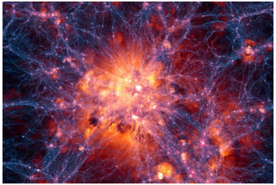
THE UNIVERSITY of EDINBURGH

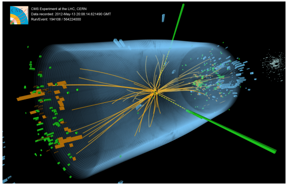
CENTRE FOR THEORETICAL PHYSICS · Higgs

# Amplitude Bootstrap Methods

Andrew McLeod

Galaxies Meet QCD
February 23, 2024






THE ROYAL SOCIETY

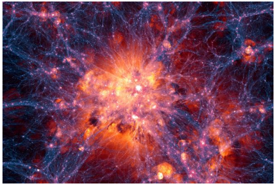
THE UNIVERSITY *of* EDINBURGH

CENTRE FOR THEORETICAL PHYSICS · Higgs

# ~~Amplitude Bootstrap Methods~~
# Building Special Functions

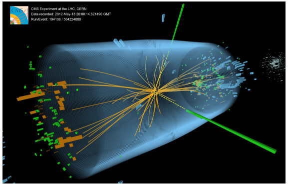
Andrew McLeod

Galaxies Meet QCD
February 23, 2024



↘

↙




THE ROYAL SOCIETY

# Outline

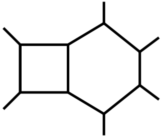
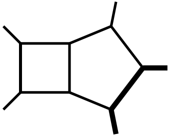
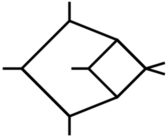
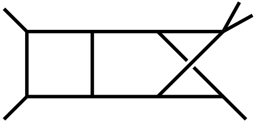
1) **Perturbative bootstrap methods**
   - $\Rightarrow$ (a historical) motivation and introduction
   - $\Rightarrow$ bootstrap calculations at **large particle multiplicities** and **high loop orders**
   - $\Rightarrow$ **distilling key lessons** from these bootstrap calculations

2) The **analytic properties** of **polylogarithmic Feynman integrals**
   - $\Rightarrow$ **singular points** and how to characterize them
   - $\Rightarrow$ **algebraic** versus **logarithmic** branch cuts
   - $\Rightarrow$ building **single-valued functions**

3) **Hermeneutical lessons** from amplitude calculations

# The Integration Bottleneck

- The technology for **reducing** the computation of scattering amplitudes (and related quantities) to the evaluation of a **small basis of master integrals** has advanced enormously in recent years
- Even so, **our ability to evaluate these integrals analytically remains limited**

[Henn, Peraro, Xu, Zhang (2022)]

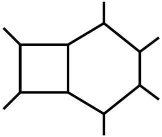
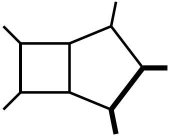
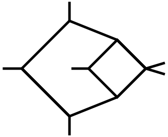
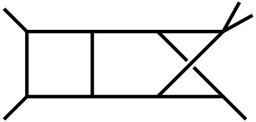
[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia (2023)]

[Badger, Becchetti, Chaubey, Marzucca (2023)]

[Henn, Lim, Bobadilla (2023)]

# The Integration Bottleneck

- The technology for **reducing** the computation of scattering amplitudes (and related quantities) to the evaluation of a **small basis of master integrals** has advanced enormously in recent years
- Even so, **our ability to evaluate these integrals analytically remains limited**

[Henn, Peraro, Xu, Zhang (2022)]

[Badger, Becchetti, Chaubey, Marzucca (2023)]

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia (2023)]

[Henn, Lim, Bobadilla (2023)]

- **Perturbative bootstrap methods** ask the following question:

> Do we know enough about the mathematical properties of amplitudes (or similar quantities) to **avoid integration** and construct them directly?

