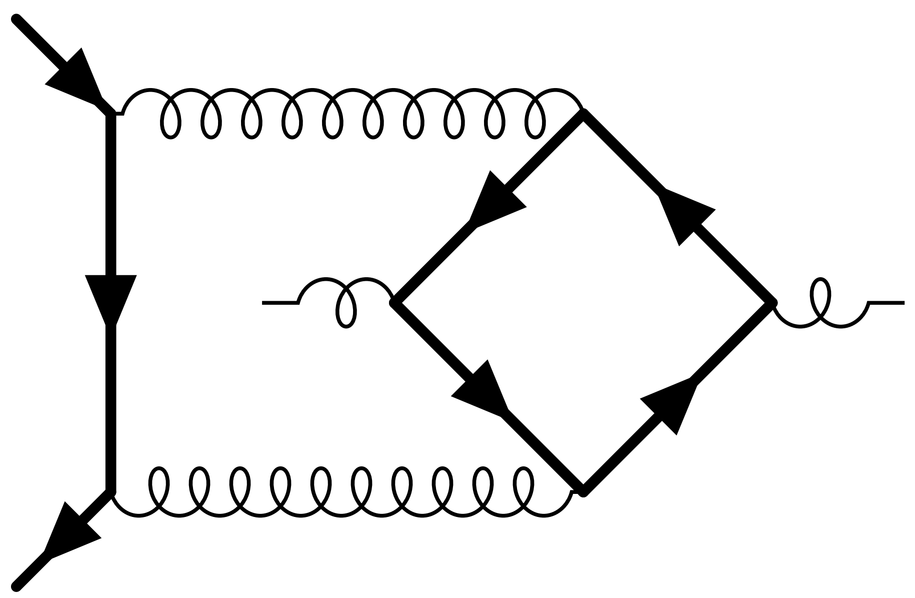
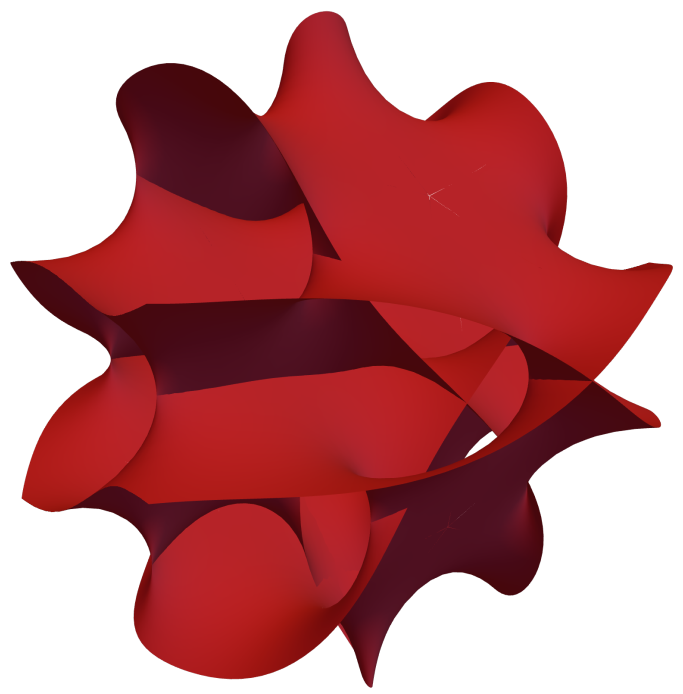
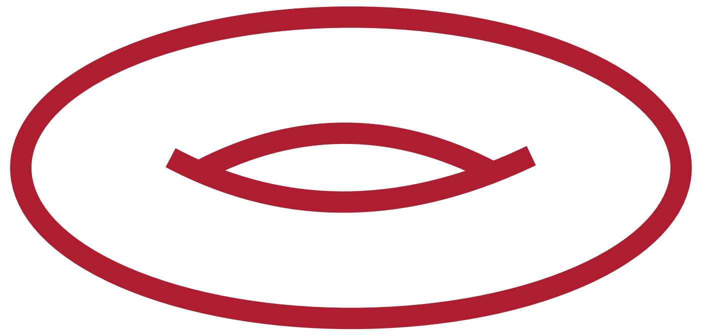
Special Functions in Feynman Integrals

Sebastian Pögel, University of Mainz
Galaxies meet QCD 2024
23rd February 2024, ETH Zurich



