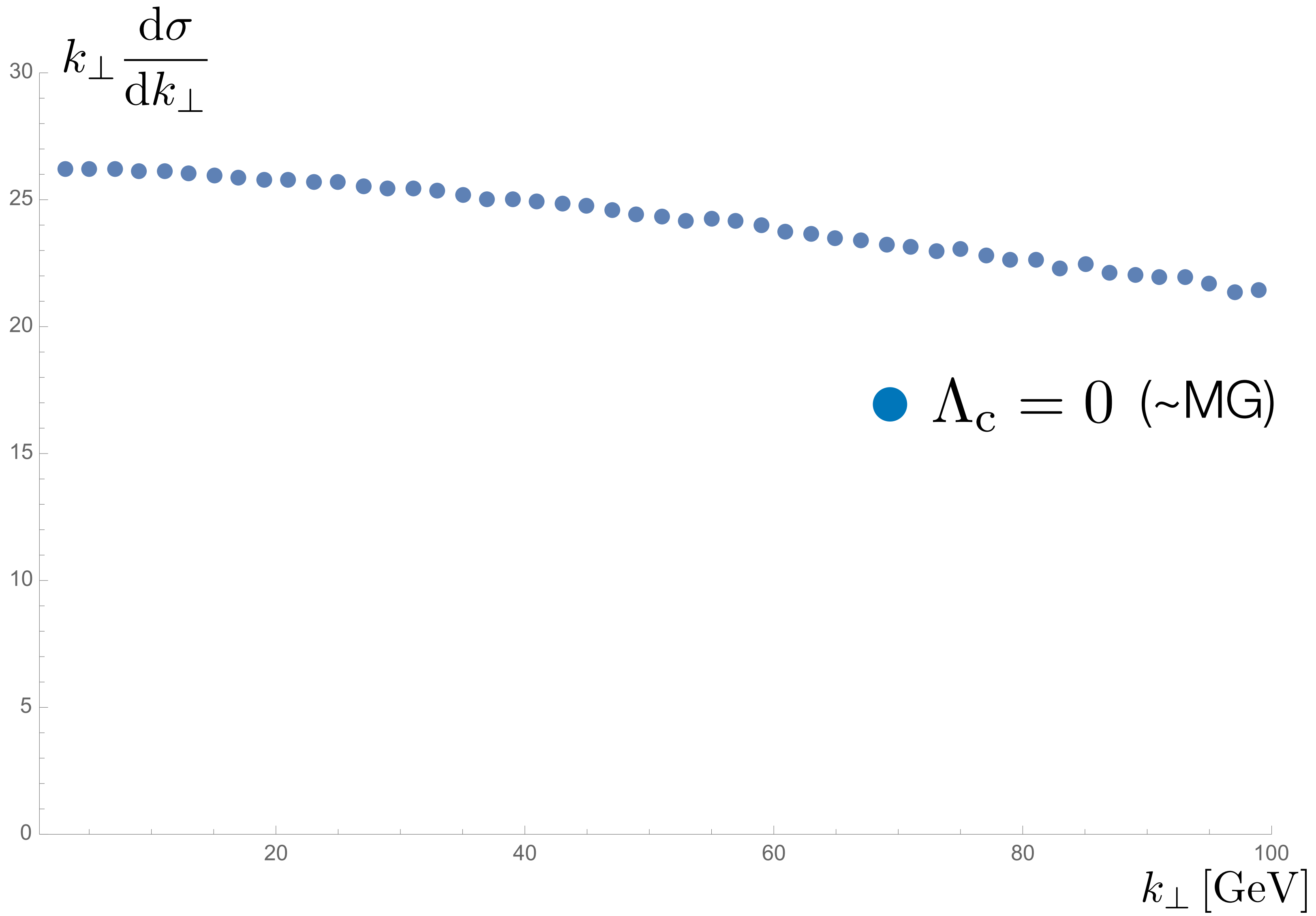
Numerical example result for this finite sum of two interference diagrams:



Always included!
This is the usual contribution.

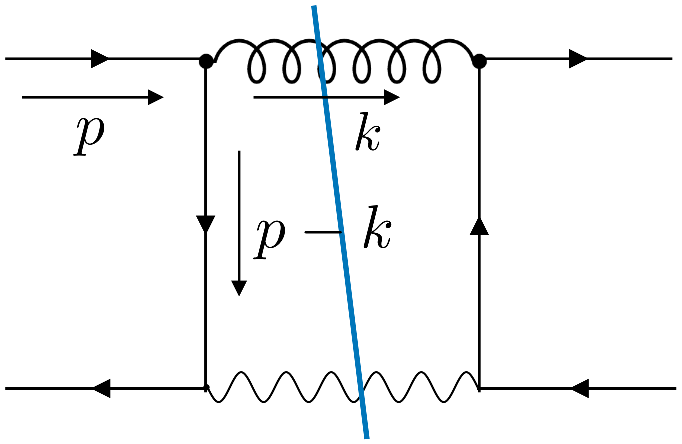


Included only when two initial-state quarks with collinearity < Λ_c

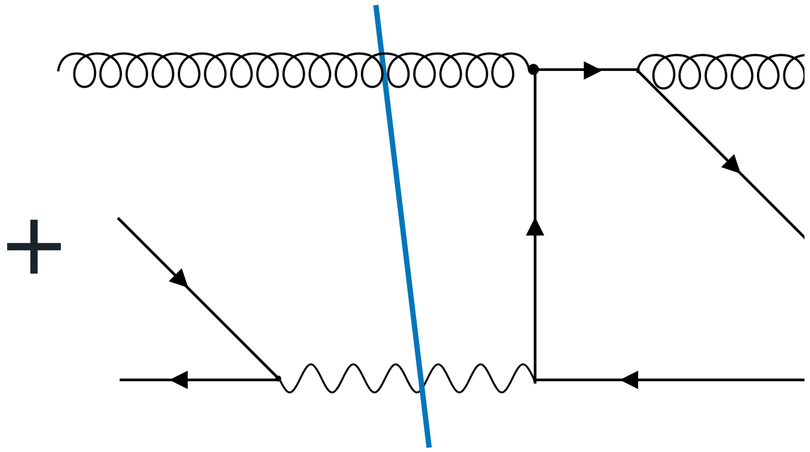


INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

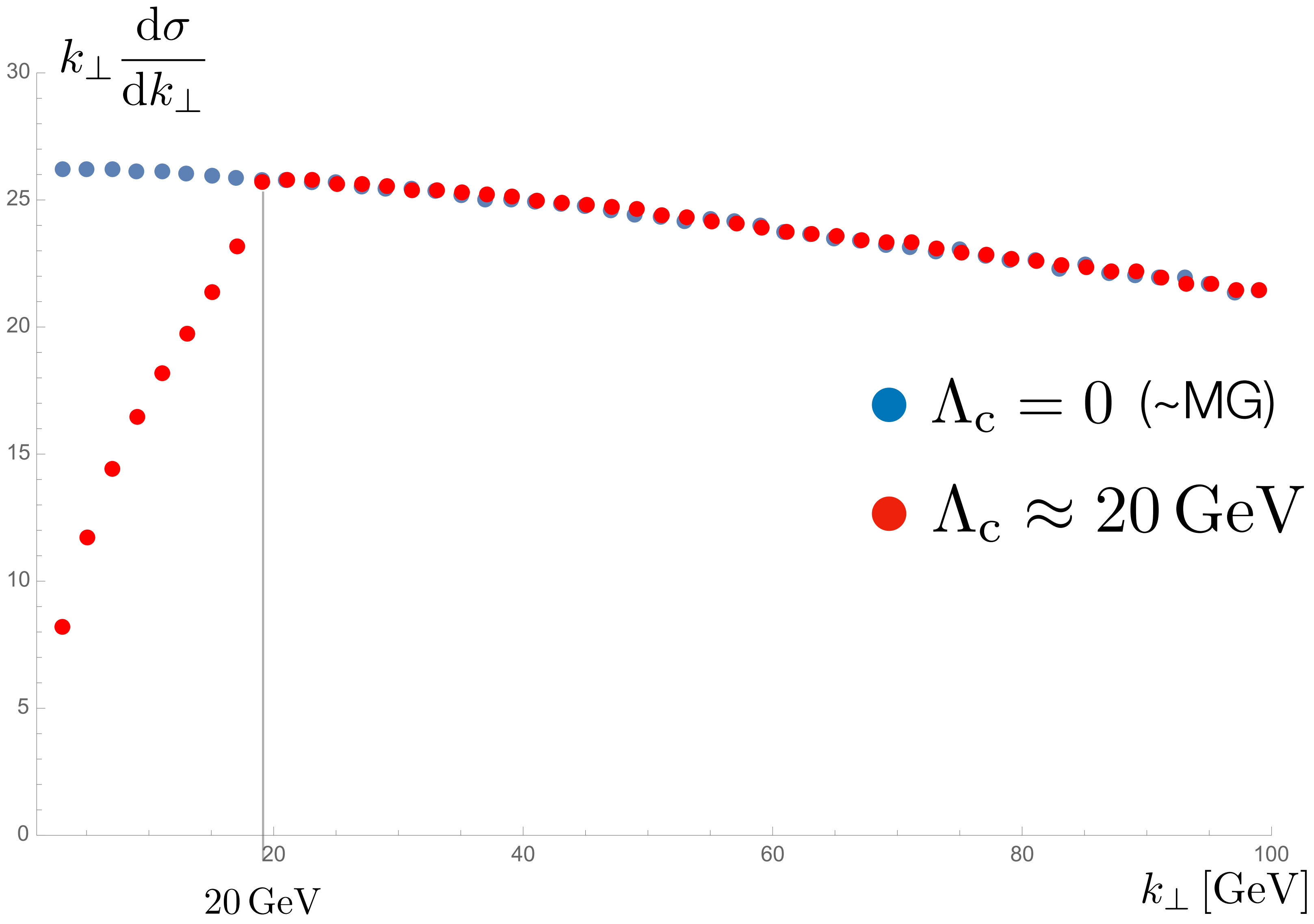
Numerical example result for this finite sum of two interference diagrams:



Always included!
This is the usual contribution.

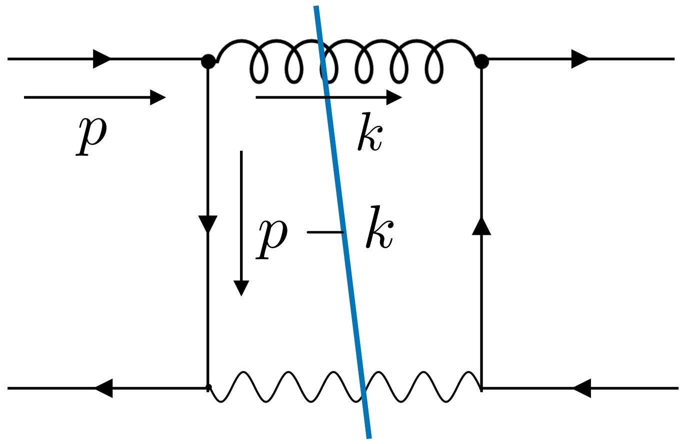


Included only when two initial-state quarks with collinearity < Λ_c

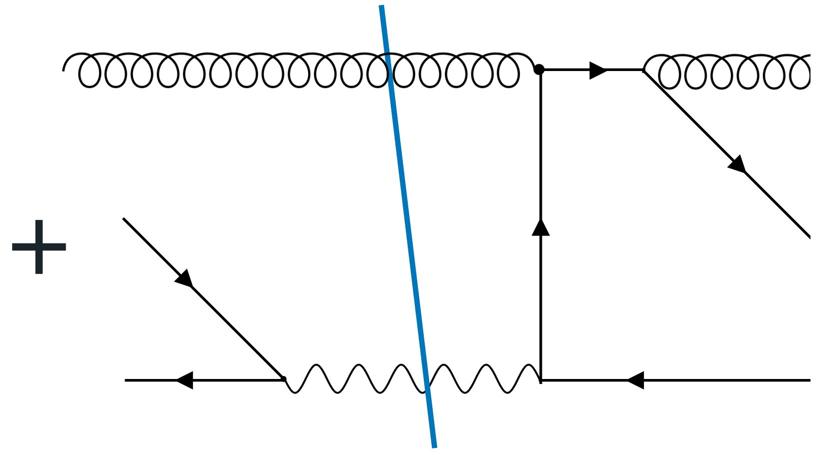


INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

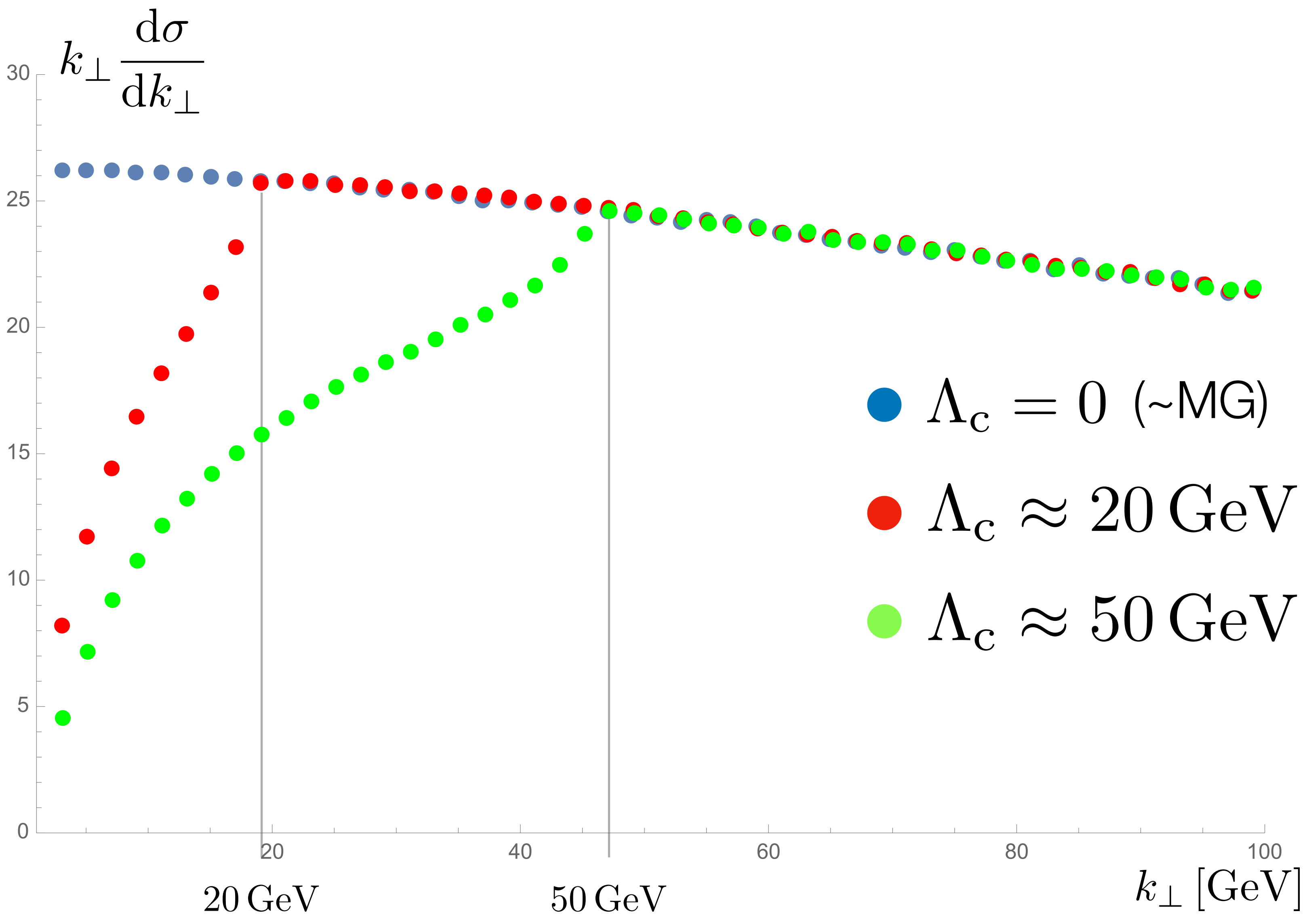
Numerical example result for this finite sum of two interference diagrams:



Always included!
This is the usual contribution.