# The Surprising Simplicity of Amplitudes

This is a natural question to ask—**despite their computational difficulty**, amplitudes are often found to evaluate to **strikingly simple expressions**

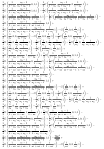
- The paradigmatic (loop-level) example is given by the first **two-loop six-particle amplitude** calculated in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

# The Surprising Simplicity of Amplitudes

This is a natural question to ask—**despite their computational difficulty**, amplitudes are often found to evaluate to **strikingly simple expressions**

- The paradigmatic (loop-level) example is given by the first **two-loop six-particle amplitude** calculated in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

$+ \cdots +$



first computed as a 17 page expression

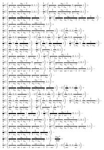
[Del Duca, Duhr, Smirnov (2009)]

# The Surprising Simplicity of Amplitudes

This is a natural question to ask—**despite their computational difficulty**, amplitudes are often found to evaluate to **strikingly simple expressions**

- The paradigmatic (loop-level) example is given by the first **two-loop six-particle amplitude** calculated in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

$+ \cdots +$



$$\Rightarrow \quad R_6^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3} \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \mathrm{Li}_4(1 - 1/u_i) \right) - \frac{1}{8} \left( \sum_{i=1}^{3} \mathrm{Li}_2(1 - 1/u_i) \right)^2 + \frac{1}{24} J^4 + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72}. \quad (3)$$

first computed as a 17 page expression, later simplified to a two-line expression

[Del Duca, Duhr, Smirnov (2009)] [Goncharov, Spradlin, Vergu, Volovich (2010)]

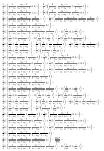
# The Surprising Simplicity of Amplitudes

This is a natural question to ask—**despite their computational difficulty**, amplitudes are often found to evaluate to **strikingly simple expressions**

(once the right theoretical language is found)

- The paradigmatic (loop-level) example is given by the first **two-loop six-particle amplitude** calculated in planar $\mathcal{N}=4$ supersymmetric Yang-Mills theory

$+ \cdots +$



$\Rightarrow$

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}. \quad (3)$$

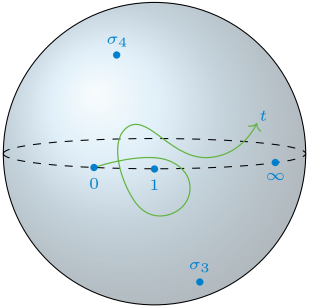
first computed as a 17 page expression, later simplified to a two-line expression

[Del Duca, Duhr, Smirnov (2009)] [Goncharov, Spradlin, Vergu, Volovich (2010)]

# Analytic Properties

Several striking features were made clear in this example by the simplified formula:

- the special functions that appear are all drawn from a highly restricted class of **generalized polylogarithms** (or, iterated integrals over the punctured Riemann sphere)




$$\int_0^t \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} \int_0^{t_2} \frac{dt_3}{t_3 - c_3} \int \ldots$$

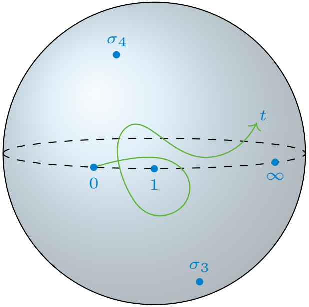
where the integration endpoint $t$ and punctures $c_i \in \{0, 1, \sigma_3, \ldots\}$ are algebraic functions of Mandelstam variables

- logarithmic branch points only appear at nine locations
- each term also involves precisely four logarithmic integrals

# Analytic Properties

Several striking features were made clear in this example by the simplified formula:

- the special functions that appear are all drawn from a highly restricted class of **generalized polylogarithms** (or, iterated integrals over the punctured Riemann sphere)




$$\int_0^{t} \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} \int_0^{t_2} \frac{dt_3}{t_3 - c_3} \int \cdots$$

where the integration endpoint $t$ and punctures $c_i \in \{0, 1, \sigma_3, \dots\}$ are algebraic functions of Mandelstam variables