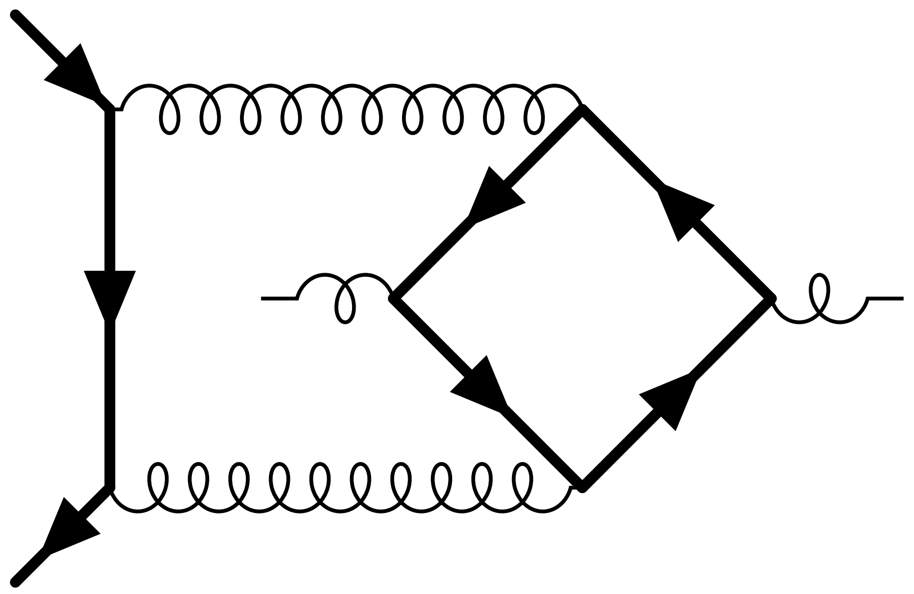




Feynman Integrals

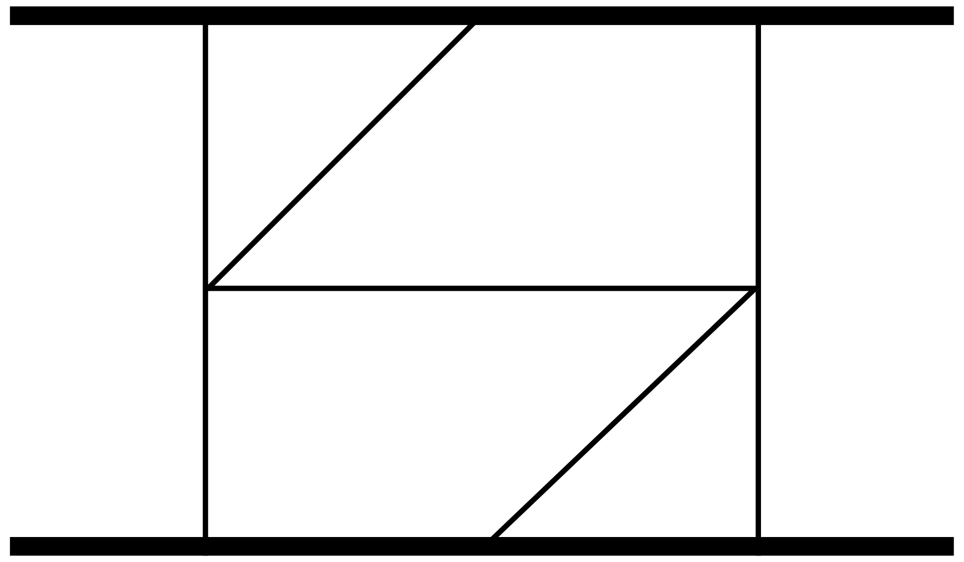
Disclaimer:
All examples given are in **even** dimensions

QCD



$gg \to t\bar{t}$

Gravity



post-Minkowskian potential corrections

Theory independent building blocks capturing most loop-level information

$$I_{\nu_1 \ldots \nu_m} = \int \prod_i \frac{\mathrm{d}^d \ell_i}{i\pi^{d/2}} \frac{1}{\prod_j D_j^{\nu_j}}$$

Using IBPs and symmetry $\longrightarrow$ **Basis of Master Integrals**

Complexity driven by #scales and #loops

Differential Equations

"Main" tool to evaluate Feynman Integrals

Basis of Master Integrals $I = \{I_1, \ldots, I_n\}$ **depending on kinematic variables** x **in dimension** $d = d_0 - 2\varepsilon$

Matrix of differential 1-forms

$$dI = \mathcal{A}(x, \varepsilon) I$$

Find "good" basis J **such that** ε **factorizes**

$$dJ = \varepsilon \tilde{\mathcal{A}}(x) J$$

Solution given by path-ordered exponential

$$J = \mathbb{P}\exp\left(\varepsilon \int \tilde{\mathcal{A}}\right) J_0$$

Let $\mathcal{C}(t)$ be an integration contour
with $t \in [0,1]$ $\mathcal{C}(0) = x_0$ $\mathcal{C}(1) = x$ and $\tilde{\mathcal{A}} = \tilde{A}(t)$

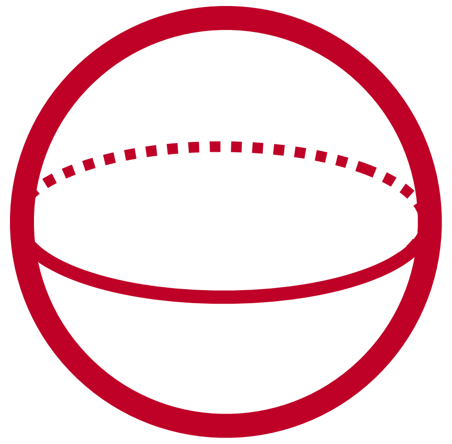
$$J = \varepsilon^0 J_0 + \varepsilon^1 \int_0^1 \mathrm{d}t \tilde{A}(t) J_0 + \varepsilon^2 \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \tilde{A}(t_1)\tilde{A}(t_2) J_0 + \mathcal{O}(\varepsilon^3)$$