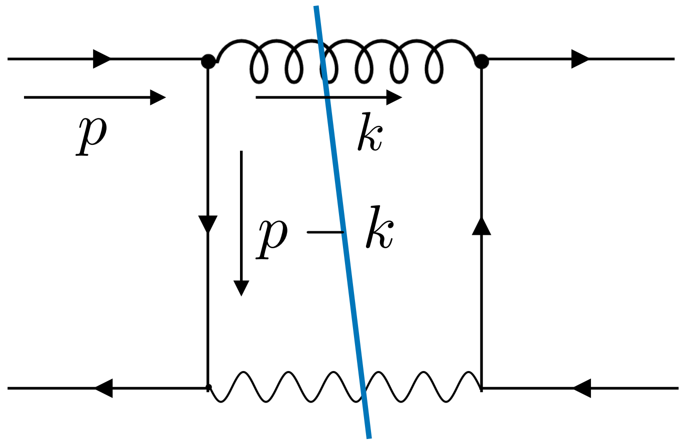


Included only when two initial-state quarks with collinearity < Λ_c

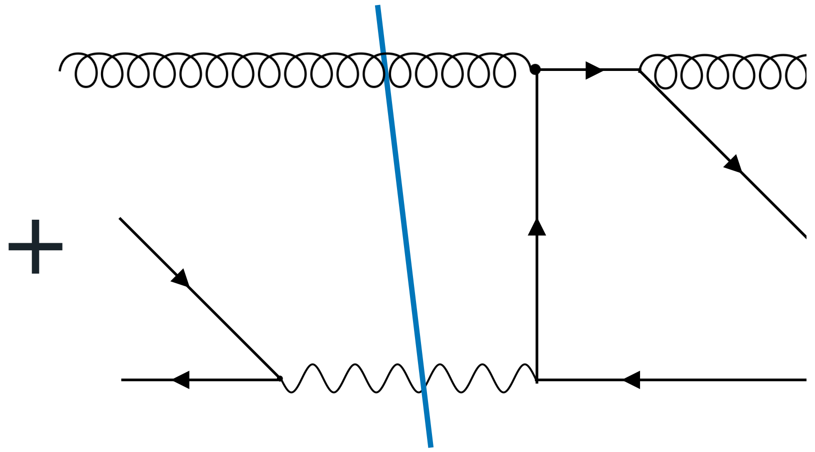


INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

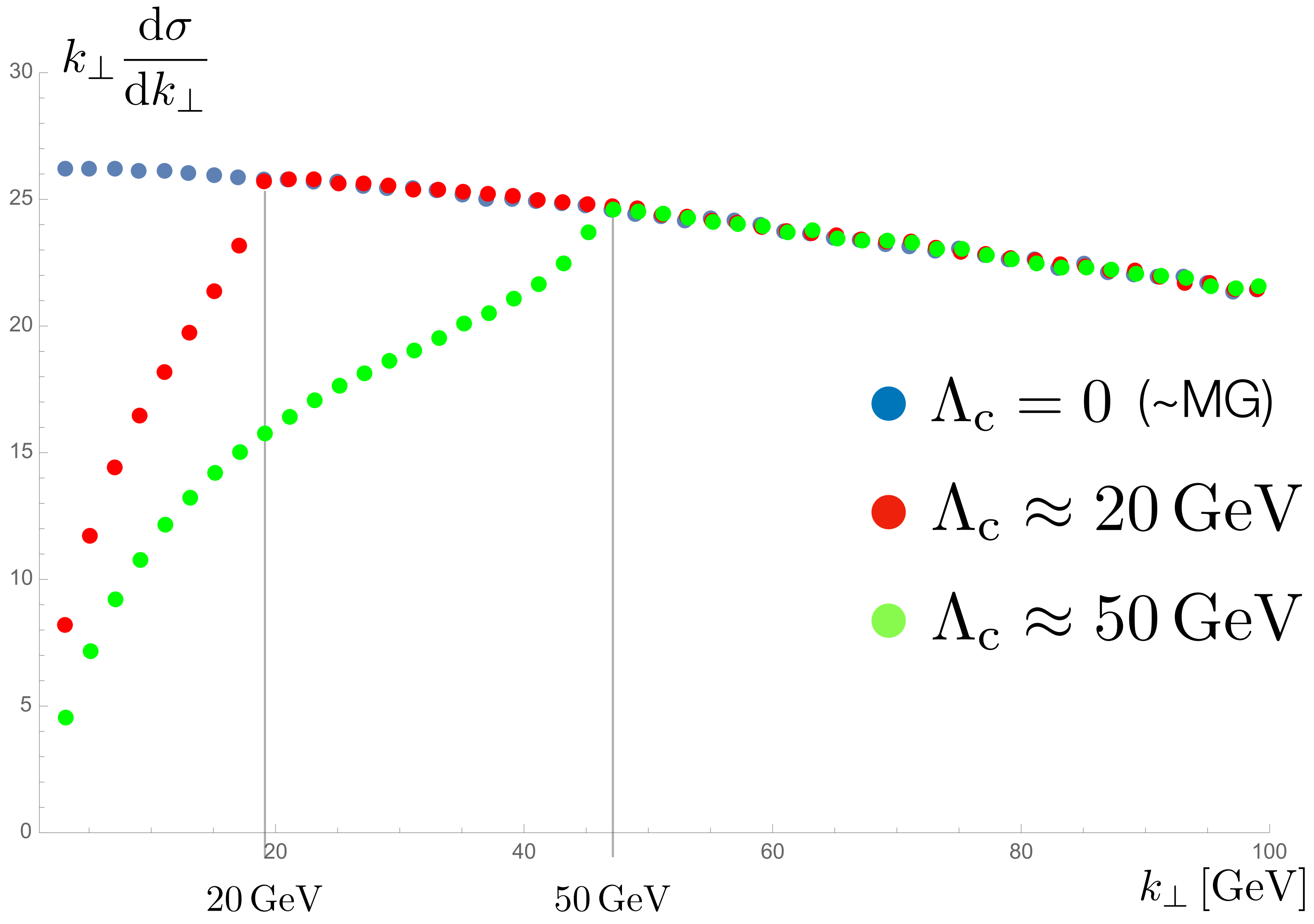
Numerical example result for this finite sum of two interference diagrams:



Always included!
This is the usual contribution.



Included only when two initial-state quarks with collinearity < Λ_c



Note that : for $\Lambda_c > 50$ GeV
the distribution does not change anymore
because highest separation of two partons
in a jet is of **order of Z mass**

INITIAL-STATE SINGULARITIES: SCALE DEPENDENCE

INITIAL-STATE SINGULARITIES: SCALE DEPENDENCE

Consider now the inclusive integral

$$I(\Lambda_\mathrm{c}) = \int_0^{100} \mathrm{d}k_\perp \frac{\mathrm{d}\sigma}{\mathrm{d}k_\perp}(\Lambda_\mathrm{c})$$

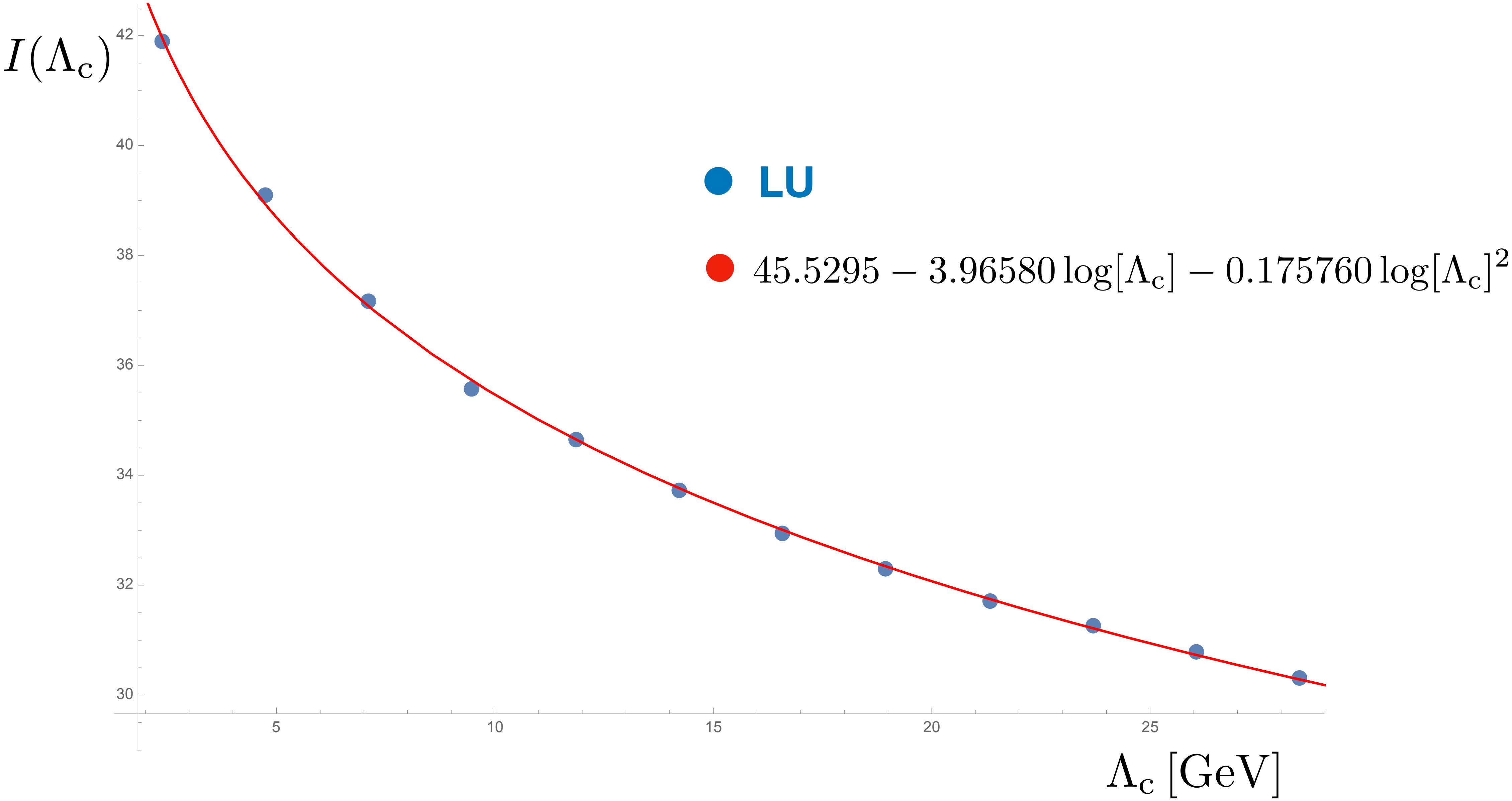
as a function of the “factorisation scale”

INITIAL-STATE SINGULARITIES: SCALE DEPENDENCE

Consider now the inclusive integral

$$I(\Lambda_c) = \int_0^{100} dk_\perp \frac{d\sigma}{dk_\perp}(\Lambda_c)$$

as a function of the “factorisation scale”



SUBTRACTION VS LOCAL UNITARITY

- One can introduce the following local (in x) **counterterm**:

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left(\left[\int_0^{1} dy \frac{-e^{-y}}{y}\right] + \left[\int_0^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

SUBTRACTION VS LOCAL UNITARITY

- One can introduce the following local (in x) **counterterm**:

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left(\left[\int_0^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_0^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

- And a **regulator** to evaluate the divergent integrals

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \lim_{\epsilon\to 0}\left(\left[\int_\epsilon^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_\epsilon^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

SUBTRACTION VS LOCAL UNITARITY

- One can introduce the following local (in x) **counterterm**:

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left(\left[\int_0^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_0^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

- And a **regulator** to evaluate the divergent integrals

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \lim_{\epsilon\to 0}\left(\left[\int_\epsilon^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_\epsilon^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

- To finally arrive at a **finite** result, differential in $x \in [0, 10]$

$$= \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \lim_{\epsilon\to 0}\left(\cancel{\log}(\epsilon) + \gamma - \mathrm{Ei}(-1) + \log(10) - \cancel{\log}(\epsilon)\right)\mathcal{J}(0)$$

SUBTRACTION VS LOCAL UNITARITY

- One can introduce the following local (in x) **counterterm**:

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left(\left[\int_0^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_0^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

- And a **regulator** to evaluate the divergent integrals

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \lim_{\epsilon\to 0}\left(\left[\int_\epsilon^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_\epsilon^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

- To finally arrive at a **finite** result, differential in $x \in [0, 10]$

$$= \int_0^{10} dx \underbrace{\left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right]}_{\text{NUMERICAL}} + \lim_{\epsilon\to 0}\left(\cancel{\log}(\epsilon) + \gamma - \mathrm{Ei}(-1) + \log(10) - \cancel{\log}(\epsilon)\right)\mathcal{J}(0)$$

SUBTRACTION VS LOCAL UNITARITY

- One can introduce the following local (in x) **counterterm**:

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left(\left[\int_0^{1} dy \frac{-e^{-y}}{y}\right] + \left[\int_0^{10} dx \frac{1}{x}\right]\right) \mathcal{J}(0)$$

- And a **regulator** to evaluate the divergent integrals

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \lim_{\epsilon \to 0} \left(\left[\int_\epsilon^{1} dy \frac{-e^{-y}}{y}\right] + \left[\int_\epsilon^{10} dx \frac{1}{x}\right]\right) \mathcal{J}(0)$$

- To finally arrive at a **finite** result, differential in $x \in [0, 10]$

$$= \int_0^{10} dx \underbrace{\left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right]}_{\text{NUMERICAL}} + \lim_{\epsilon \to 0} \underbrace{\left(\cancel{\log(\epsilon)} + \gamma - \text{Ei}(-1) + \log(10) - \cancel{\log(\epsilon)} \right)}_{\text{ANALYTIC}} \mathcal{J}(0)$$

SUBTRACTION VS LOCAL UNITARITY

- One can introduce the following local (in x) **counterterm**:

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left(\left[\int_0^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_0^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

- And a **regulator** to evaluate the divergent integrals

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \lim_{\epsilon\to 0}\left(\left[\int_\epsilon^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_\epsilon^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

- To finally arrive at a **finite** result, differential in $x \in [0, 10]$

$$= \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \lim_{\epsilon\to 0}\left(\cancel{\log}(\epsilon) + \gamma - \mathrm{Ei}(-1) + \log(10) - \cancel{\log}(\epsilon)\right)\mathcal{J}(0)$$

NUMERICAL

ANALYTIC

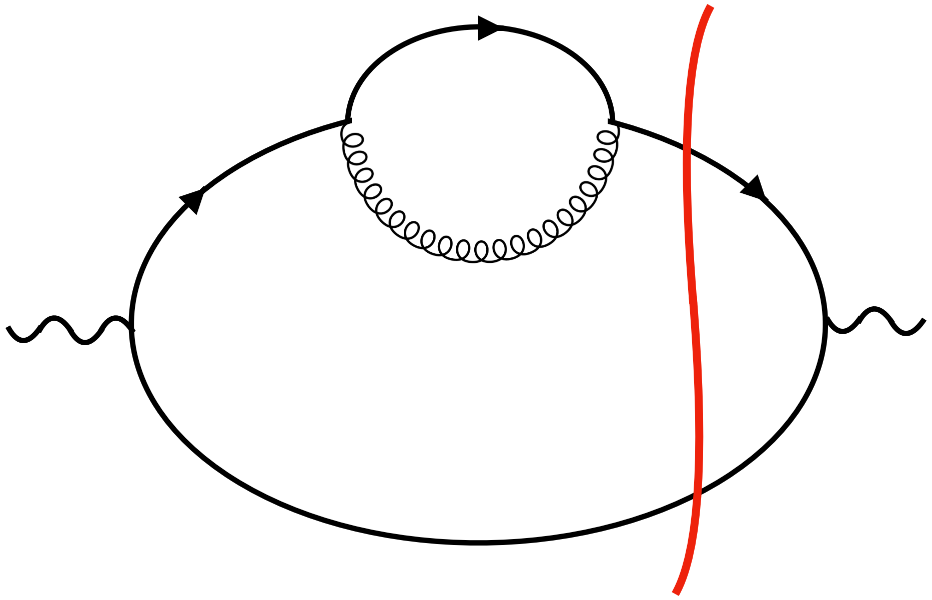
- **Local Unitarity** aligns the measure and combines "real and virtual":