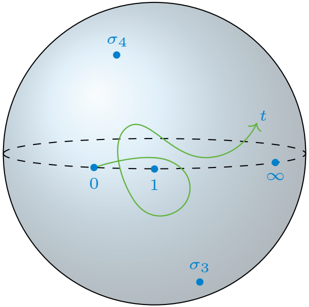
- logarithmic branch points only appear at nine locations
- each term also involves precisely four logarithmic integrals

**Could the amplitude exhibit these same properties at higher loop orders?**

# Analytic Properties

Several striking features were made clear in this example by the simplified formula:

- the special functions that appear are all drawn from a highly restricted class of **generalized polylogarithms** (or, iterated integrals over the punctured Riemann sphere)




$$\int_0^{t} \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} \int_0^{t_2} \frac{dt_3}{t_3 - c_3} \int \cdots$$

where the integration endpoint $t$ and punctures $c_i \in \{0, 1, \sigma_3, \ldots\}$ are algebraic functions of Mandelstam variables

- logarithmic branch points only appear at nine locations
- each term also involves precisely ~~four~~ $2L$ logarithmic integrals

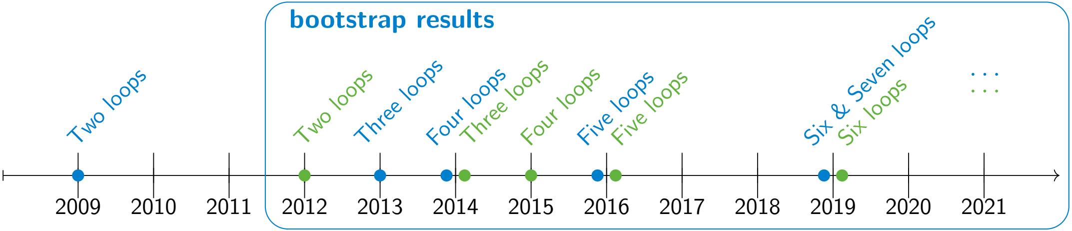
**Could the amplitude exhibit these same properties at higher loop orders?**

# Bootstrap Methods

Starting from the conjecture that the $L$-loop amplitude ‘lives’ in this space, we can try to **bootstrap** it directly by looking for a function that exhibits all the expected properties

# Bootstrap Methods

Starting from the conjecture that the $L$-loop amplitude 'lives' in this space, we can try to **bootstrap** it directly by looking for a function that exhibits all the expected properties




MHV [Del Duca, Duhr, Smirnov (2009)] [Dixon, Drummond, Henn (2011)] [Dixon, Drummond, von Hippel, Pennington (2013)] [Dixon, Drummond, Duhr, Pennington (2014)] [Caron-Huot, Dixon, AJM, von Hippel (2016)] [Caron-Huot, Dixon, Dulat, von Hippel, AJM, Papathanasiou (2019)]

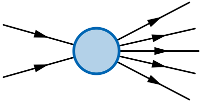
NHMV [Dixon, Drummond, Henn (2012)] [Dixon, von Hippel (2014)] [Dixon, von Hippel, AJM (2015)] [Caron-Huot, Dixon, AJM, von Hippel (2016)] [Caron-Huot, Dixon, Dulat, von Hippel, AJM, Papathanasiou (2019)]

- each of these results is unique, and satisfies a number of nontrivial cross-checks
- thus, for the six-particle amplitude, we can **bypass integration altogether**

# Successful Bootstrap Examples

The same methods have been successfully now to many examples

seven-particle amplitude



4 loops

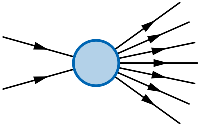
[Dixon, Drummond, Harrington, AJM, Papathanasiou, Spradlin (2016)] [Drummond, Foster, Gürdoğan, Papathanasiou (2018)]

. . .

. . .

. . .

