Loop soups and solution of O(n) Conformal Field Theories in 2D

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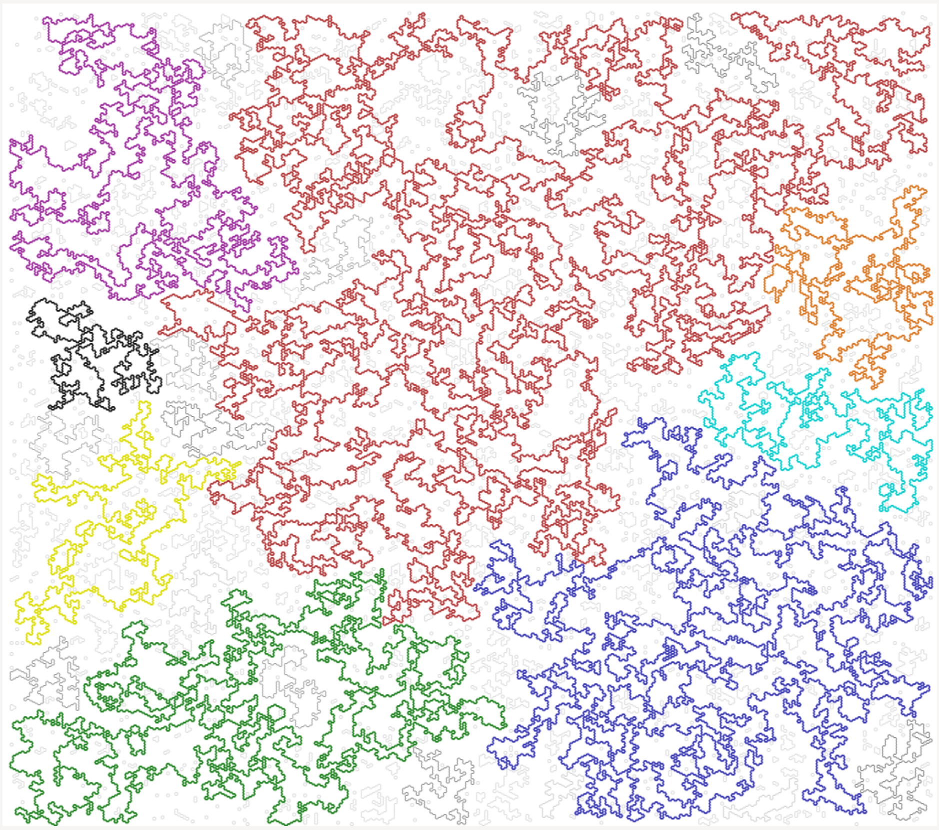
A problem at least 40 years old (Nienhuis, Den Nijs) with many historic contributions (from Symanzik, Brydges-Fröhlich—Spencer to Affleck Seiberg Schwimmer to Dotsenko Fateev to Schramm, Loewner, Werner, Smirnov and many more)

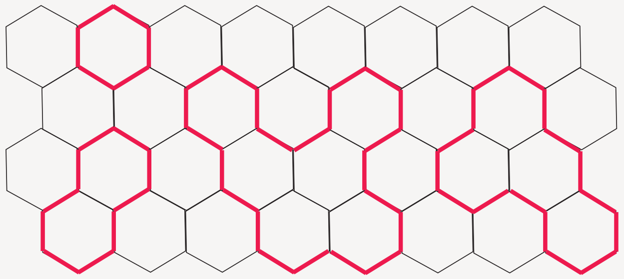
A review based on recent work by the (mostly) Saclay group:
Y.He, L. Grans-Samuelsson, J.L. Jacobsen, R. Nivesvivat, S. Ribault, H. Saleur

based on earlier progress by:
N. Read, H. Saleur; A. Gainutdinov, N. Read, H. Saleur;
B. Estienne, Y. Ikhlef; M. Picco, R. Santachiara;
V. Gorbenko, S. Rychkov, B. Zan;
G. Delfino, J. Viti

Loop soups

(ensembles of self-avoiding mutually avoiding loops
with fugacities per loop and bond)





- Are standard in many problems of statistical physics ($O(n)$ model, Q-state Potts model, disordered free electrons models (plateau transitions...)
- Are a big thing in probability theory (SLE evolution)

While many properties (like critical exponents) have been known for decades, first "phenomenologically" by physicists (Coulomb gas constructions, Bethe-ansatz) (den Nijs, Nienhuis, Dotsenko Fateev, Duplantier Saleur...) then rigorously by mathematicians (W. Werner, S. Smirnov, H. Dominil Copin...)

Understanding of the full CFT (OPEs, 4-point functions etc) has eluded us until very recently a bit of (field theoretic) context....