Geometries of Feynman Integrals

Integrals associated to geometries ⟶ **Determines suitable function space**

Complicated geometry $\hat{=}$ complicated functions

Riemann Sphere

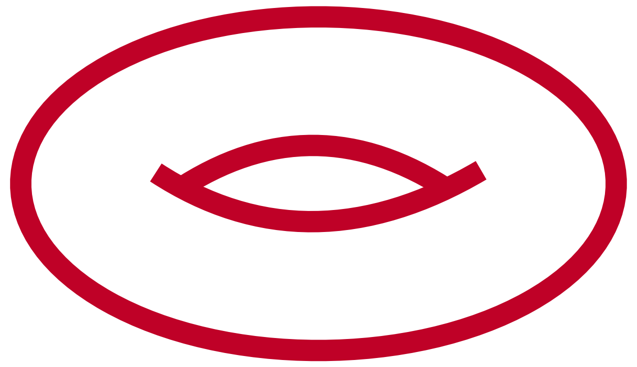


↓

Polylogarithms

"Well understood"

Elliptic curve

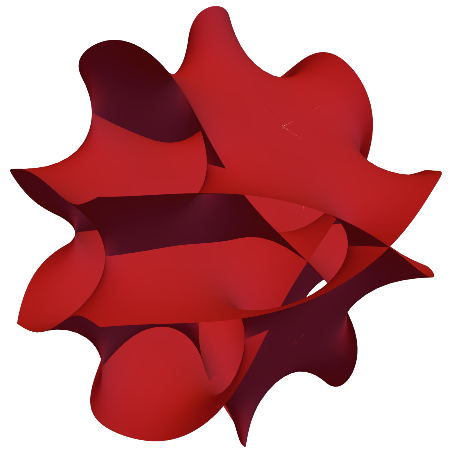


↓

Elliptic Integrals, modular forms, Elliptic Polylogarithms

Feasible, but technically challenging

Calabi—Yau manifolds

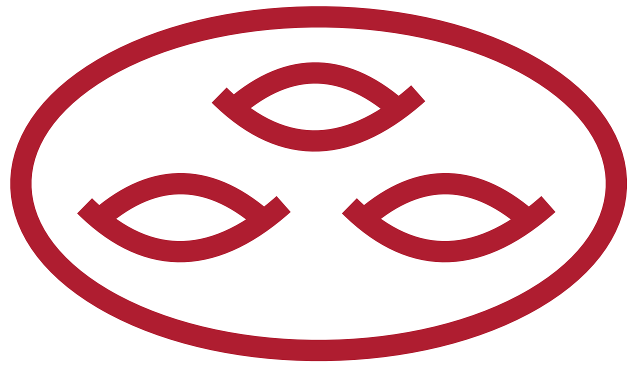


↓

??
Series expansions

Only simplest cases feasible

Higher-genus curves



↓

???
Higher genus polylogarithms in active research

Currently impossible

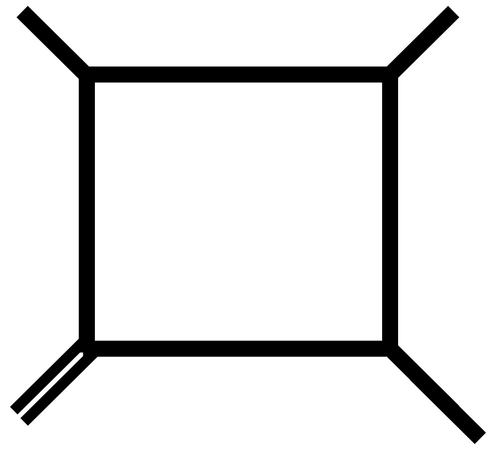
Simple examples: Logarithms

Starting at one-loop, all we need are logarithms and dilogarithms

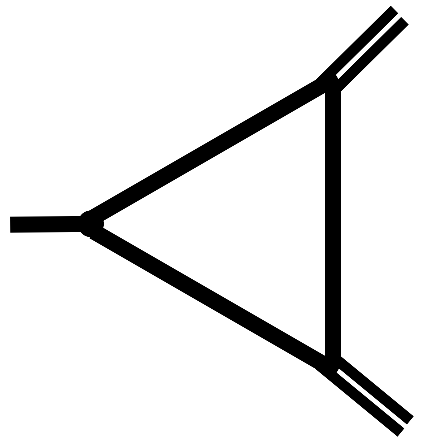
$$\log(x) = \int_0^x \frac{\mathrm{d}t}{t}$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\mathrm{d}t}{t}\log(1-t) = -\int_0^x \frac{\mathrm{d}t_1}{t_1}\int_0^{1-t_1}\frac{\mathrm{d}t_2}{t_2}$$