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) + \frac{-e^{-x}}{x}\mathcal{J}(0)\Theta(1-x)\right]$$

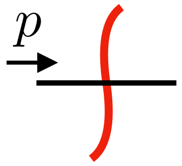
LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

In **LU**, we cannot consider ***truncated*** amplitudes only :



Traditional Cutkosky rule



$$= -2\pi \mathrm{i} \frac{\delta(p^0 - E(\vec{p}))}{2E(\vec{p})}$$

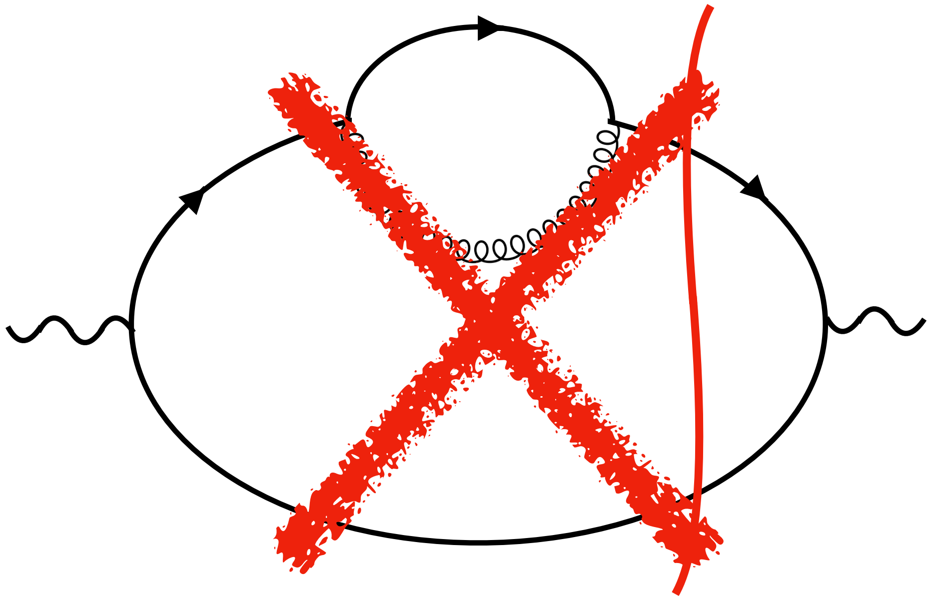
$$E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$$

would not apply here !

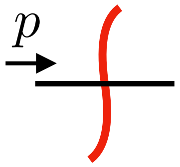
LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

In **LU**, we cannot consider ***truncated*** amplitudes only :



Traditional Cutkosky rule

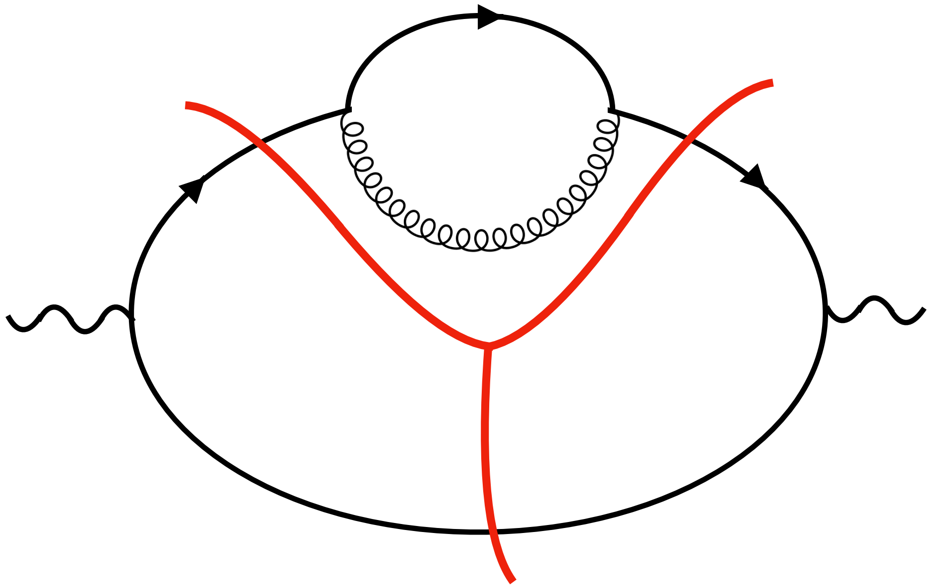


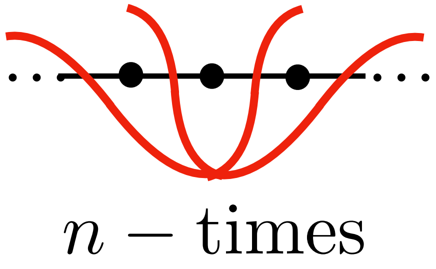
$$= -2\pi \mathrm{i} \frac{\delta(p^0 - E(\vec{p}))}{2E(\vec{p})}$$

$$E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$$

would not apply here !

So consider this Cutkosky cut as a **higher-order residue** → **Generalised cutting rule**





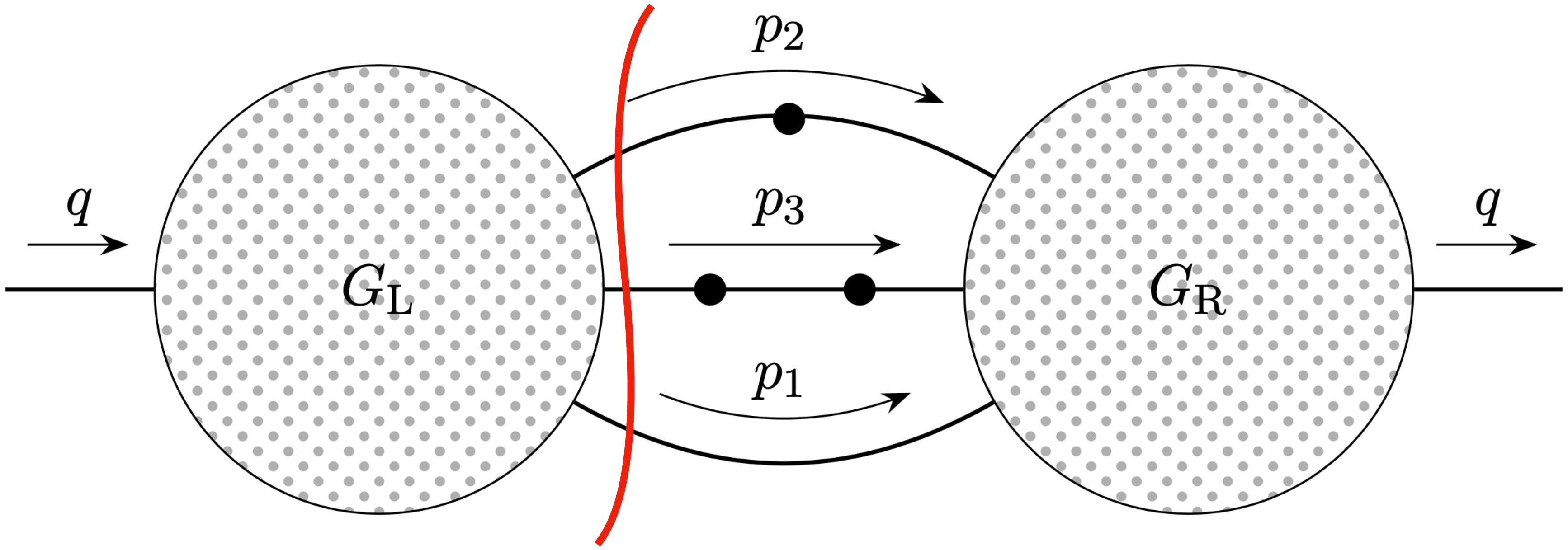
$$= -2\pi \mathrm{i} \frac{\delta^{(n)}\left[p^0 - E(\vec{p})\right]}{(p^0 + E(\vec{p}))^2}$$

$$\int \mathrm{d}x\, \delta^{(n+1)}[x] f(x) = \frac{1}{n!} \frac{\mathrm{d}^n f}{\mathrm{d}x^n}\bigg|_{x=0}$$

LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

This is well understood for **raised loop propagators**,
but for **raised external propagators** of supergraphs, there are subtleties :

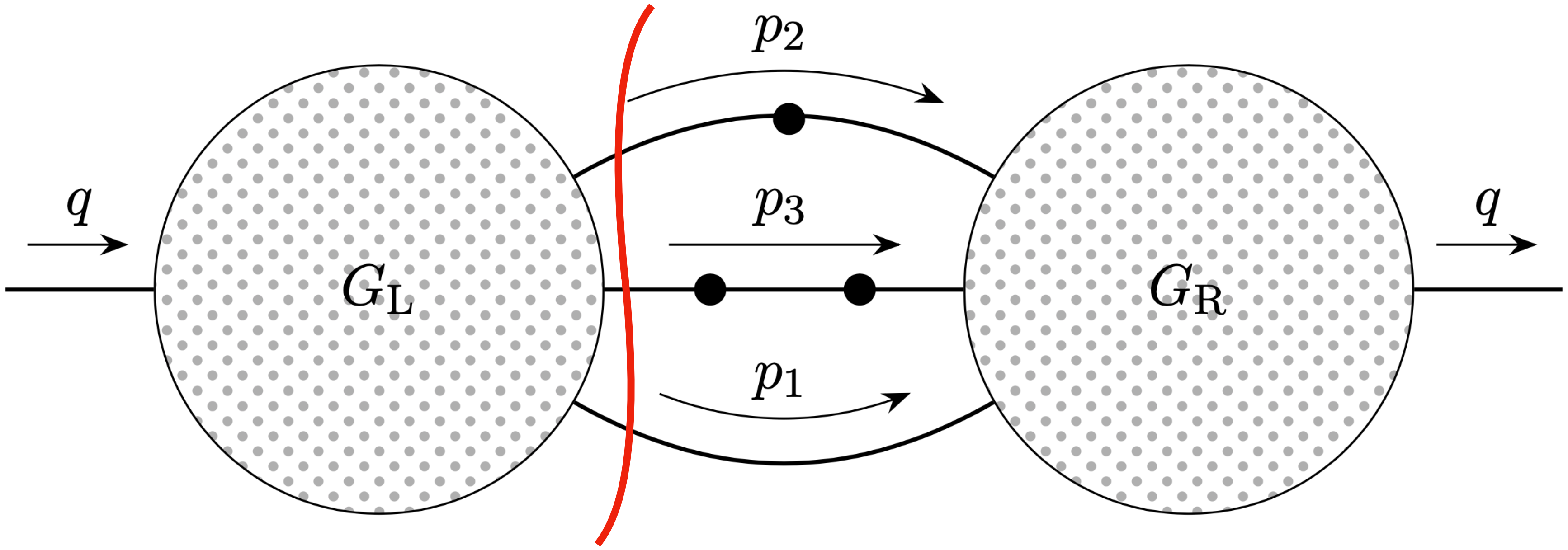


$$\propto \delta^{(1)}[p_1^0 - E(\vec{p_1}\ )]\delta^{(2)}[p_2^0 - E(\vec{p_2}\ )]\delta^{(3)}[p_3^0 - E(\vec{p_3}\ )]$$

LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

This is well understood for **raised loop propagators**,
but for **raised external propagators** of supergraphs, there are subtleties :



$$\propto \delta^{(1)}[p_1^0 - E(\vec{p_1})]\delta^{(2)}[p_2^0 - E(\vec{p_2})]\delta^{(3)}[p_3^0 - E(\vec{p_3})]$$

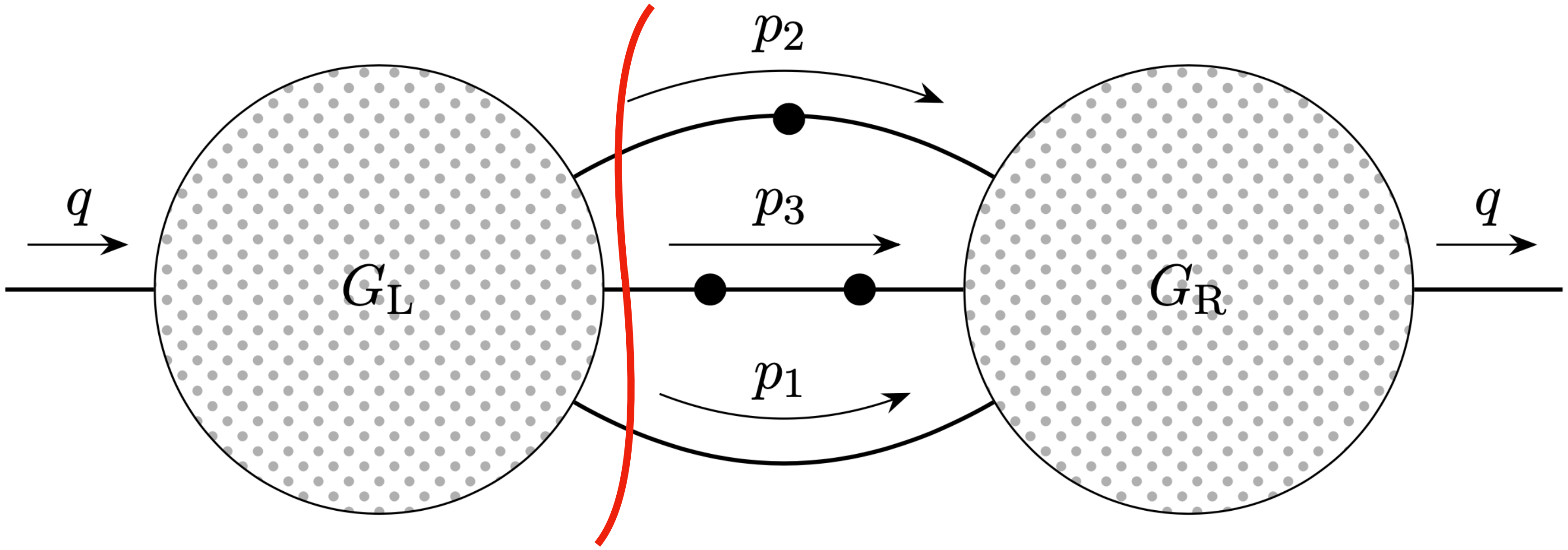
Derivatives can act on each other because: $p_3 = q - p_1 - p_2$

$$\propto \delta^{(1)}[p_1^0 - E(\vec{p_1})]\delta^{(2)}[p_2^0 - E(\vec{p_2})]\delta^{(3)}[q^0 - p_1^0 - p_2^0 - E(t\,\vec{p_1}) - E(t\,\vec{p_2})]$$

LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

This is well understood for **raised loop propagators**,
but for **raised external propagators** of supergraphs, there are subtleties :



$$\propto \delta^{(1)}[p_1^0 - E(\vec{p_1}\,)]\delta^{(2)}[p_2^0 - E(\vec{p_2}\,)]\delta^{(3)}[p_3^0 - E(\vec{p_3}\,)]$$