The $O(n)$ model

- $O(n)$ Landau-Ginzburg model in 2D escapes the Mermin-Wagner theorem : has second-order phase transition for $n < 2$

- Alternatively the NLσM

$$\mathcal{L} = \frac{1}{2g_\sigma^2}\partial_\mu\phi \cdot \partial_\mu\phi \qquad (\phi.\phi = 1)$$

flows to weak coupling for $n < 2$

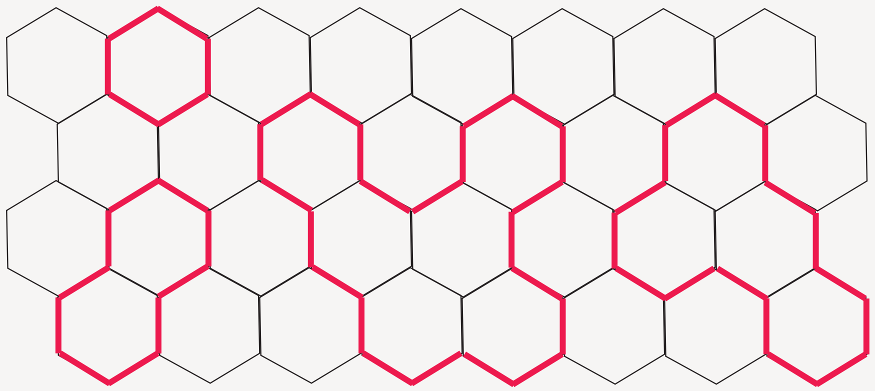
$n = 1$ is Ising (discrete symmetry)

$$\beta(g_\sigma^2) = (n-2)g_\sigma^4 + O(g_\sigma^6)$$

and admits a critical point

- A simple lattice regularization (Affleck, Nienhuis, Schwimmer...)

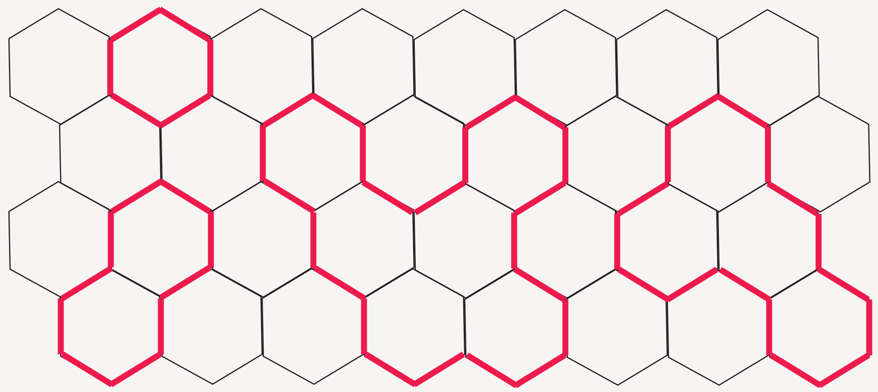
$n = 0$: SAW



n-component vectors $\vec{S}_i$ with $O(n)$ symmetric $\vec{S}_i.\vec{S}_j$ coupling

$$\left(Z \propto \int \prod_i d\vec{S}_i \prod_{<ij>} \left(1 + K\vec{S}_i.\vec{S}_j\right) \right)$$

$$Z = \sum_{\text{dilute loop gas}} K_c^B n^L$$



Dilute loop gas

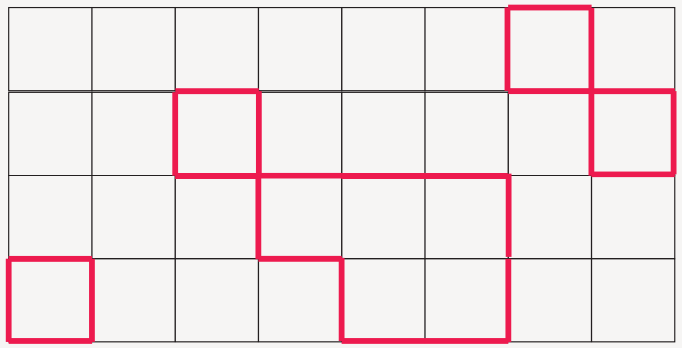
Conformal loop ensemble

Note that n can be extended to $\mathbb{C}$

Related with the Izergin-Korepin spin chain

Very robust universality class:

Crossings don't matter



- The natural questions:

what is the corresponding CFT?
what happens to the continuous symmetry?
what does n non integer mean?