Some one-loop integrals in $d = 4 - 2\varepsilon$



$$\sim \frac{1}{\varepsilon^2}\left((-s_{23})^{-\varepsilon} + (-s_{34})^{-\varepsilon} - (-K_1^2)^{-\varepsilon}\right) - \mathrm{Li}_2\left(1-\frac{K_1^2}{s_{23}}\right) - \mathrm{Li}_2\left(1-\frac{K_1^2}{s_{34}}\right) - \frac{1}{2}\log^2\left(\frac{-s_{12}}{-s_{23}}\right) - \frac{\pi^2}{6} + \mathcal{O}(\varepsilon)$$



$$\sim \frac{1}{\varepsilon^2}\frac{\left(-K_1^2\right)^{-\varepsilon} - \left(-K_2^2\right)^{-\varepsilon}}{(-K_1^2) - (-K_2^2)}$$

Generalized Polylogs (GPLs)

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{\mathrm{d}x_1}{x_1 - a_1} G(a_2, \ldots, a_n; x_1)$$

Length is called transcendental weight

Example of class of functions called Chen iterated integrals with kernels $\omega = \frac{\mathrm{d}x}{x-a} = \mathrm{d}\log(x-a)$

Closed under integration:

$$\int (\text{rational function}) \times (\text{GPL}) = \sum (\text{GPLs})$$

(Partial fraction + IBP)

Contains classical polylogs:

$$\log(x)^n = n! G(\underbrace{0, \ldots, 0}_{n}; x)$$

$$\mathrm{Li}_n(x) = -G(\underbrace{0, \ldots, 0}_{n-1}, 1; x)$$

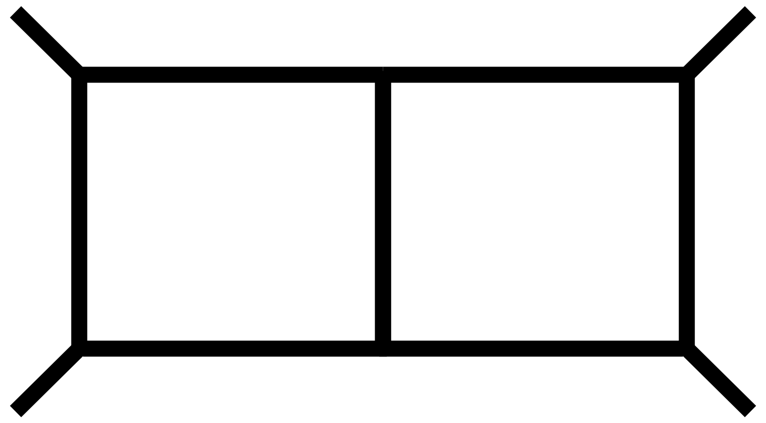
$$dJ = \varepsilon \tilde{A}(x) J$$

with $\tilde{A}$ made up of $\mathrm{d}\log$ forms

↓

ε-expansion of J has GPLs as coefficients

Example: Four-Point Double Box



One kinematic scale $x = \frac{s_{23}}{s_{12}}$ **with 8 Master Integrals** $I = \{I_1, \ldots, I_8\}$ **in** $d = 4 - 2\varepsilon$

With "good" basis J **obtain** ε**-factorized differential equation**

$$\mathrm{d}J = \varepsilon \left(\frac{\mathrm{d}x}{x} A + \frac{\mathrm{d}x}{1+x} B \right) J \qquad A, B \in \mathbb{Q}^{8\times 8}$$

$\mathrm{d}\log(x)$ $\mathrm{d}\log(x+1)$

$$J = \mathbb{P}\exp\left[\varepsilon \int (A\,\mathrm{dlog}(x) + B\,\mathrm{dlog}(x))\right] J_0$$

Solution given in Harmonic Polylogarithms

$G(a_1, a_2, \ldots, a_n; x)$
$a_i \in \{1, 0, -1\}$

Some Properties of GPLs

- **Satisfy shuffle Algebra** $G(\vec{a};x)G(\vec{b};x) = \sum_{\vec{c}\in\vec{a}\sqcup\!\sqcup\vec{b}} G(\vec{c};x)$

- **The Symbol ~ rough approximate of GPL through argument of** $\mathrm{d}\log$ **forms**

  Examples: $S(\log(x)) = x \quad S(\mathrm{Li}_n(x)) = -\underbrace{x\otimes\ldots\otimes x}_{n-1}\otimes(1-x)$

  Compatible with shuffle product: $S(I\cdot J) = S(I)\sqcup\!\sqcup S(J)$

  Useful for simplification and finding identities

$$S(\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)) = -S(\log(x)\log(1-x))$$

$$\Rightarrow \quad \mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)+\log(x)\log(1-x)\sim 0$$

Full identity requires ζ_2

Constants with transcendental weight
$\pi\sim 1$
$\varepsilon\sim 1$
$\zeta_{m_1\ldots m_n}\sim\sum m_i$

Implementation
PolylogTools

- **Can be evaluated efficiently**

  Massaging into fast convergent region + series expansion

Fast implementations available
(e.g. Ginac, handyG, FastGPL)

Beyond Polylogs

Fantastic Geometries
and where to find them

How do we identify geometry of integrals?