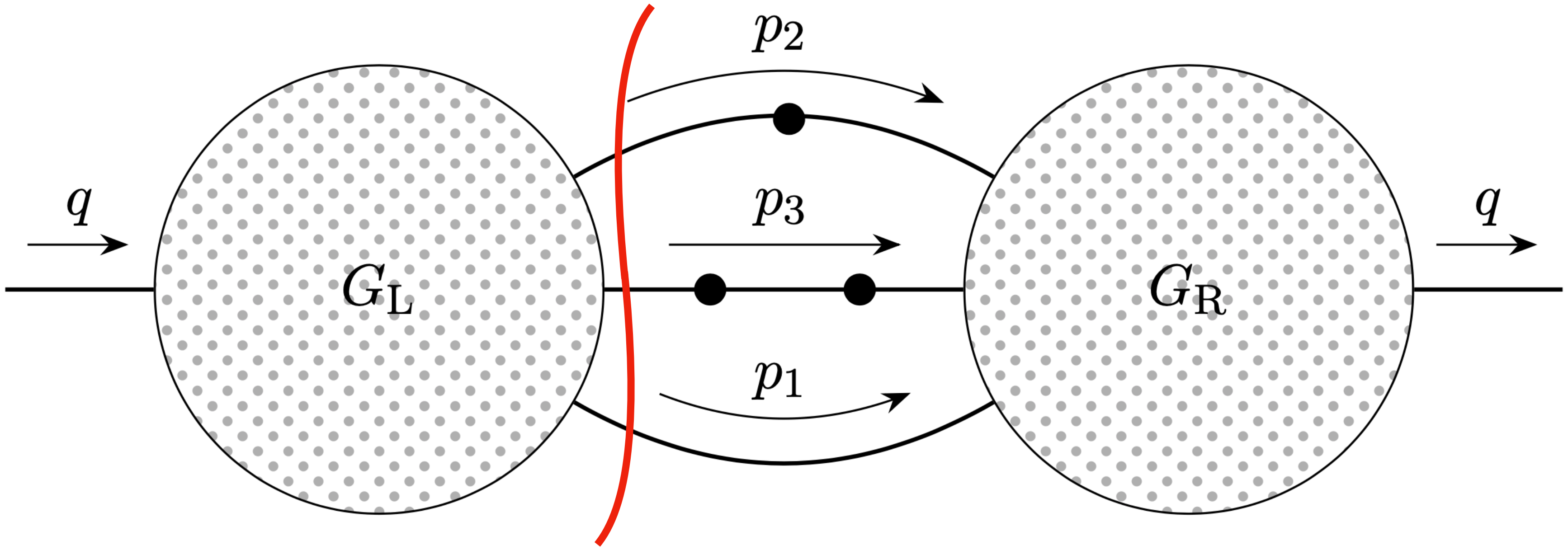
Derivatives can act on each other because: $p_3 = q - p_1 - p_2$

$$\propto \delta^{(1)}[\underline{p_1^0} - E(\vec{p_1}\,)]\delta^{(2)}[p_2^0 - E(\vec{p_2}\,)]\delta^{(3)}[q^0 - \underline{p_1^0} - p_2^0 - E(t\,\vec{p_1}\,) - E(t\,\vec{p_2}\,)]$$

LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

This is well understood for **raised loop propagators**,
but for **raised external propagators** of supergraphs, there are subtleties :



$$\propto \delta^{(1)}[p_1^0 - E(\vec{p_1}\,)]\delta^{(2)}[p_2^0 - E(\vec{p_2}\,)]\delta^{(3)}[p_3^0 - E(\vec{p_3}\,)]$$

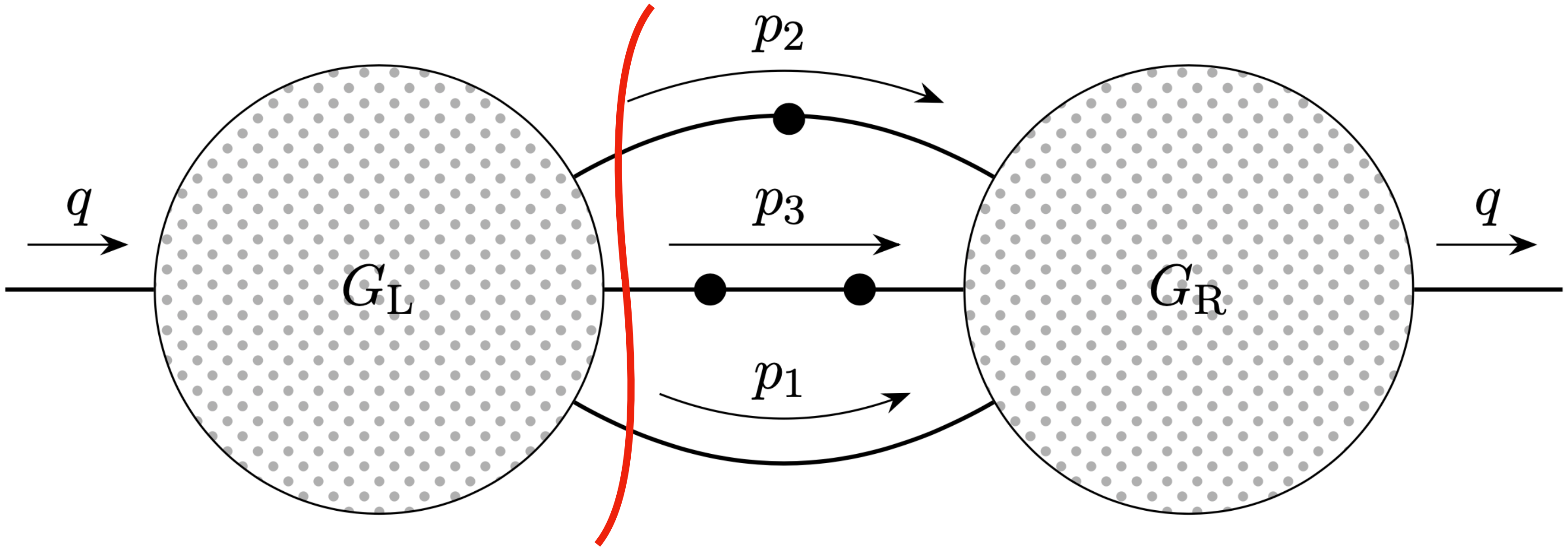
Derivatives can act on each other because: $p_3 = q - p_1 - p_2$

$$\propto \delta^{(1)}[\underline{p_1^0} - E(\vec{p_1}\,)]\delta^{(2)}[\underline{p_2^0} - E(\vec{p_2}\,)]\delta^{(3)}[q^0 - \underline{p_1^0} - \underline{p_2^0} - E(t\,\vec{p_1}\,) - E(t\,\vec{p_2}\,)]$$

LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

This is well understood for **raised loop propagators**,
but for **raised external propagators** of supergraphs, there are subtleties :



$$\propto \delta^{(1)}[p_1^0 - E(\vec{p_1}\ )]\delta^{(2)}[p_2^0 - E(\vec{p_2}\ )]\delta^{(3)}[p_3^0 - E(\vec{p_3}\ )]$$

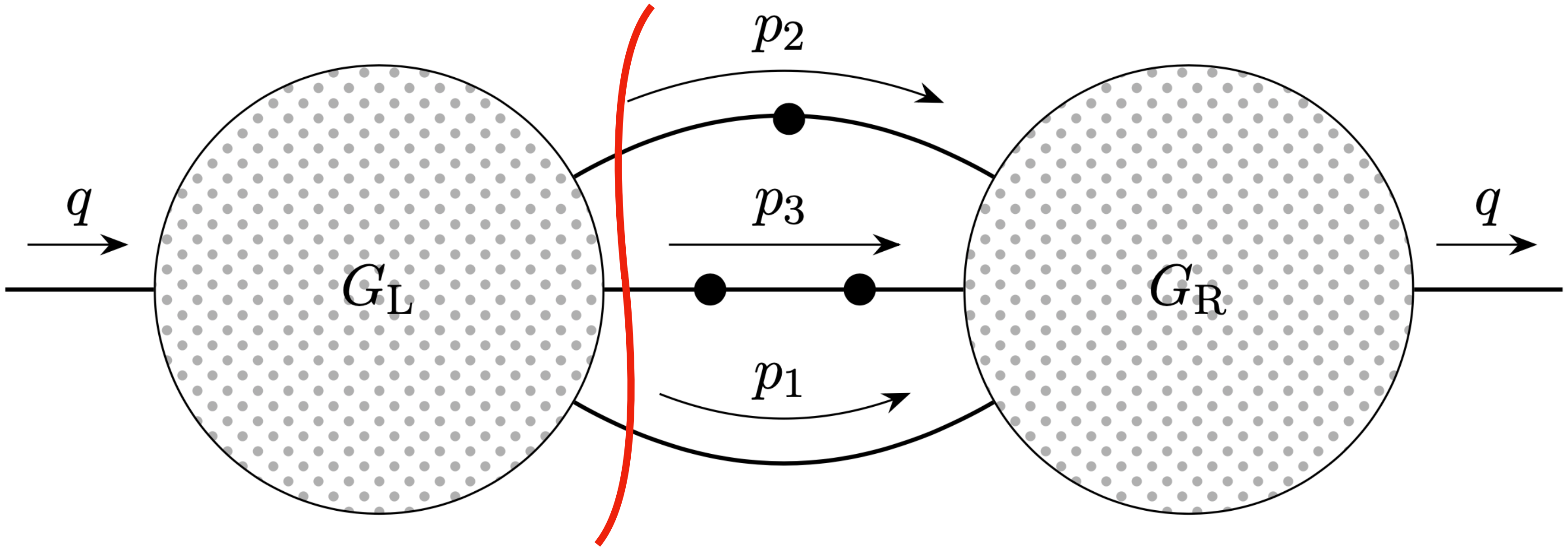
Derivatives can act on each other because: $p_3 = q - p_1 - p_2$

$$\propto \delta^{(1)}[\underline{p_1^0} - E(\vec{p_1}\ )]\delta^{(2)}[\underline{p_2^0} - E(\vec{p_2}\ )]\delta^{(3)}[q^0 - \underline{p_1^0} - \underline{p_2^0} - E(\underline{t}\,\vec{p_1}\ ) - E(\underline{t}\,\vec{p_2}\ )]$$

LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

This is well understood for **raised loop propagators**,
but for **raised external propagators** of supergraphs, there are subtleties :



$$\propto \delta^{(1)}[p_1^0 - E(\vec{p_1}\,)]\delta^{(2)}[p_2^0 - E(\vec{p_2}\,)]\delta^{(3)}[p_3^0 - E(\vec{p_3}\,)]$$

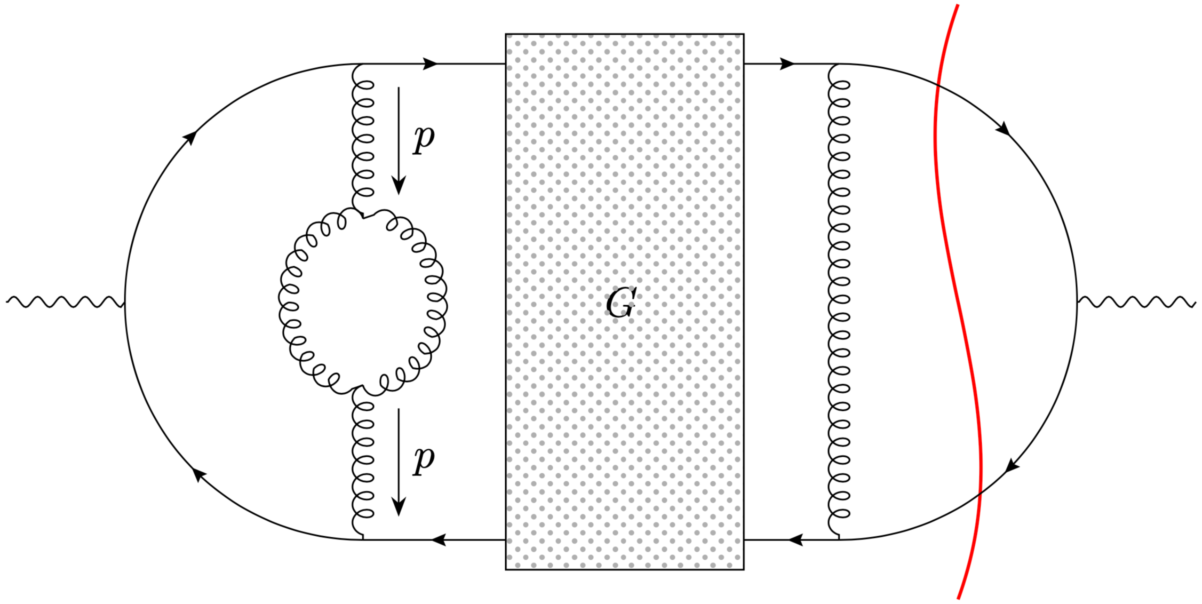
Derivatives can act on each other because: $p_3 = q - p_1 - p_2$

$$\propto \delta^{(1)}[\underline{p_1^0} - E(\vec{p_1}\,)]\delta^{(2)}[\underline{p_2^0} - E(\vec{p_2}\,)]\delta^{(3)}[q^0 - \underline{p_1^0} - \underline{p_2^0} - E(\underline{t}\,\vec{p_1}\,) - E(\underline{t}\,\vec{p_2}\,)]$$

Use **multivariate dual numbers** (auto-differentiation) in order to **efficiently compute amplitude derivatives** of G_L and G_R in p_2^0 and t (in this example)

SPURIOUS SOFT SINGULARITIES

[Capatti, VH, Ruijl, arxiv : 2203.11038]



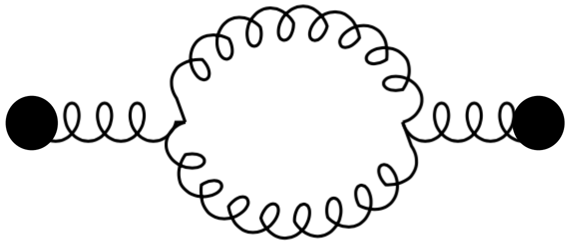
$\propto \frac{1}{(p^2)^2}$

For $p = 0$

this induces a spurious soft divergence whose cancellation has nothing to do with KLN!

Only beyond NLO (needs a soft propagator dressed with a self-energy correction)

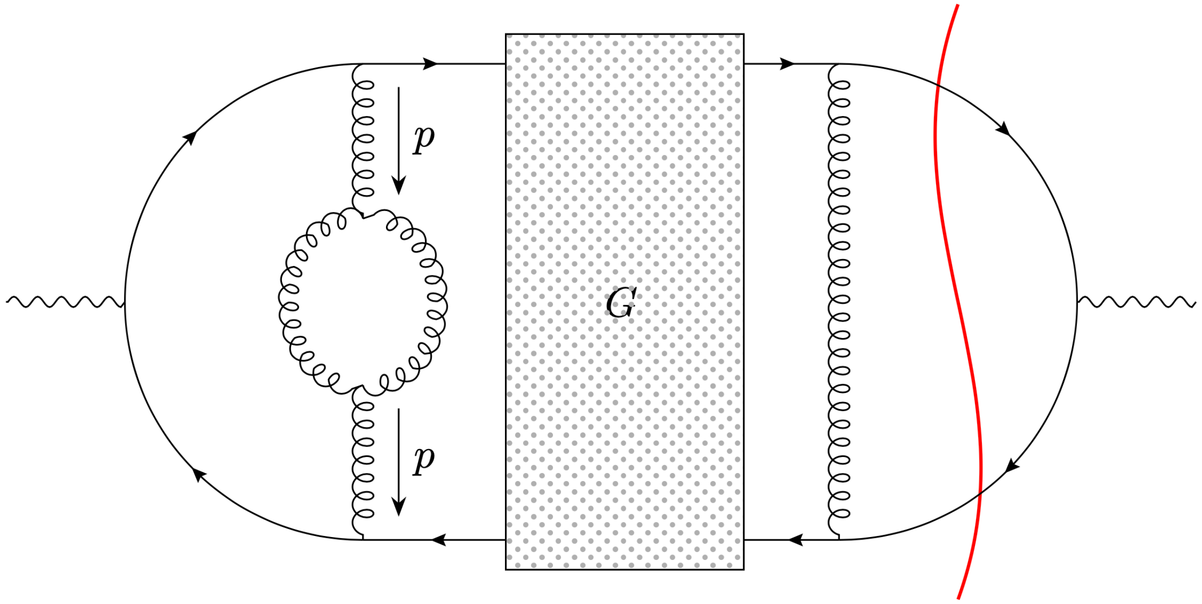
At the integrated level, we have



$$\propto \frac{1}{p^2}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\frac{1}{p^2}$$

SPURIOUS SOFT SINGULARITIES

[Capatti, VH, Ruijl, arxiv : 2203.11038]

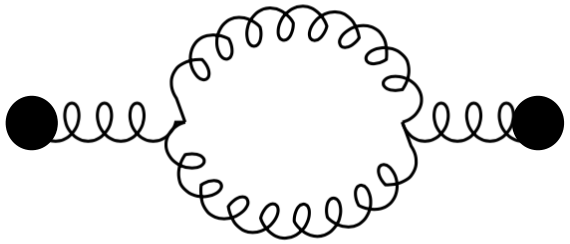


$\propto \frac{1}{(p^2)^2}$

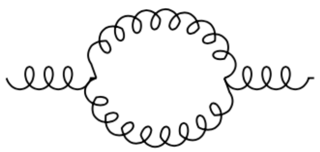
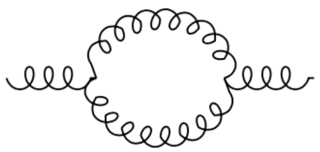
For $p = 0$

this induces a spurious soft divergence whose cancellation has nothing to do with KLN!