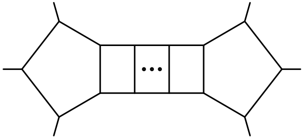
all-multiplicity amplitudes



2 loops

[Caron-Huot (2011)]
[He, Li, Zhang (2020)]

special classes of integrals



$\infty$ loops

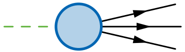
[Caron-Huot, Dixon, von Hippel, AJM, Papathanasiou (2018)]

- Takeaway: once we learn the right theoretical language in which to formulate perturbative quantities in QFT, rapid progress can be made

# Technology for Multiple Polylogarithms

Having applied bootstrap methods at such high loop orders, we have extremely well-developed technology for working with functions such as multiple polylogarithms

- As an illustration, the ‘simplest’ quantities that have been bootstrapped are supersymmetric three-particle form factors



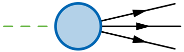
8 loops

[Dixon, Gürdoğan, AJM, Wilhelm (2022)]

# Technology for Multiple Polylogarithms

Having applied bootstrap methods at such high loop orders, we have extremely well-developed technology for working with functions such as multiple polylogarithms

- As an illustration, the 'simplest' quantities that have been bootstrapped are supersymmetric three-particle form factors



$\Rightarrow$ 1,671,656,292 "words"

8 loops

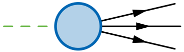
[Dixon, Gürdoğan, AJM, Wilhelm (2022)]

- However, even these form factors become highly nontrivial at high loop orders

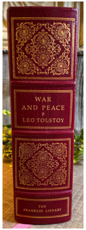
# Technology for Multiple Polylogarithms

Having applied bootstrap methods at such high loop orders, we have extremely well-developed technology for working with functions such as multiple polylogarithms

- As an illustration, the 'simplest' quantities that have been bootstrapped are supersymmetric three-particle form factors



$\Rightarrow$ 1,671,656,292 "words" $\sim 2800 \times$




8 loops

[Dixon, Gürdoğan, AJM, Wilhelm (2022)]

- However, even these form factors become highly nontrivial at high loop orders

# Building Special Functions

We can only work with such large functions because we **directly build them to have the properties we want**

- respect the symmetries of the problem
- expected behavior in special kinematic limits
- logarithmic and algebraic branch cuts that start in physical locations

# Building Special Functions

We can only work with such large functions because we **directly build them to have the properties we want**

- respect the symmetries of the problem
- expected behavior in special kinematic limits
- logarithmic and algebraic branch cuts that start in physical locations

There has in particular been a resurgence of interest—and progress—in understanding the **analytic properties** of scattering amplitudes

# Building Special Functions

We can only work with such large functions because we **directly build them to have the properties we want**

- respect the symmetries of the problem
- expected behavior in special kinematic limits
- logarithmic and algebraic branch cuts that start in physical locations

There has in particular been a resurgence of interest—and progress—in understanding the **analytic properties** of scattering amplitudes

- **Even in single-valued functions**—in which all branch cuts cancel—**a great deal of information is encoded in the analytic structure**

# Landau Analysis

The body of techniques that have developed for studying the analytic structure of Feynman integrals is often referred to as **Landau analysis**.

# Landau Analysis

The body of techniques that have developed for studying the analytic structure of Feynman integrals is often referred to as **Landau analysis**. It address questions such as:

- **where** in the space of masses and external momenta Feynman integrals can become singular and develop branch cuts

# Landau Analysis

The body of techniques that have developed for studying the analytic structure of Feynman integrals is often referred to as **Landau analysis**. It address questions such as:

- **where** in the space of masses and external momenta Feynman integrals can become singular and develop branch cuts
- how Feynman integrals behave near these singular surfaces (for instance, do they develop a **pole**, an **algebraic branch cut**, or a **logarithmic branch cut**)

# Landau Analysis

The body of techniques that have developed for studying the analytic structure of Feynman integrals is often referred to as **Landau analysis**. It address questions such as:

- **where** in the space of masses and external momenta Feynman integrals can become singular and develop branch cuts
- how Feynman integrals behave near these singular surfaces (for instance, do they develop a **pole**, an **algebraic branch cut**, or a **logarithmic branch cut**)
- where specific singularities can appear within **iterated integral representations**

# Landau Analysis

The body of techniques that have developed for studying the analytic structure of Feynman integrals is often referred to as **Landau analysis**. It address questions such as:

- **where** in the space of masses and external momenta Feynman integrals can become singular and develop branch cuts
- how Feynman integrals behave near these singular surfaces (for instance, do they develop a **pole**, an **algebraic branch cut**, or a **logarithmic branch cut**)
- where specific singularities can appear within **iterated integral representations**
- what **sequences of discontinuities** are consistent with causality

# The General Idea

All of the interesting analytic structure that appears in Feynman integrals can be traced back to singularities that occur along the contour of integration

- For instance, if we are interested in studying a function

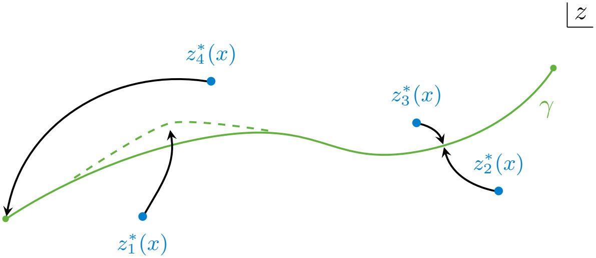
$$f(x) = \int_{\gamma} dz \, \frac{g(x,z)}{h(x,z)}$$

  where $g(x,z)$ and $h(x,z) = (z - z_1^*(x)) \cdots (z - z_n^*(x))$ are polynomials, we can learn a lot from how the points $\{z_i(x)\}$ interact with the contour $\gamma$ as we vary $x$

# The General Idea

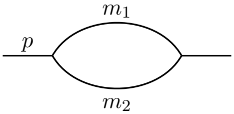
All of the interesting analytic structure that appears in Feynman integrals can be traced back to singularities that occur along the contour of integration

- For instance, if we are interested in studying a function
$$f(x) = \int_\gamma dz \, \frac{g(x,z)}{h(x,z)}$$
where $g(x,z)$ and $h(x,z) = (z - z_1^*(x)) \cdots (z - z_n^*(x))$ are polynomials, we can learn a lot from how the points $\{z_i(x)\}$ interact with the contour $\gamma$ as we vary $x$



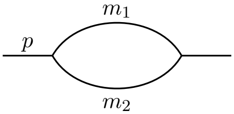
# The Bubble

To see how this technology can be used in practice, let's look at the bubble diagram in two and three dimensions:



# The Bubble

To see how this technology can be used in practice, let's look at the bubble diagram in two and three dimensions:

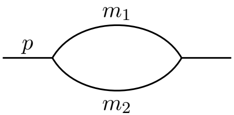



- By solving the **Landau equations**, we can identify all the kinematic surfaces where interesting analytic structure can appear:

$$\{m_1^2 = 0, \quad m_2^2 = 0, \quad p^2 = (m_1 \pm m_2)^2 = r_\pm, \quad p^2 = 0\}$$

# The Bubble

To see how this technology can be used in practice, let's look at the bubble diagram in two and three dimensions:




- By solving the **Landau equations**, we can identify all the kinematic surfaces where interesting analytic structure can appear:

$$\{m_1^2 = 0, \quad m_2^2 = 0, \quad p^2 = (m_1 \pm m_2)^2 = r_\pm, \quad p^2 = 0\}$$

- We can also show that **certain (sequences of) discontinuities are not allowed**

$$\mathrm{Disc}_{p^2=r_-}(I) = 0$$

$$\mathrm{Disc}_{m_1^2=0}\big(\mathrm{Disc}_{m_2^2=0}(I)\big) = \mathrm{Disc}_{m_2^2=0}\big(\mathrm{Disc}_{m_1^2=0}(I)\big) = 0$$

# The Bubble

- Finally, we can predict how the bubble will behave near each of these singular points (using for instance the **method of regions** seen in Andrea's talk)

$$I(m_i^2 \to 0) \sim \begin{cases} \log m_i^2 & \text{in } D = 2 \\ \sqrt{m_i^2} & \text{in } D = 3 \end{cases} \qquad I(p^2 \to 0) \sim \begin{cases} \text{absent} & \text{in } D = 2 \\ 1/\sqrt{p^2} & \text{in } D = 3 \end{cases}$$

$$I(p^2 \to (m_1 \pm m_2)^2) \sim \begin{cases} 1/\sqrt{p^2 - r_\pm} & \text{in } D = 2 \\ \log\left(p^2 - r_\pm\right) & \text{in } D = 3 \end{cases}$$