Maximal Cuts

Skeletonized version of integral
Much simpler to extract geometry

Homogeneous solution to differential equation of full integral

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r} + \sum_{i=0}^{r-1} c_r(x) \frac{\mathrm{d}^i}{\mathrm{d}x^i}$$

$$\mathrm{MaxCut}(I) \sim \int \frac{\prod \mathrm{d}\ell_j^{d_0}}{\prod_i D_i} /.\{D_i \to \delta(D_i)\}$$

Reduces number of integrations

For polylogarithmic integrals: rational expression

$$\frac{\mathrm{d}I}{\mathrm{d}x} = A(x) I \quad \longrightarrow \quad \mathcal{L}^{(r)} I_i = (\mathrm{inhom.})$$

Linear system → **High order differential operator**

Simplify via $(\mathrm{inhom.}) = 0$ and $d = d_0$

For polylogarithmic integrals: Rationally factorizes

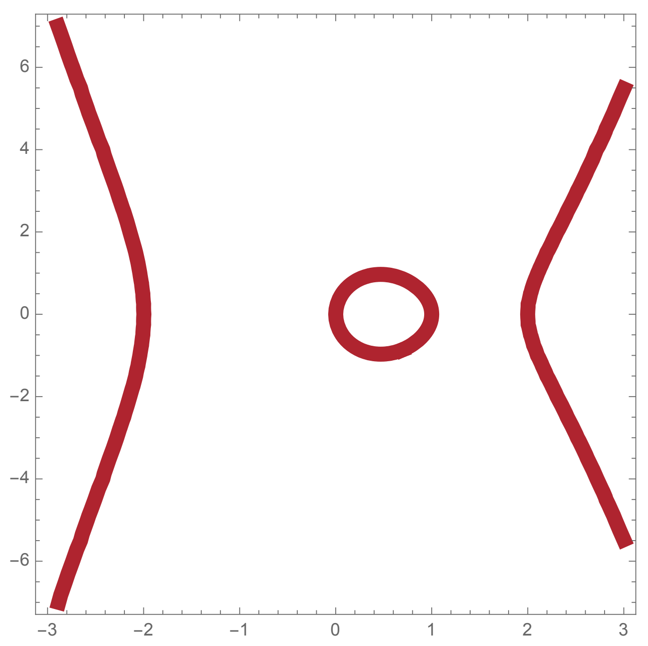
Elliptic Curves

Algebraic Curve: **Polynomial** $f \in \mathbb{C}[y, z]$ $\quad y, z \in \mathbb{C}$ **such that** $f(y, z) = 0$