Only beyond NLO (needs a soft propagator dressed with a self-energy correction)

At the integrated level, we have  $\propto \frac{1}{p^2}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\frac{1}{p^2}$

but not at the local level; we must introduce **spurious soft counterterms** :

 $-\tilde{T}_1\Bigg($  $\Bigg)$, $\tilde{T}_{\text{soft\_dod}}(\gamma) = \sum_{j=0}^{\text{soft\_dod}(\gamma)} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\gamma(\lambda\, p)\Big|_{\lambda=0}$, $[\tilde{T}] = 0$

COMBINED UV AND SPURIOUS IR FOREST

Eurekâ moment:

- Remarkably, we always have : $\mathrm{soft\_dod} = \mathrm{UV\_dod} - 1$
- Spurious soft expansion also valid as UV counter term.
- Spurious soft IR forest similar to the one produced by the R-operation

so that we can combine the UV and spurious soft subtraction as one !

$$\boxed{\hat{T}_{\mathrm{dod}} = T_{\mathrm{dod}} + \tilde{T}_{\mathrm{dod}-1} - T_{\mathrm{dod}}\tilde{T}_{\mathrm{dod}-1}}$$

until we realised that we had just re-invented the wheel: [J. H. Lowenstein, 1976]

COMBINED UV AND SPURIOUS IR FOREST

Eurekâ moment:

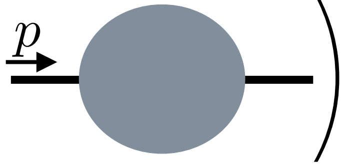
- Remarkably, we always have : $\text{soft\_dod} = \text{UV\_dod} - 1$
- Spurious soft expansion also valid as UV counter term.
- Spurious soft IR forest similar to the one produced by the R-operation

so that we can combine the UV and spurious soft subtraction as one !

$$\boxed{\hat{T}_{\text{dod}} = T_{\text{dod}} + \tilde{T}_{\text{dod}-1} - T_{\text{dod}}\tilde{T}_{\text{dod}-1}}$$

until we realised that we had just re-invented the wheel: [J. H. Lowenstein, 1976]

Novelty though: **automatic renormalisation** of fermion **masses in the OS scheme:**



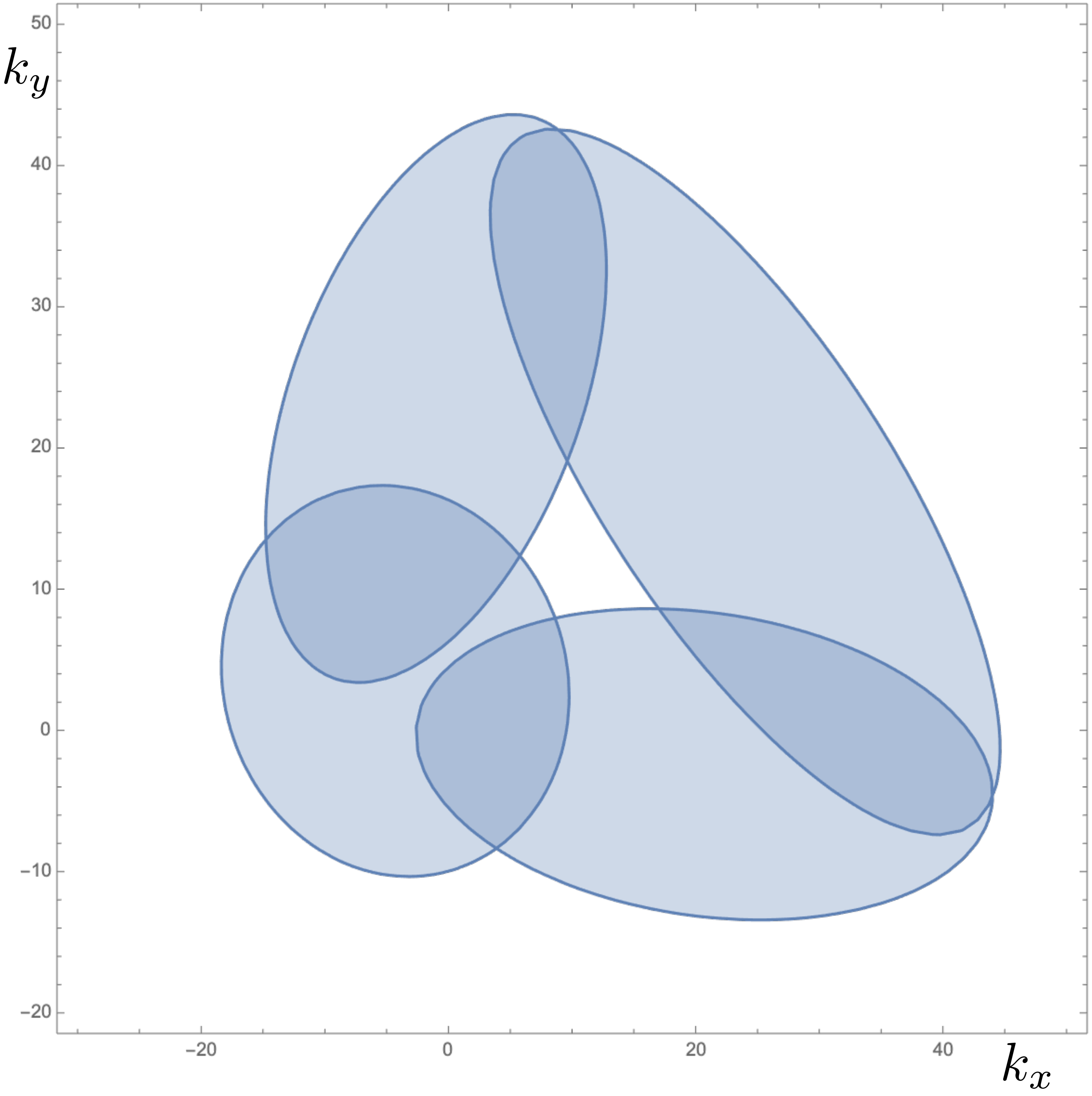
$$T^{\text{os}\pm}\left(\Sigma = \;\;\right) = (1 \pm \gamma^0)\Sigma(p = \pm p^{\text{os}})\,, \quad p^{\text{os}} = (m,0,0,0)$$

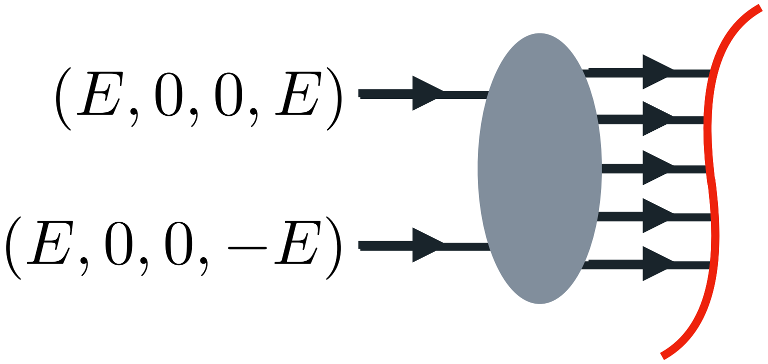
$$\frac{1}{2}\left(\left[T^{\text{os}+}(\Sigma)\right] + \left[T^{\text{os}-}(\Sigma)\right]\right) = \delta m^{\text{os}}$$

Implying that our local UV counterterm T^{os} automatically generates the OS mass renormalisation counterterm !

LOCALITY UNITARITY: CAUSAL FLOW

The **rescaling** change of variables is however **not general** : (**but always sufficient in practice**!)



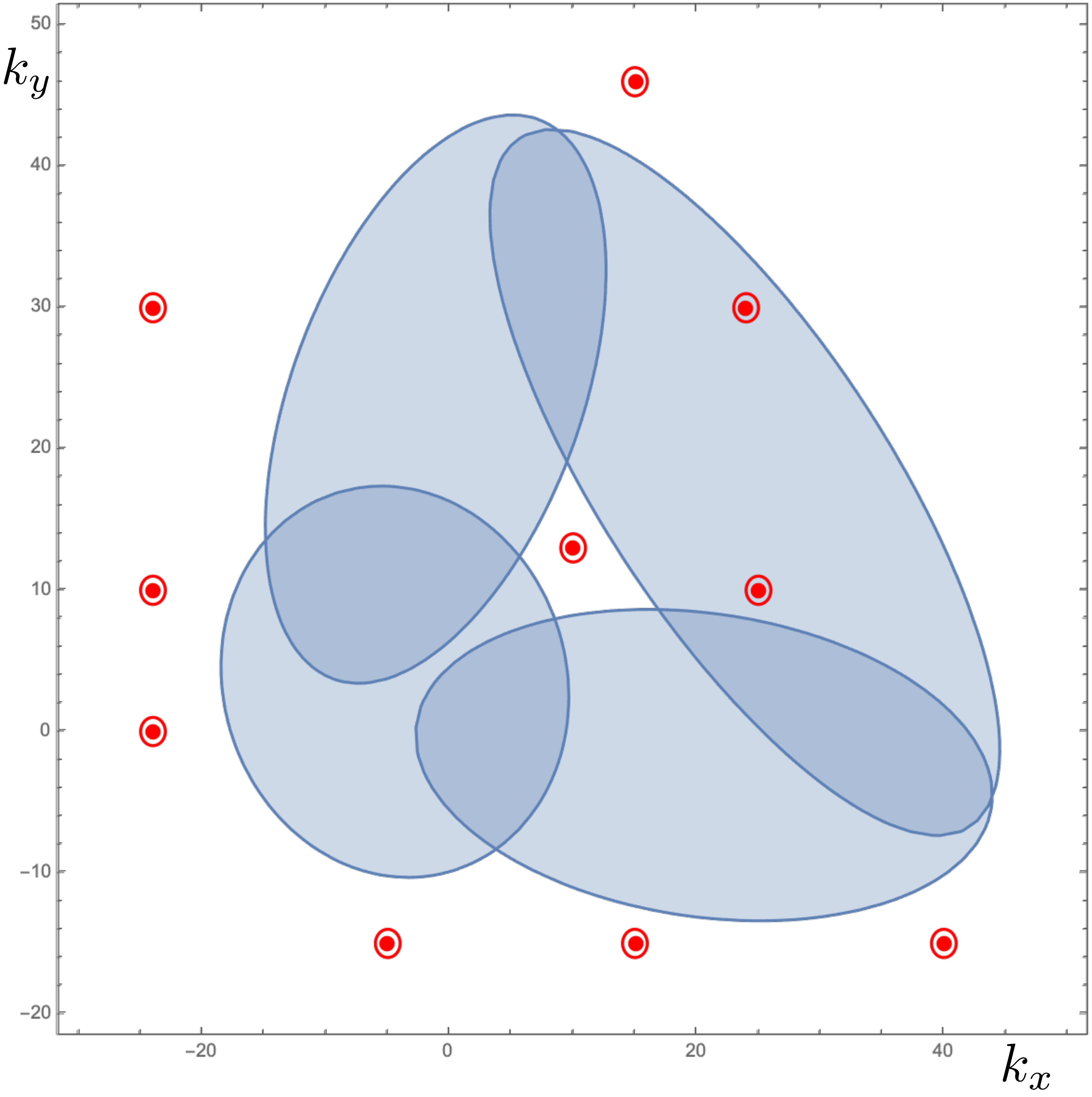


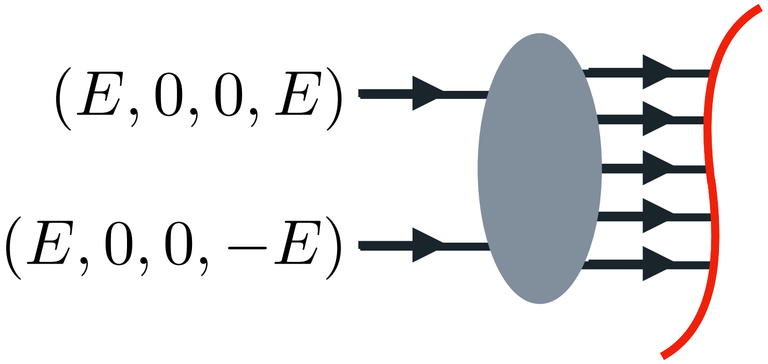
Ex: **Box_4E** from sect. 3.1 of

[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

LOCALITY UNITARITY: CAUSAL FLOW

The **rescaling** change of variables is however **not general** : (**but always sufficient in practice**!)



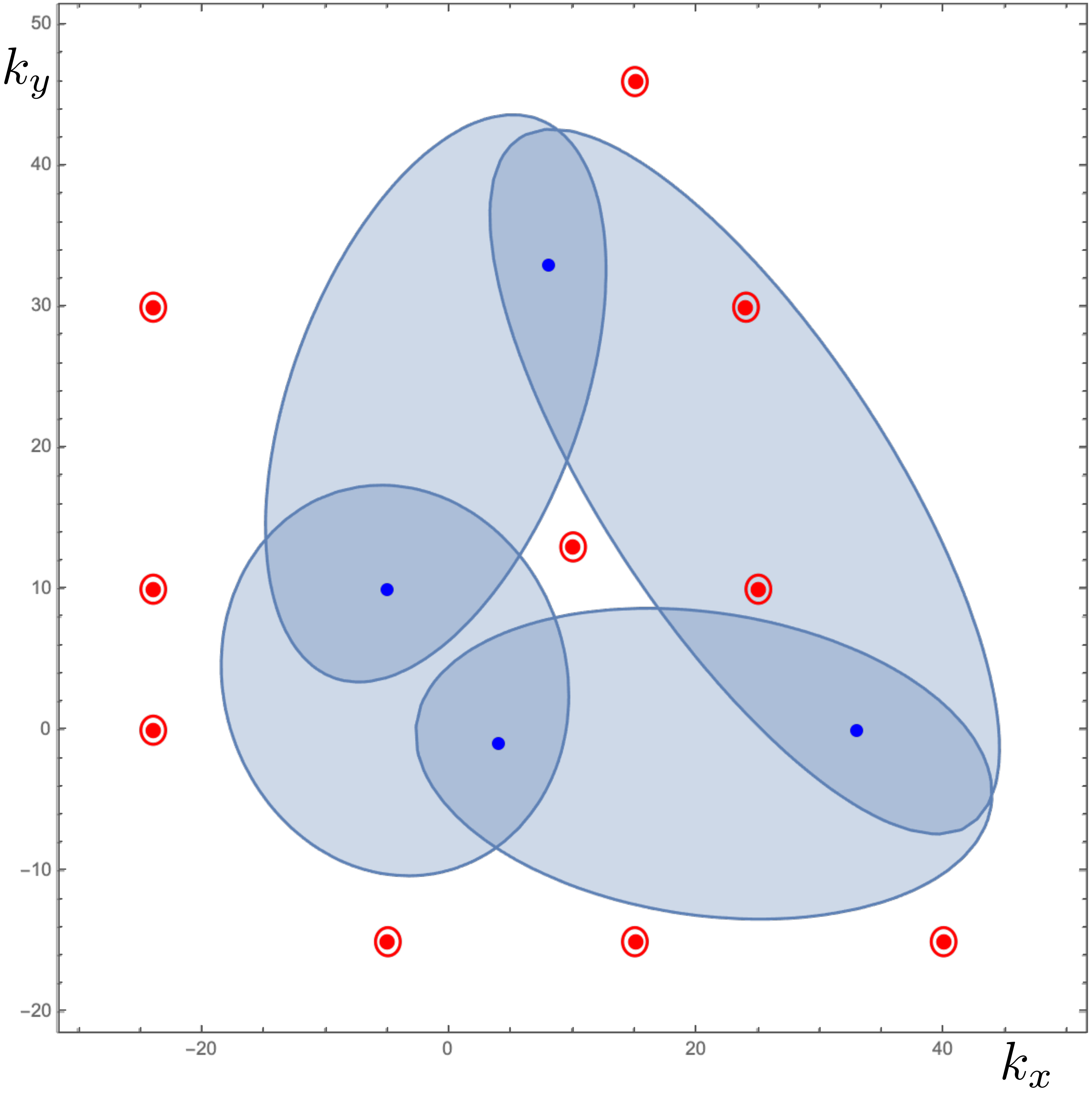


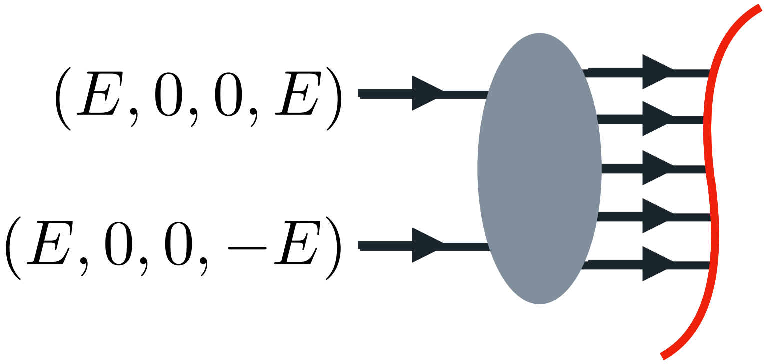
Ex: Box_4E from sect. 3.1 of

[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

LOCALITY UNITARITY: CAUSAL FLOW

The **rescaling** change of variables is however **not general** : (**but always sufficient in practice**!)