# The Bubble

- Finally, we can predict how the bubble will behave near each of these singular points (using for instance the **method of regions** seen in Andrea's talk)

$$I(m_i^2 \to 0) \sim \begin{cases} \log m_i^2 & \text{in } D = 2 \\ \sqrt{m_i^2} & \text{in } D = 3 \end{cases} \qquad I(p^2 \to 0) \sim \begin{cases} \text{absent} & \text{in } D = 2 \\ 1/\sqrt{p^2} & \text{in } D = 3 \end{cases}$$

$$I(p^2 \to (m_1 \pm m_2)^2) \sim \begin{cases} 1/\sqrt{p^2 - r_\pm} & \text{in } D = 2 \\ \log\left(p^2 - r_\pm\right) & \text{in } D = 3 \end{cases}$$

These constraints **uniquely determine** the functional form of the bubble integral:

$$I_{\text{2D}} \sim \frac{1}{\sqrt{p^2 - r_+}\sqrt{p^2 - r_-}} \log\left(\frac{\sqrt{p^2 - r_+} + \sqrt{p^2 - r_-}}{\sqrt{p^2 - r_+} - \sqrt{p^2 - r_-}}\right)$$

$$I_{\text{3D}} \sim \frac{1}{\sqrt{p^2}} \log\left(\frac{\sqrt{m_1^2} + \sqrt{m_2^2} + \sqrt{p^2}}{\sqrt{m_1^2} + \sqrt{m_2^2} - \sqrt{p^2}}\right)$$

# Single-Valued Functions

Even in functions in single-valued functions in which all the branch cuts have been hidden, a lot of information can be learned from these techniques

- although these branch cuts have been hidden, the locations and nature of these singular points still control the behavior of this function
- the mechanism for 'hiding' different types of branch cuts are rather different

$\Rightarrow$ **logarithmic branch cuts** can be hidden by adding non-holomorphic contributions

$$\log(x) \to \frac{1}{2}\Big(\log(x) + \log(x^*)\Big)$$

$\Rightarrow$ **square root branch cuts** can be hidden by imposing a Galois symmetry

$$f(x) \xrightarrow{\sqrt{\bullet} \to -\sqrt{\bullet}} f(x)$$

# Single-Valued Functions

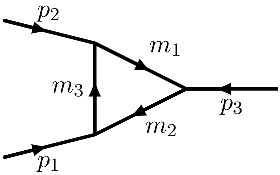
In particular, the locations of where interesting things are happening in the 3D bubble and triangle integrals are dictated by the values of the masses

- the bubble exhibits logarithmic behavior near

$$p^2 = (m_1 \pm m_2)^2$$

- the triangle integral exhibits square-root-type behavior where

$$1 - y_{12}^2 - y_{23}^2 - y_{13}^2 - 2y_{12}\, y_{23}\, y_{13} = 0$$




where

$$y_{ij} = \frac{(p_i + p_j)^2 - m_i^2 - m_j^2}{2m_i m_j}$$

# Single-Valued Functions

In particular, the locations of where interesting things are happening in the 3D bubble and triangle integrals are dictated by the values of the masses