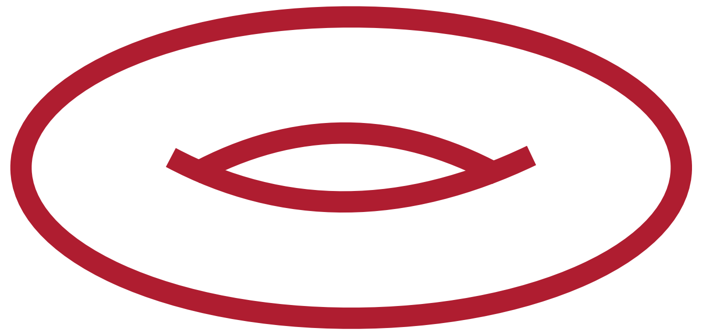
Simplest Example:
Elliptic Curves

$$f(y, z) = y^2 - (z - a_1)(z - a_2)(z - a_3)(z - a_4) = 0$$

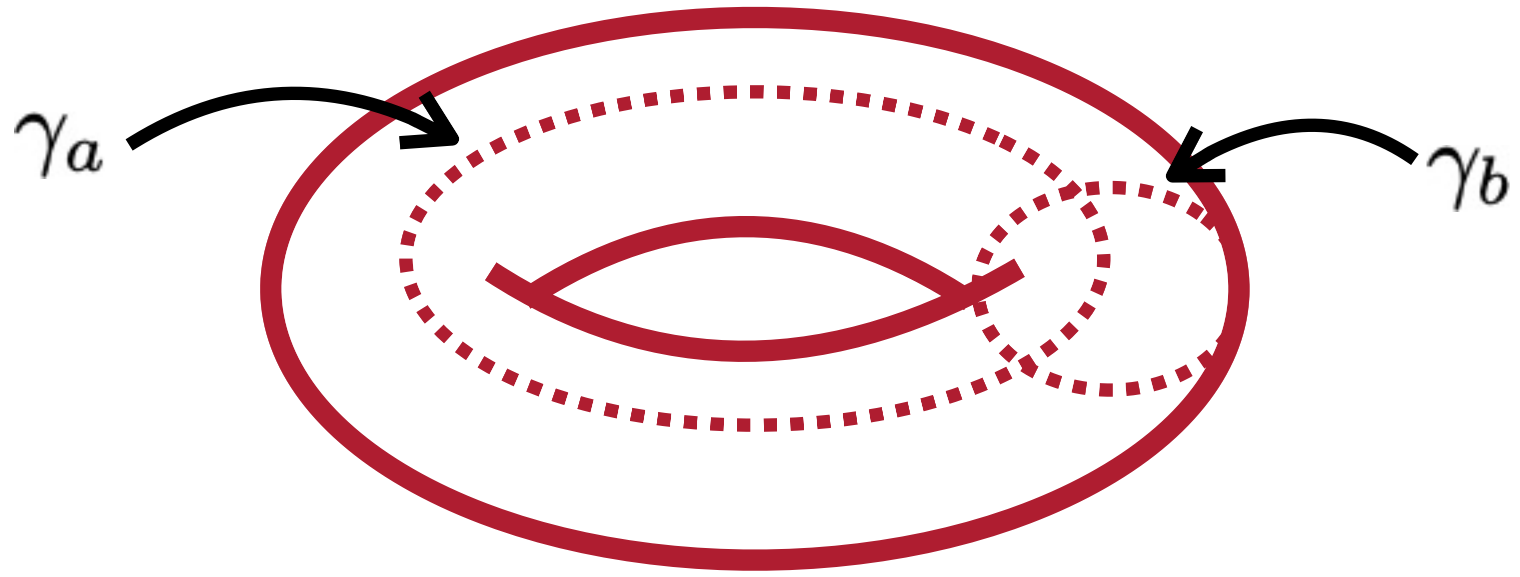
Genus 1



$\simeq$



Modular Group and Forms



Two independent cycles γ_a and γ_b

One holomorphic form $\frac{\mathrm{d}x}{y}$

Combination define periods of elliptic curve

$$\psi_1 = \int_{\gamma_a} \frac{\mathrm{d}x}{y} \qquad \psi_2 = \int_{\gamma_b} \frac{\mathrm{d}x}{y}$$

Periods span a lattice

Define ratio of periods $\tau = \frac{\psi_2}{\psi_1}$

Elliptic nome $q = e^{2\pi i \tau}$

Awesome expansion variable

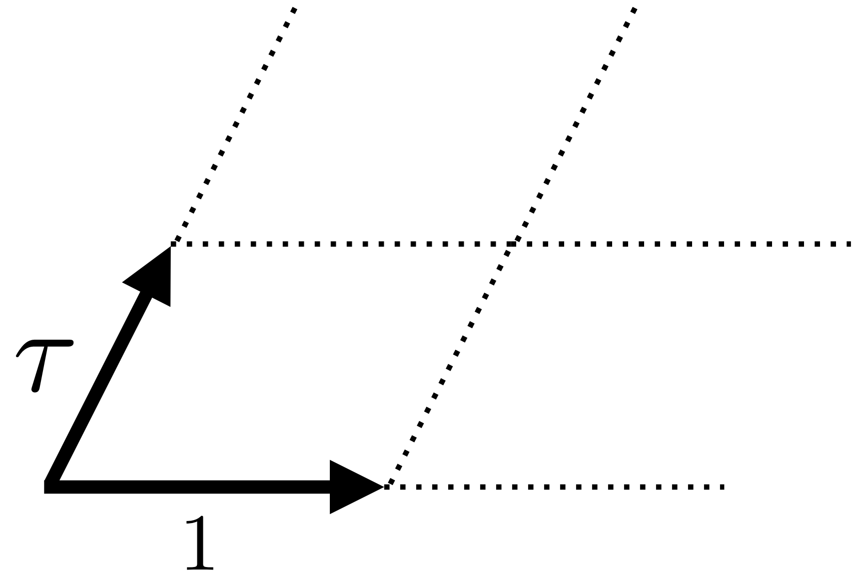
Can always choose τ such that $|q| \leq 0.0043$

Modular group acts naturally on lattice

new basis for lattice

$$\tau' = \frac{a\tau + b}{c\tau + d} \Leftrightarrow \begin{aligned} \psi_1' &= d\psi_1 + c\psi_2 \\ \psi_2' &= b\psi_1 + a\psi_2 \end{aligned}$$

modular transformation
≙
Möbius transformation



Modular forms:
Functions that transform nicely under transformations

$$\eta_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \eta_k(\tau)$$

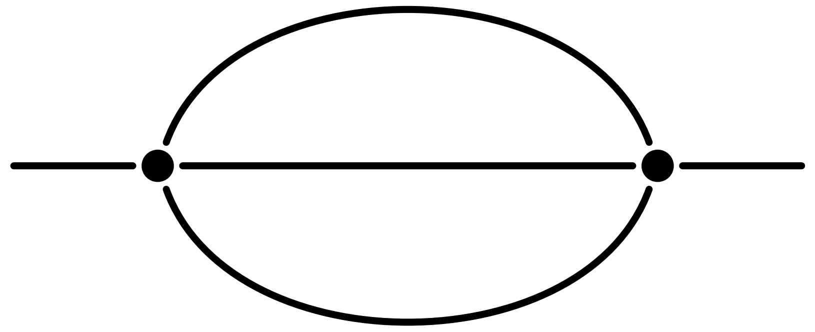
have quasi integral expansion in q

k: modular weight

e.g. $\psi_1' = (c\tau + d)\psi_1$

Elliptic Feynman Integrals

Simplest example: The Sunrise Integral

$I =$ 

Massive propagators

one scale $x = p^2/m^2$

In d=2

$$\mathrm{MaxCut}(I) \propto \int \frac{\mathrm{d}u}{\sqrt{u(u-4)\left(u^2-2u(x+1)+(x-1)^2\right)}}$$