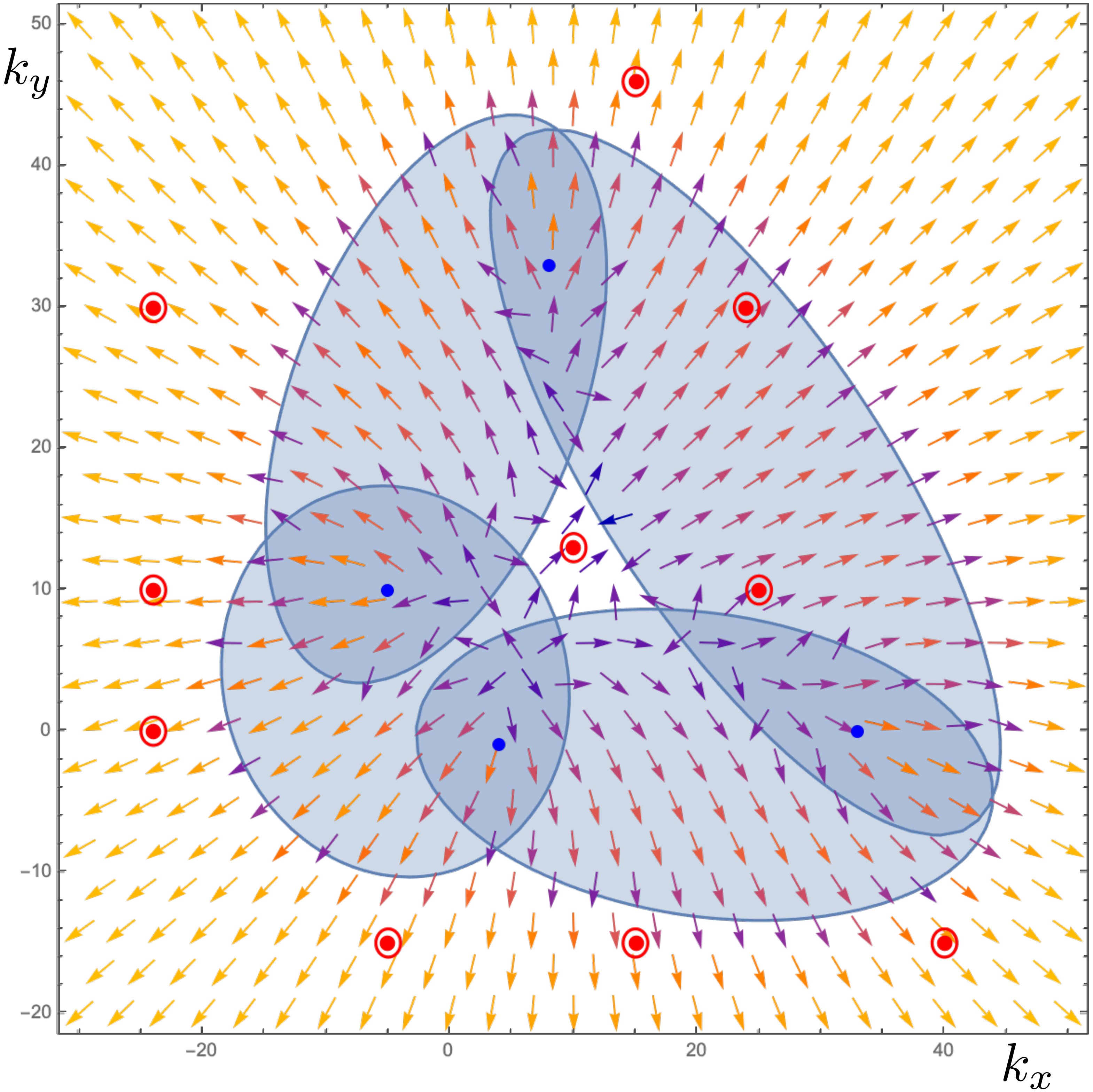


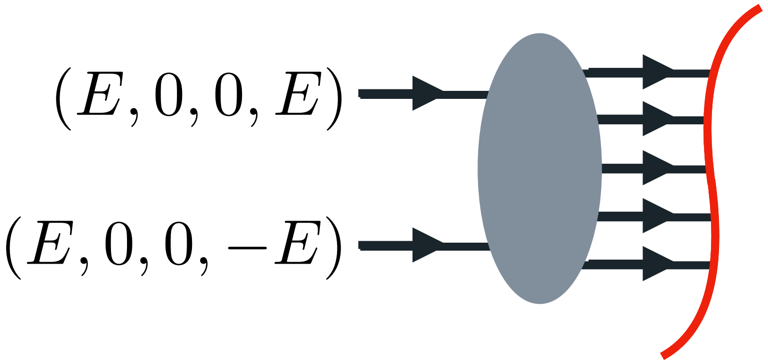
Ex: **Box_4E** from sect. 3.1 of

[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

LOCALITY UNITARITY: CAUSAL FLOW

The **rescaling** change of variables is however **not general** : (**but always sufficient in practice**!)



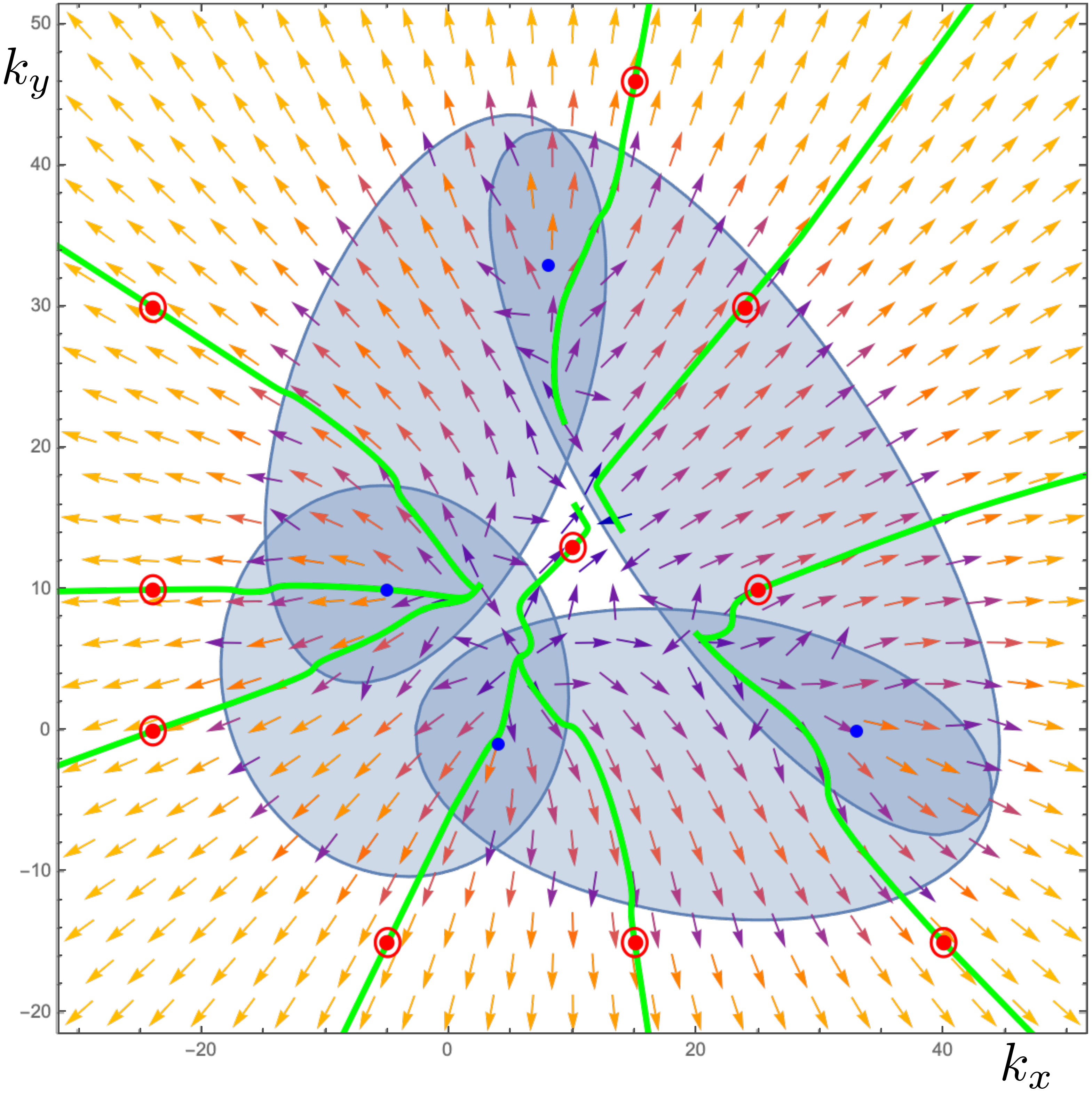


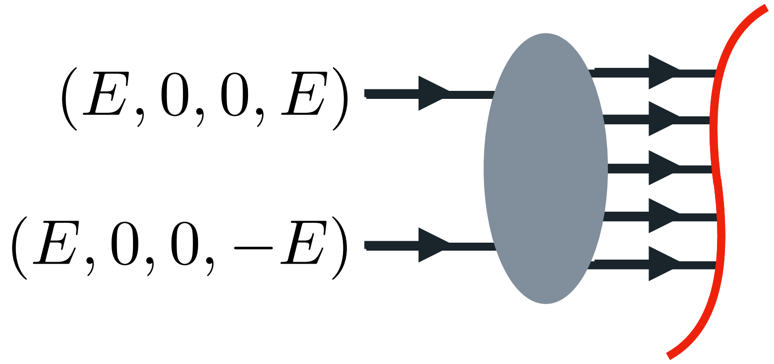
Ex: **Box_4E** from sect. 3.1 of

[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

LOCALITY UNITARITY: CAUSAL FLOW

The **rescaling** change of variables is however **not general** : (**<u>but always sufficient in practice</u>**!)





Ex: **Box_4E** from sect. 3.1 of

[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

Compute a **causal flow** $\vec{\phi}$ from our existing construction of a **deformation field** $\vec{\kappa}$:

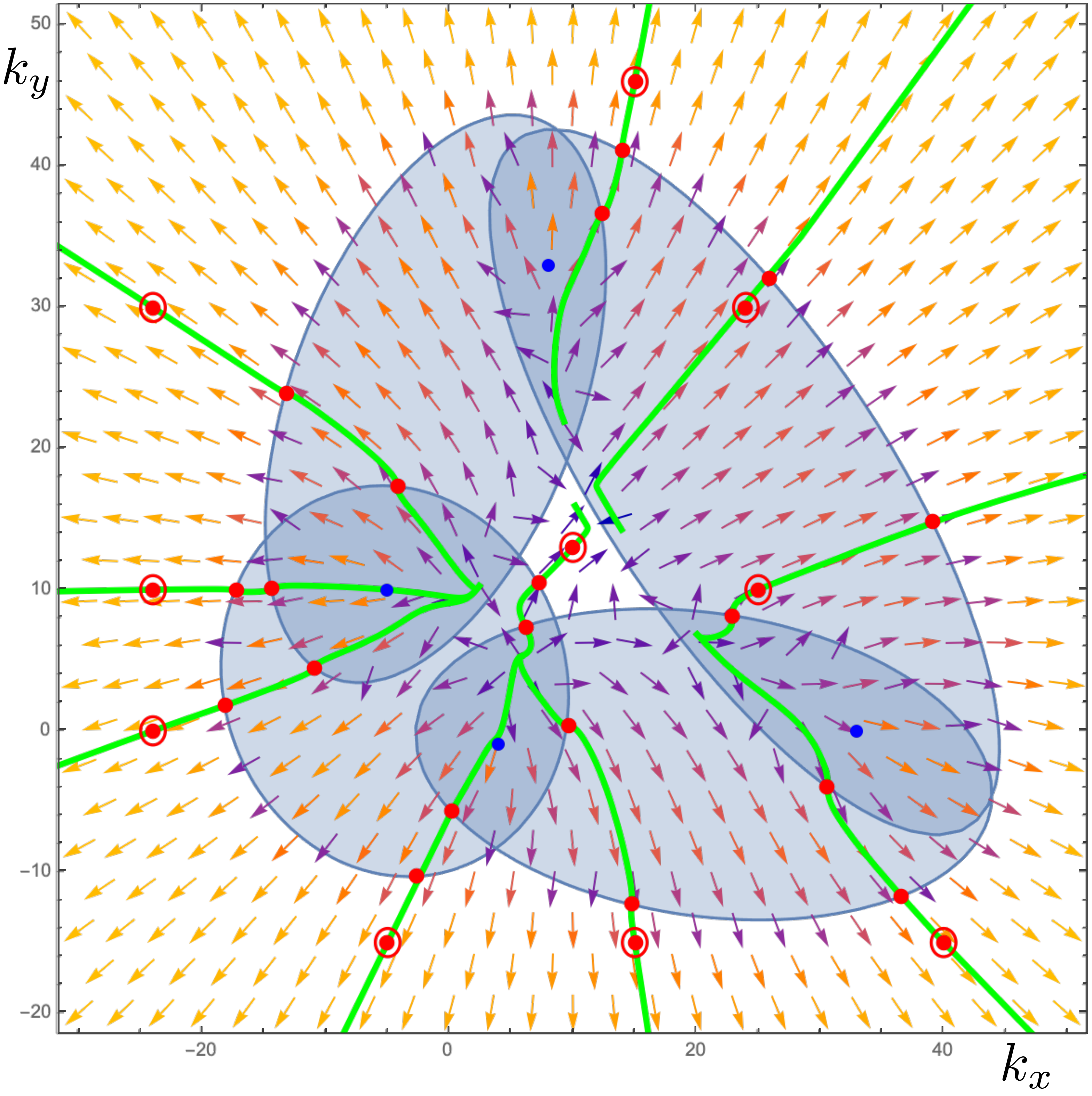
$$\partial_t \vec{\phi}(t,\vec{k}) = \vec{\kappa}(\vec{\phi}(t,\vec{k}))$$

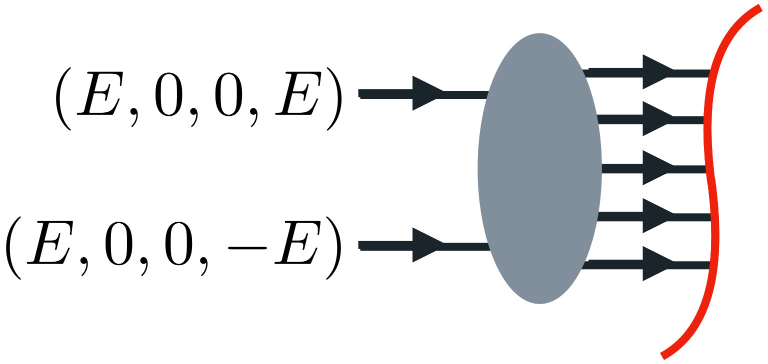
$$\vec{\phi}(0,\vec{k}) = \vec{k}$$

In general, this **ODE** can be **solved numerically**.