- the bubble exhibits logarithmic behavior near

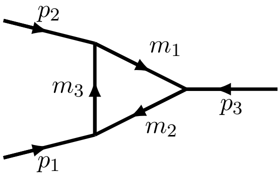
$$p^2 = (m_1 \pm m_2)^2$$

- the triangle integral exhibits square-root-type behavior where

$$1 - y_{12}^2 - y_{23}^2 - y_{13}^2 - 2y_{12}\, y_{23}\, y_{13} = 0$$

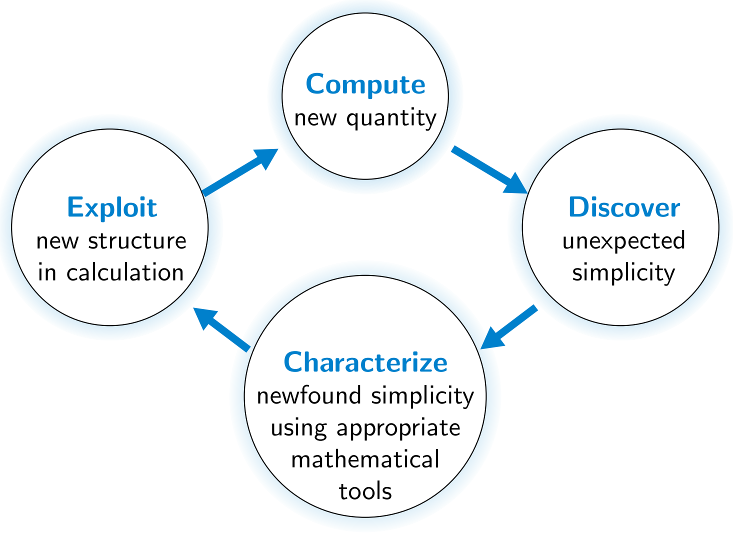
  where

$$y_{ij} = \frac{(p_i + p_j)^2 - m_i^2 - m_j^2}{2m_i m_j}$$

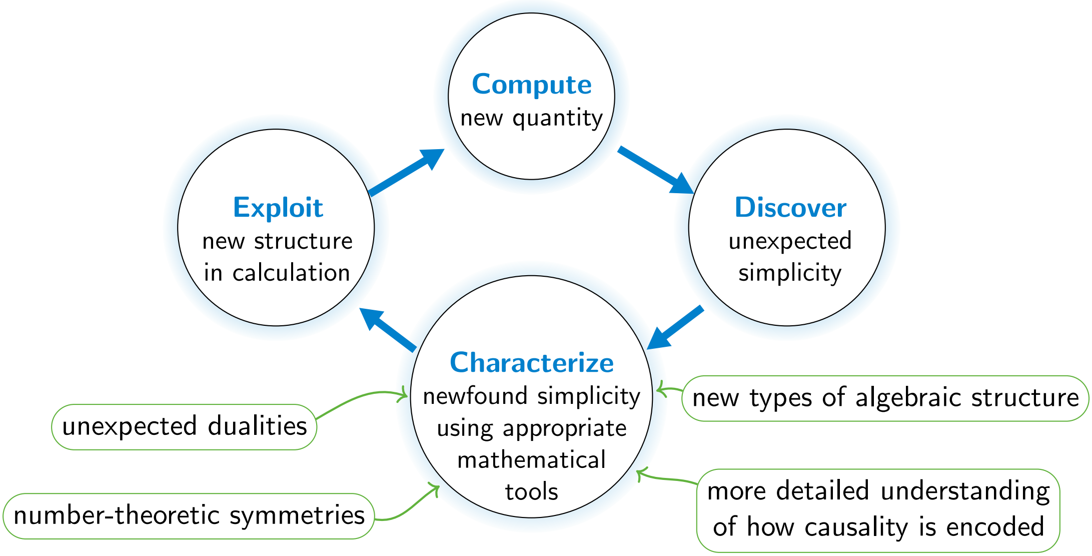



Does anything identifiably special happen at these points (in terms of cosmology) in the basis of integrals that have been used to evaluate the bispectrum of galaxies using the EFTofLSS?

# A Virtuous Cycle



# A Virtuous Cycle



# Conclusions

**A great deal of mathematical and physical structure is hidden** in many of the quantities we are interested in computing in perturbative QFT

- If this structure can be understood, it can sometimes be leveraged to develop **highly efficient computational techniques**
- One key to uncovering this structure is to identify the **right theoretical language**, or special functions, with which to work—and to try to build some of the known properties of the result into these functions directly

Thanks!