Maximal cut has algebraic obstruction: Not polylogarithmic

Root of quartic polynomial → elliptic curve

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{3x^2-20x+9}{x(x-1)(x-9)}\frac{\mathrm{d}}{\mathrm{d}x^1} + \frac{x-3}{x(x-1)(x-9)}\right]\mathrm{MaxCut}(I) = 0$$

Equivalently, differential operator does not factorize

Identify maximal cut with elliptic integrals

$$\mathrm{MaxCut}(I)|_{\gamma_1} \propto \mathrm{K}(k)$$

$$\mathrm{MaxCut}(I)|_{\gamma_2} \propto i\mathrm{K}(1-k)$$

Elliptic integral of the first kind

$$\mathrm{d}I = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & 1 \\ \eta_3 & \eta_4 & \eta_2 \end{pmatrix} I$$

Kernels are modular forms

Sunrise as iterated integrals of modular forms

Super fast converging expansion in q

Elliptic Polylogs

Several generalizations of polylogs possible on elliptic curves

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_r \\ c_1 & \dots & c_r \end{smallmatrix}; z; \tau\right)$$

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_r \\ c_1 & \dots & c_r \end{smallmatrix}; z\right)$$

$$\mathrm{ELi}_{n_1 \dots n_l; m_1, \dots m_l; 2o_1, \dots 2o_{l-1}}(x_1, \dots, x_l; y_1 \dots y_l; q)$$

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_r \\ c_1 & \dots & c_r \end{smallmatrix}; z\right)$$

$$\Gamma\left(\begin{smallmatrix} n_1 & \dots & n_r \\ a_1 & \dots & a_r \end{smallmatrix}; z\right)$$

Nice function classes with special properties
(e.g. closure under integration, as for GPLs)

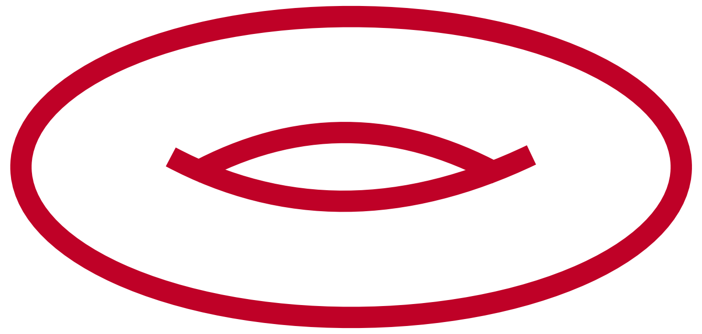
Expressing results in these functions is not trivial
(need to transform factorized DEQ to specific kernels)

Beyond Elliptics

How do we generalize from elliptic integrals?

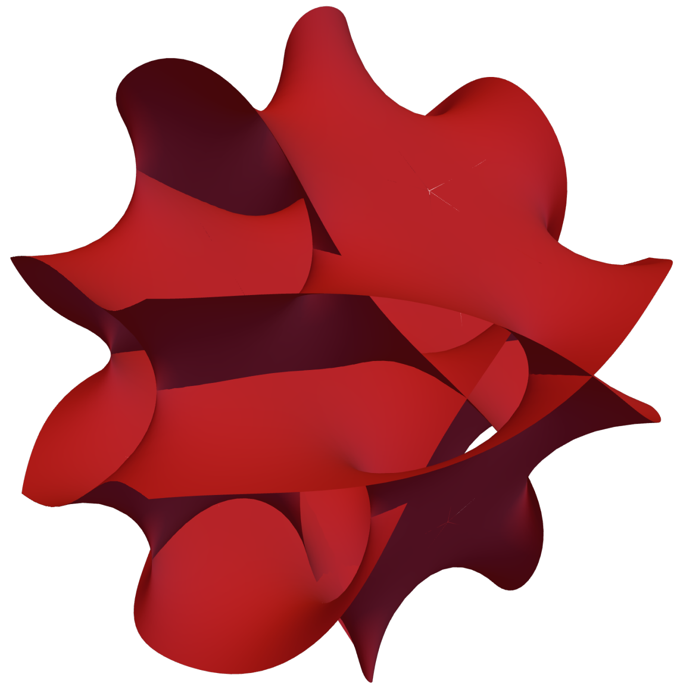
Elliptic Feynman integrals are phenomenological state of the art
What else is there?

Elliptic Curve



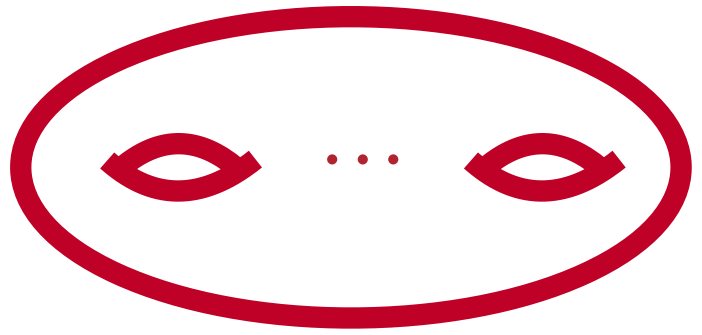
Dimension
1

Genus
1



Higher dimensional varieties

Curves of higher genus



e.g. Calabi—Yaus

e.g. Hyperelliptic curves

Calabi–Yau Feynman Integrals

An Outlook

Plenty of Calabi—Yau integrals known

Beyond one kinematic scale currently infeasible

String and mathematics literature extremely helpful
Mirror symmetry provides mirror map