LOCALITY UNITARITY: CAUSAL FLOW

The **rescaling** change of variables is however **not general** : (**<u>but always sufficient in practice</u>**!)





Ex: **Box_4E** from sect. 3.1 of

[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

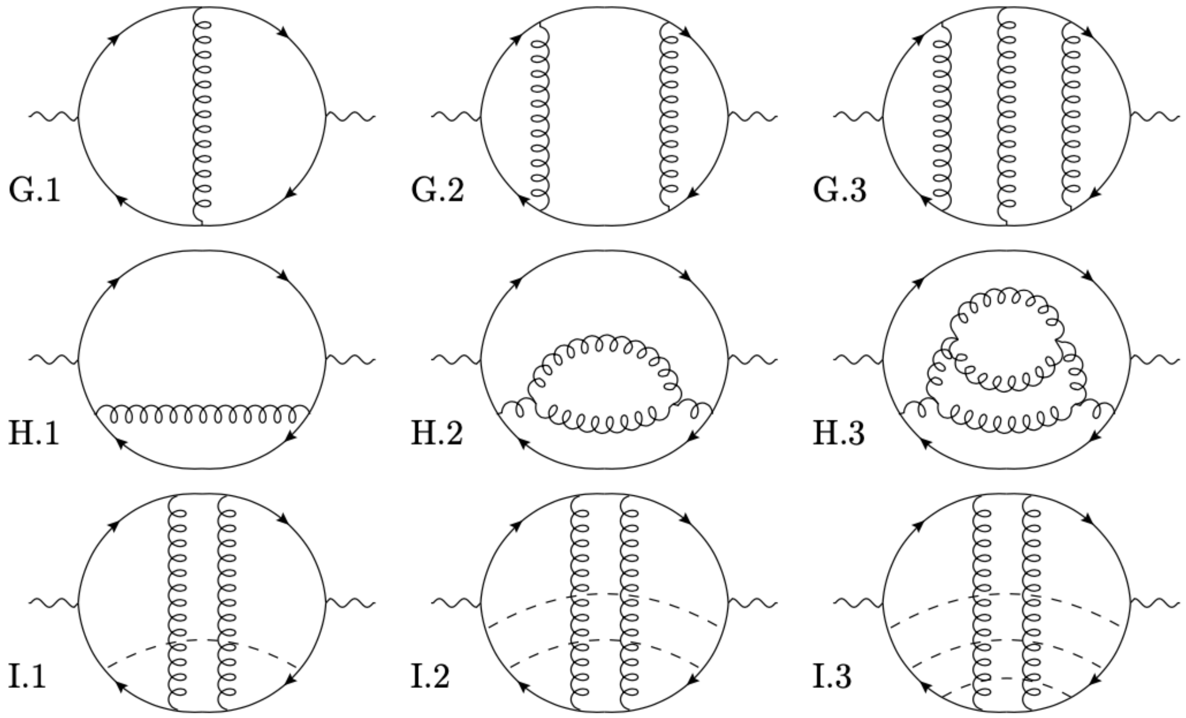
Compute a **causal flow** $\vec{\phi}$ from our existing construction of a **deformation field** $\vec{\kappa}$:

$$\partial_t \vec{\phi}(t,\vec{k}) = \vec{\kappa}(\vec{\phi}(t,\vec{k}))$$

$$\vec{\phi}(0,\vec{k}) = \vec{k}$$

In general, this **ODE** can be **solved numerically**.

IMPLEMENTATION RUN-TIME PERFORMANCE



SG	proc.	order	t_{gen} [s]	M_{disk} [MB]	N_{sg} [-]	N_{cuts} [-]	t_{eval} [ms]	$t_{\text{eval}}^{(\text{f128})}$ [ms]
G.1	$1 \to 2$	NLO	0.1	0.13	2	4	0.004	0.13
G.2	$1 \to 2$	NNLO	4.7	3.0	17	9	0.04	2.1
G.3	$1 \to 2$	N3LO	36K	509	220	16	17.6	281
H.1	$1 \to 2$	NLO	0.07	0.12	2	2	0.006	0.14
H.2	$1 \to 2$	NNLO	1.5	1.3	17	3	0.056	1.9
H.3	$1 \to 2$	N3LO	255	43	220	4	2.35	56
I.1	$1 \to 3$	NNLO	126	22	266	9	0.32	12.4
I.2	$1 \to 4$	NNLO	1.9K	120	4492	9	4.4	67
I.3	$1 \to 5$	NNLO	36K	20K	$\mathcal{O}$(100K)	9	3.6K	17.3K

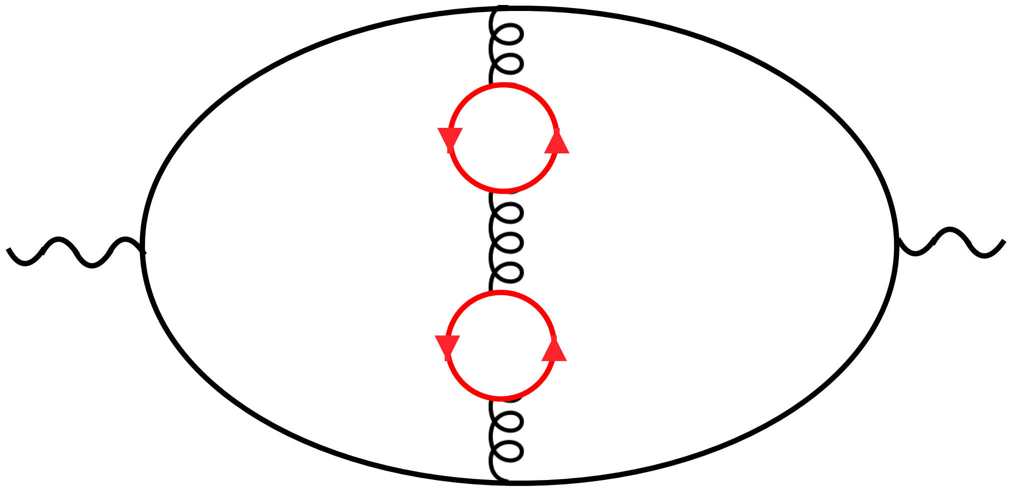
NB: these are **integrand** performance.

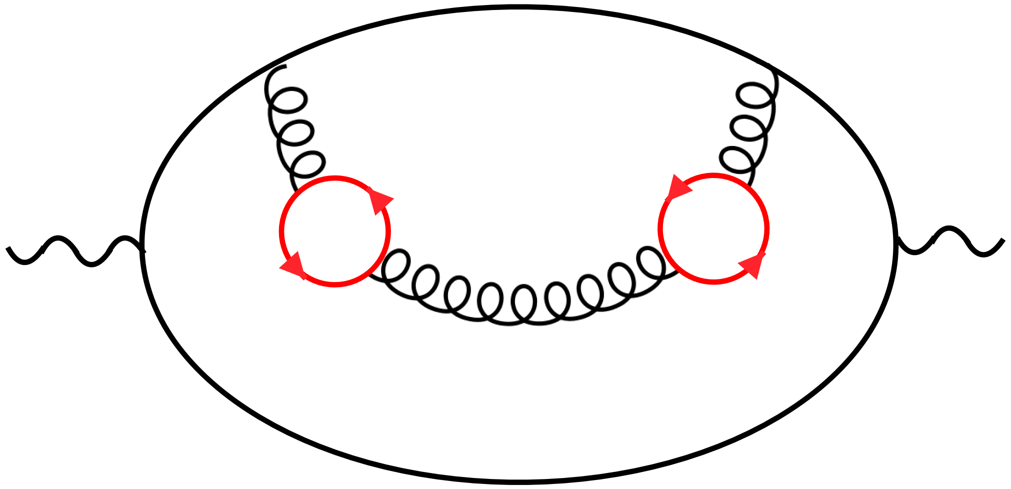
(Note: we recently found massive speedup w.r.t the above)

Integration (sampling) not optimised yet. so we do not report quantitatively on it yet.

PARTIAL N3LO RESULTS

n_f^2 contributions :





$$K_{jj}^{\text{(MC LU) I}} = 24.45(10)$$

$$K_{jj}^{\text{(MC LU) II}} = -24.80(22)$$

(Large accidental cancellation between the two graphs, but validation otherwise successful)

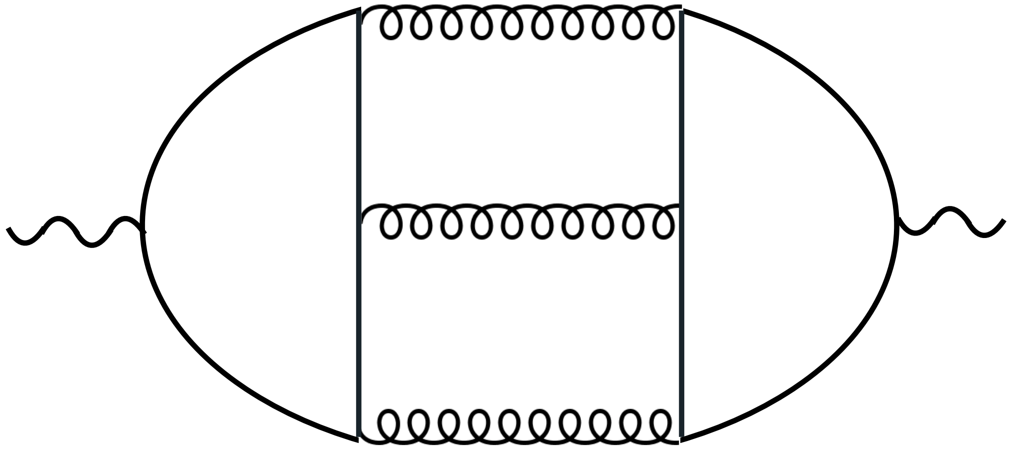
$$K_{jj}^{\text{(MC LU) I+II}} = -0.35(24)$$

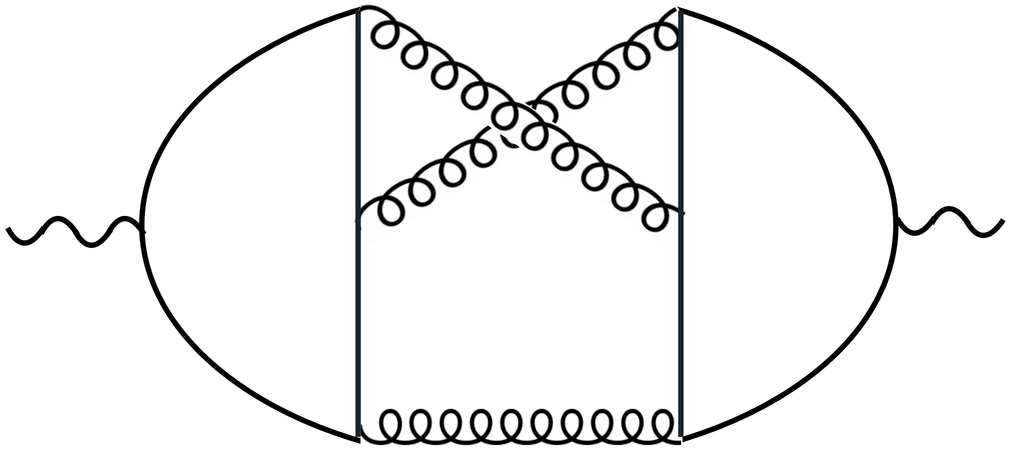
$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f^2)} = C_F\left(\frac{1208}{27} - \frac{8}{3}\zeta_2 - \frac{304}{9}\zeta_3\right) = -0.331415$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

PARTIAL N3LO RESULTS

Singlet contributions : (Results for low Monte-Carlo statistics here)





$$K_{jj}^{(\mathrm{MC\ LU})\mathrm{I}} = 48.4(1.0)$$

$$K_{jj}^{(\mathrm{MC\ LU})\mathrm{II}} = -74.0(1.1)$$

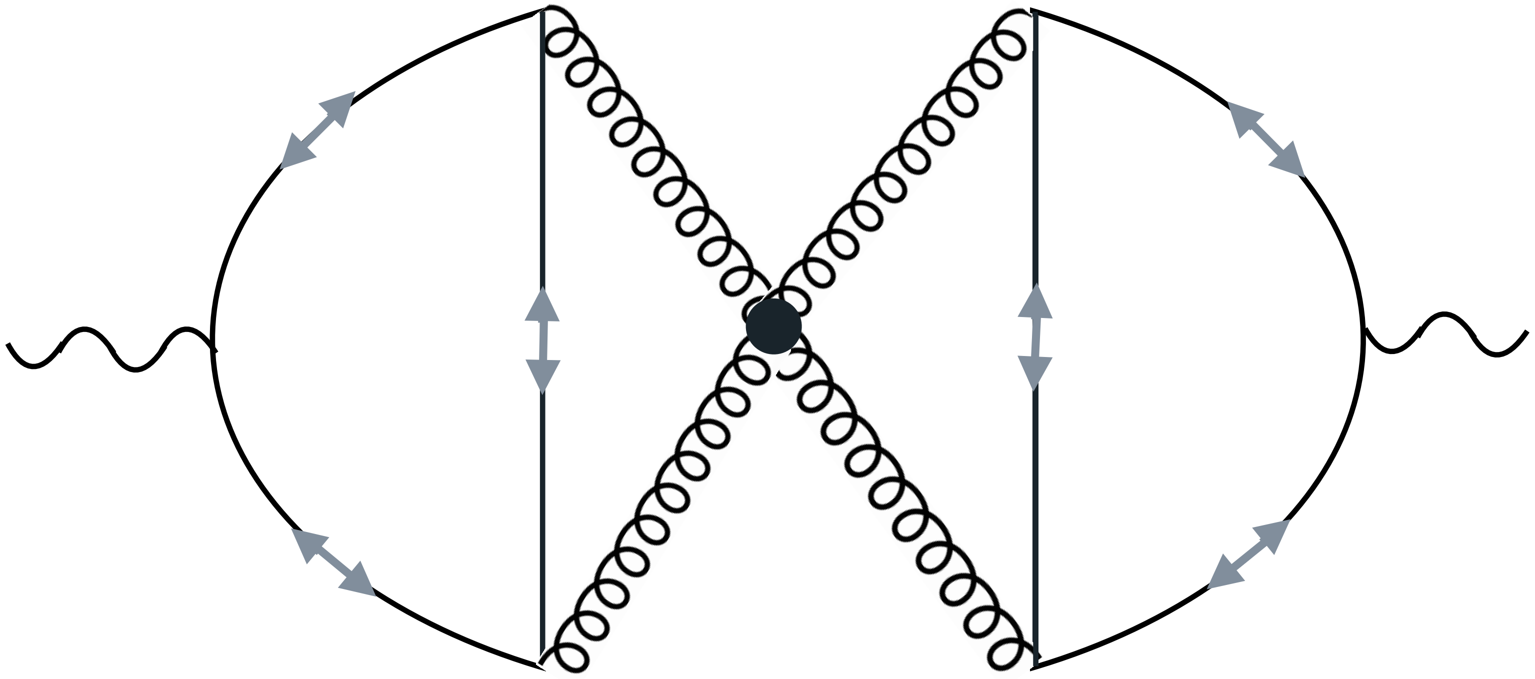
$$K_{jj}^{(\mathrm{MC\ LU})\mathrm{I+II}} = -25.6(1.5)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3),\,\mathrm{singlet}} = \frac{d_F^{abc}d_F^{abc}}{N_R}\left(\frac{176}{3} - 128\zeta_3\right) = -26.4435$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