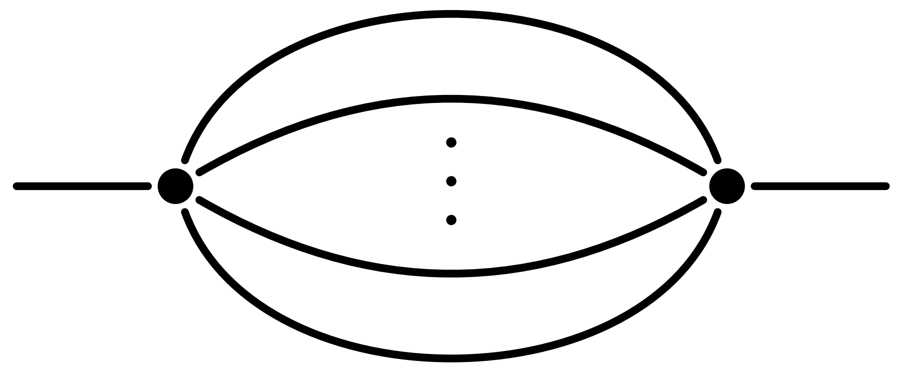
Defines τ and q, similar to elliptic case

For equal-mass Banana Integral series expansions in q

Frontier: Multi-scale Banana Integrals

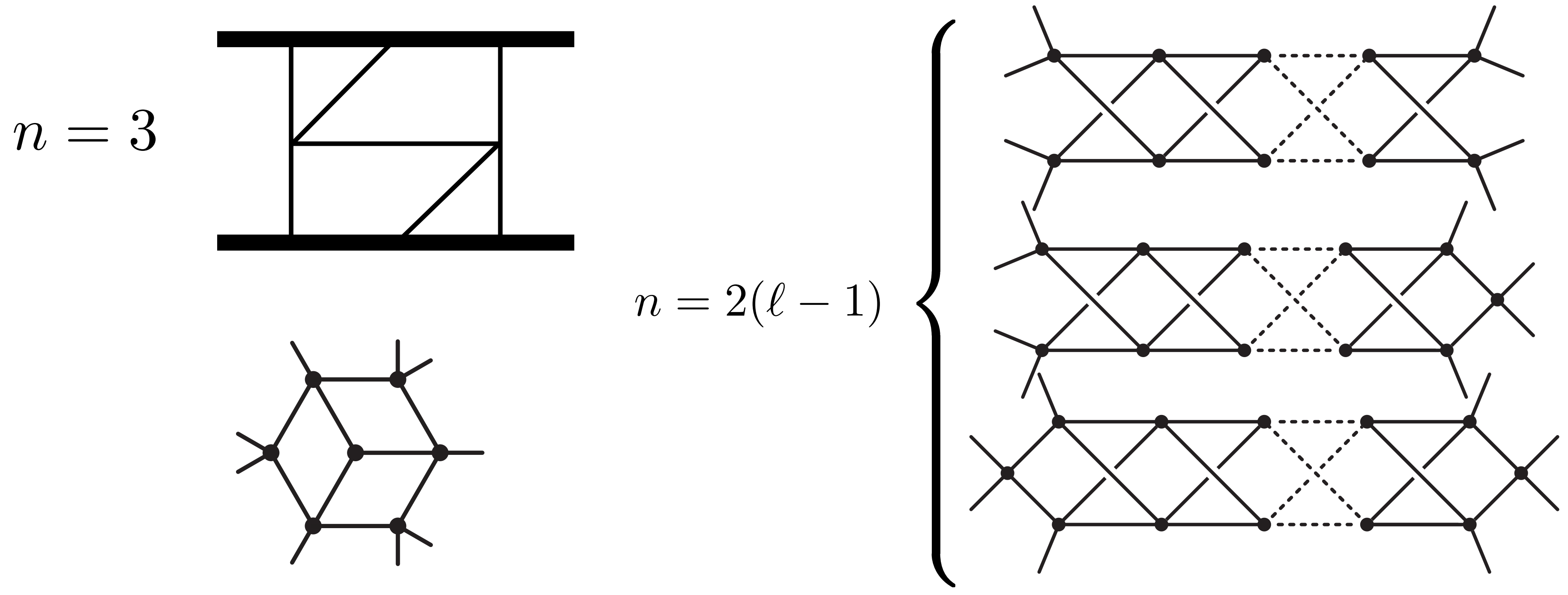
Simplest Example:

Banana Integrals



ℓ-loop Banana integral
$\hat{=}$
$(\ell-1)$-fold Calabi–Yau

Other examples:



n = dimension of manifold

Conclusions

- Feynman integrals span zoo of special functions
  - At 1-loop just Logs and Dilogs
  - Already at 2-loop, elliptic, higher-genus, and Calabi—Yaus appear
  - Size of the zoo unknown

- Polylogs well understood class
- Elliptics feasible, but still technically challenging

- In odd dimensions significantly less explored (at least from QCD side)
  - However, much of the technology should carry over to d=3