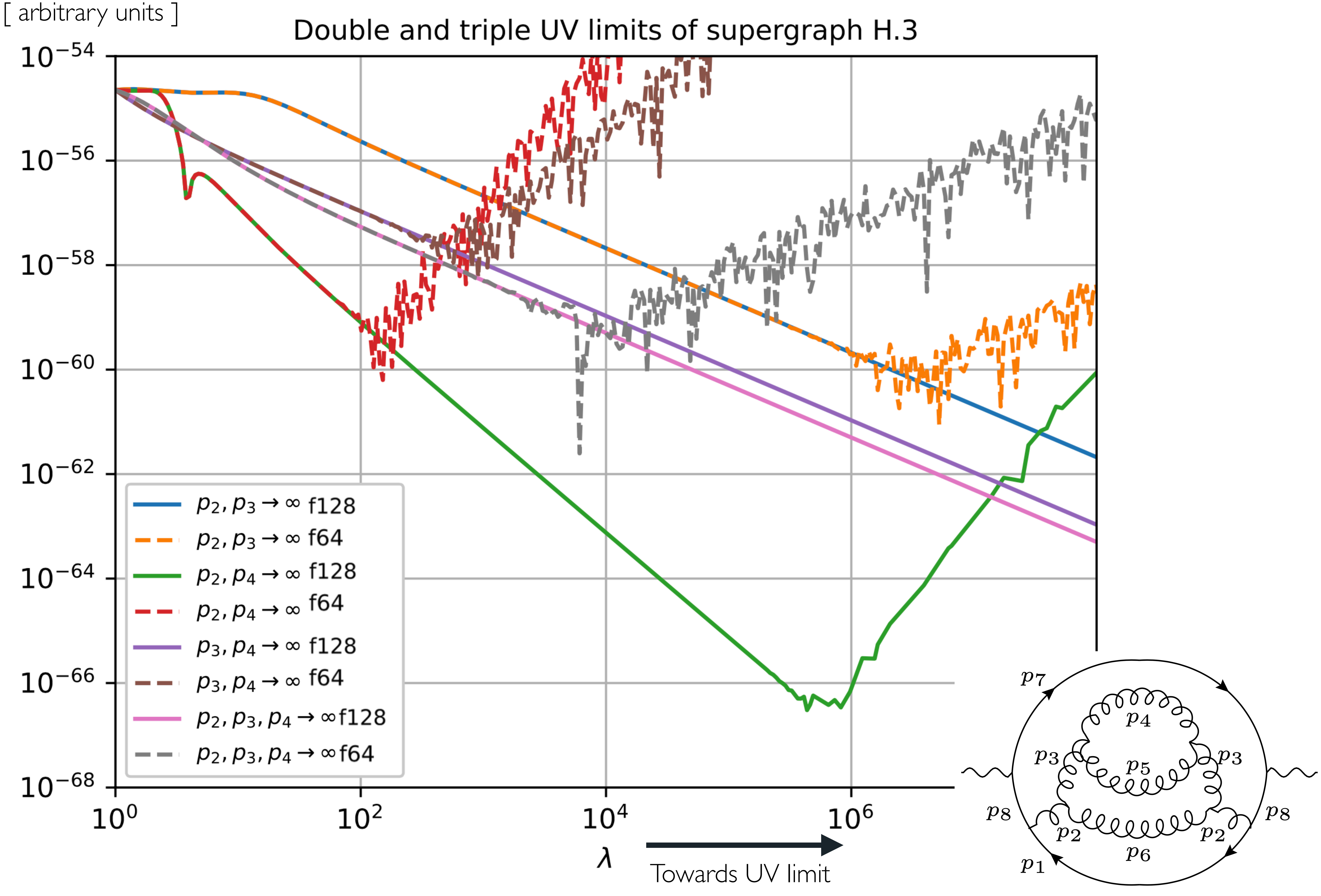
CURIOSITY

Singlet contributions : A non-obvious zero contribution…

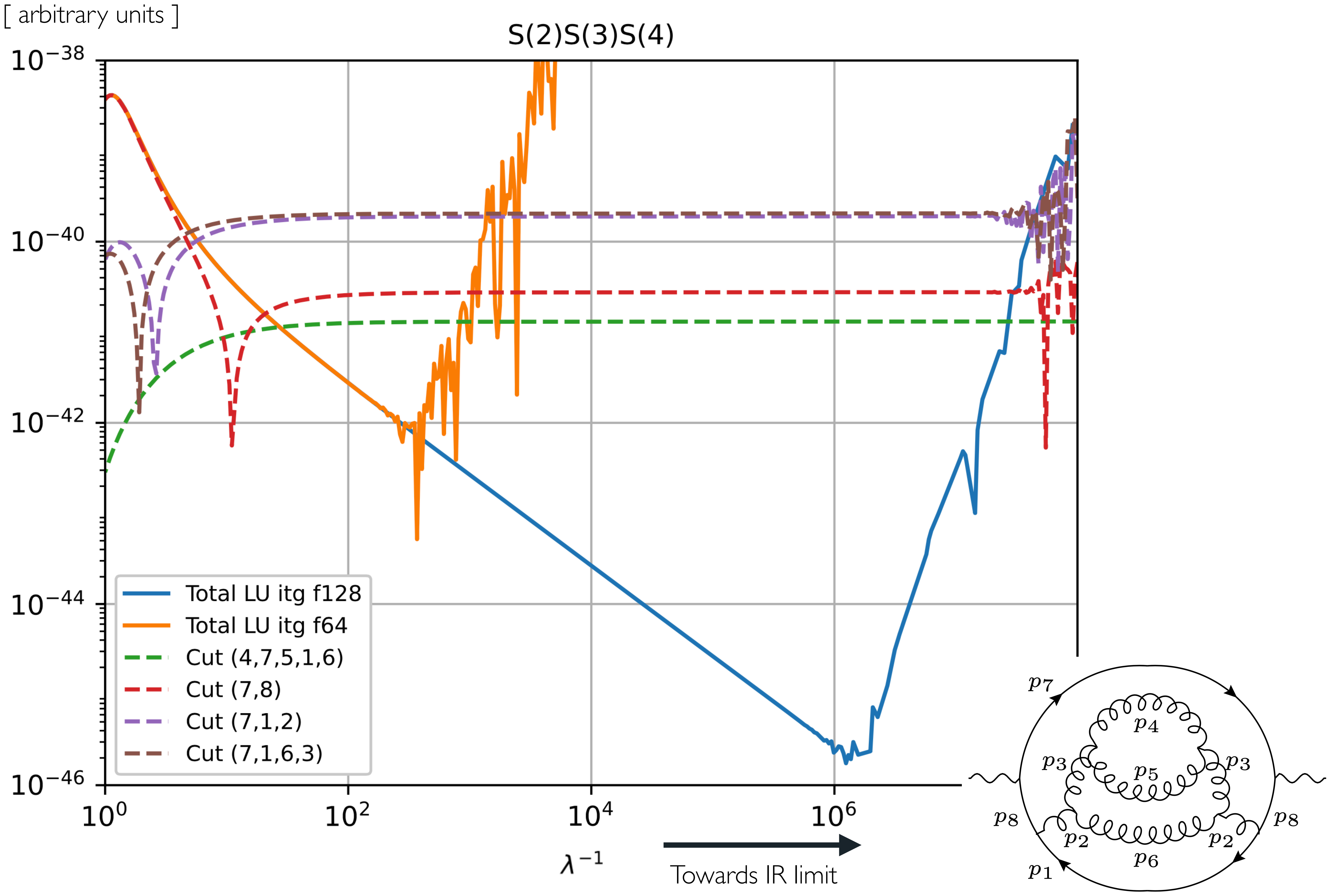


This graph gives no contribution inclusively.
I have not looked into it with any depth, but I don't see
an obvious really as to why it should be zero…

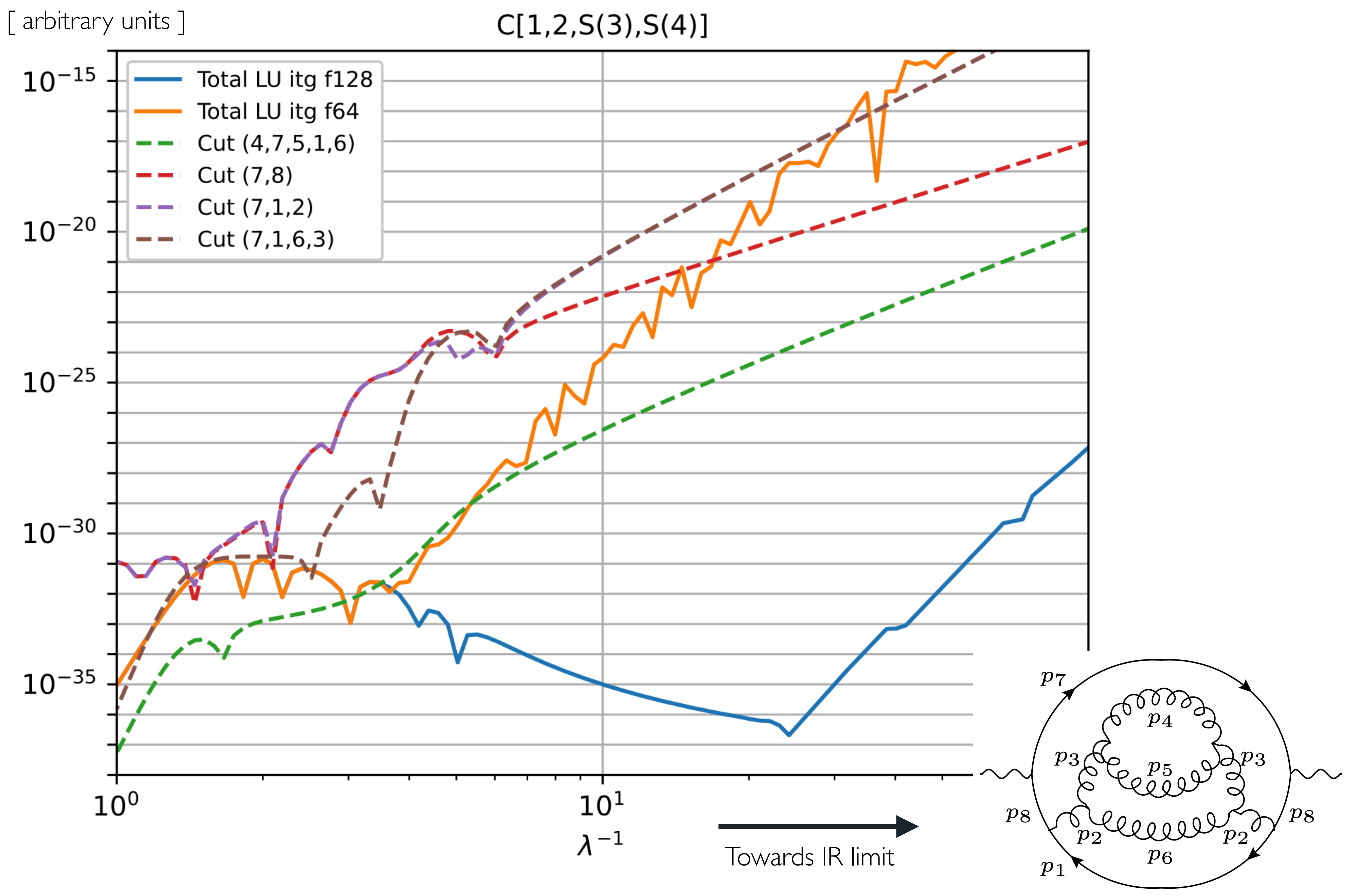
TESTING N3LO UV LIMITS



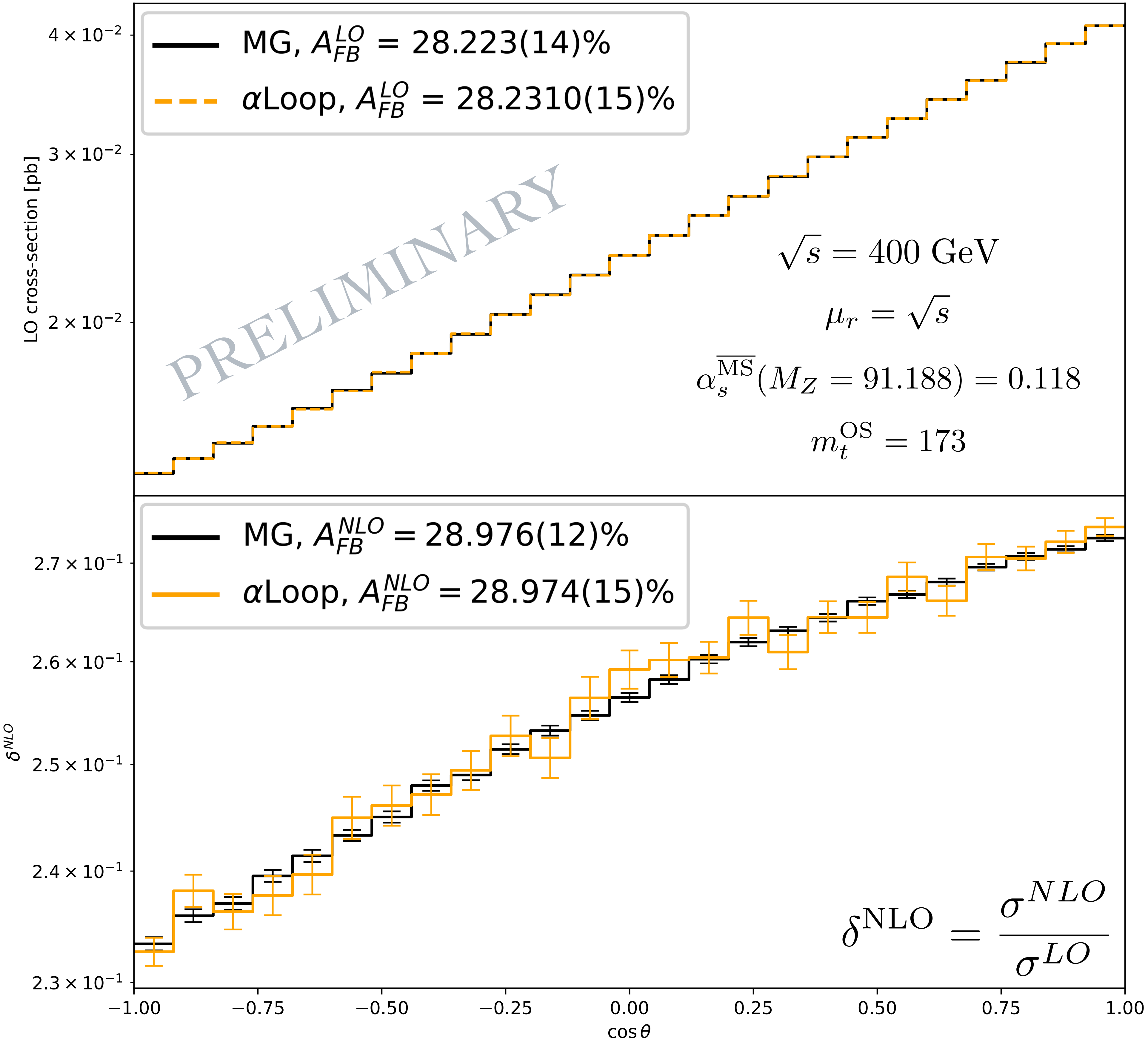
TESTING IR SOFT LIMITS



TESTING IR SOFT-COLLINEAR LIMITS



Example II : NLO AFB for $e^+e^- \to \gamma^\star/Z \to t\bar{t}$



First result in LU with γ^5 and EW-boson

Contour deformation well-behaved in this case

Credits to ETHZ student

Max Hofer