Why things are difficult

- Conformal invariance of local massless field theories in 2D leads to the Hilbert space being a representation of $\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}$$

Critical exponents ($\langle V(z,\bar{z})V(0,0)\rangle = z^{-2h}\bar{z}^{-2\bar{h}}$) are eigenvalues of $L_0, \bar{L}_0$

- Unitarity leads in particular to a full classification and solution of theories with central charge $c < 1$

- Extra symmetries (e.g. SUSY, Z_N) can easily be added to the picture

- 2D CFTs with continuous symmetries are usually described by Wess Zumino Witten (WZW) models (Knizhnik Zamolodchikov, Affleck...)

Charges Q^a give rise to a pair of chiral and antichiral local currents J^a, $\bar{J}^a$ with Kac-Moody algebra commutations

$$\left[J_n^a, J_m^b\right] = f_c^{ab} J^c + \frac{1}{2} k n \delta^{ab}\delta_{m+n}$$

where k is a (usually quantized) anomaly (level)

alas all this breaks down when unitarity is lost

Why is unitarity lost?

The mild non-locality can be traded for genuine locality at the price of introducing complex Boltzmann weights

(Some) Casimirs and dimensions become negative when $n \notin \mathbb{N}^*$

Why is is important?

Relevant Virasoro representation theory is ... wild!

- Only properties like the central charge and (some) critical exponents have been known for a long time. E.g. (Dotsenko Fateev):

$$n \in [-2, 2]; n = 2\cos\frac{\pi}{x}, \;\; x \in [1, \infty)$$

$$c = 1 - \frac{6}{x(x+1)} \;\; \in [-2, 1]$$

In the SLE language $\kappa = \frac{4x}{x+1}, \;\; \kappa = \frac{4(x+1)}{x}$

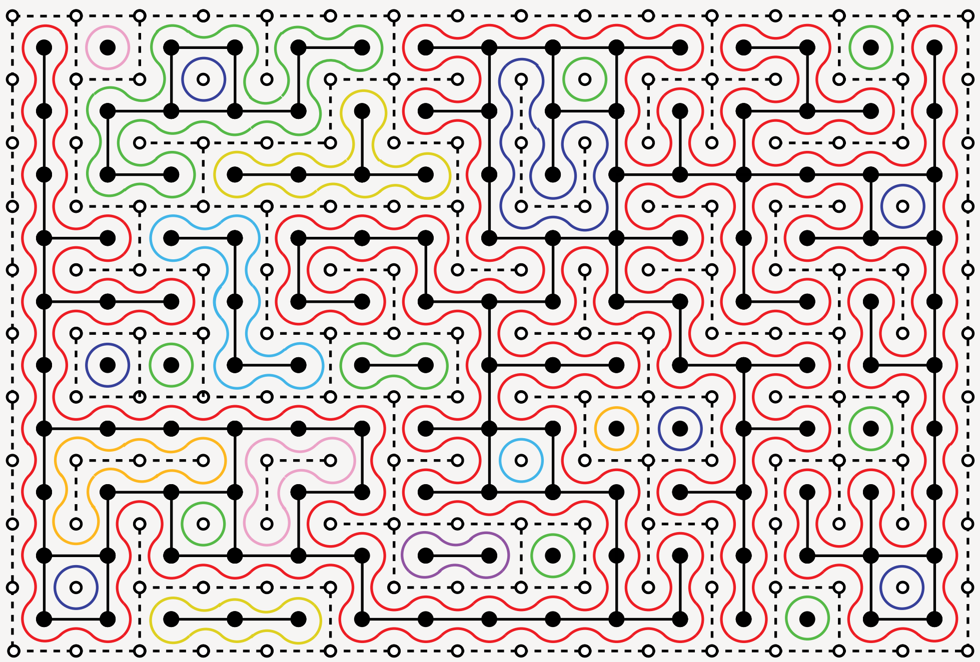
but now the CFTs are fully solved!

Note on a close cousin: the Q-state Potts model

- LG universality class is discrete S_Q symmetry   Second order phase transition for $Q \leq 4$

$$\frac{\mathscr{H}}{kT} = \int d^d x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{1}{3!} q_0 Q_{ijk}\phi_i\phi_j\phi_k + \frac{1}{4!}(u_0 S_{ijkl} + f_0 F_{ijkl})\phi_i\phi_j\phi_k\phi_l \right)$$

- Lattice discretization:



Discrete spins $\sigma = 1, \ldots, Q$ with S_Q invariant $\delta_{\sigma_i \sigma_j}$ coupling

$$Z = \sum_{\text{clusters}} (e^{K_c} - 1)^B Q^C = \sum_{\text{dense loop gas}} \sqrt{Q}^L$$

$Q = 1$: percolation

By duality it is (almost) the same as a dense loop gas   (properties of boundaries vs insides of clusters)

Related with the XXZ spin chain   (And also CP^{m-1}, $m = \sqrt{Q}$, $\theta = \pi$)   (Read Saleur)

bits and pieces of a long story

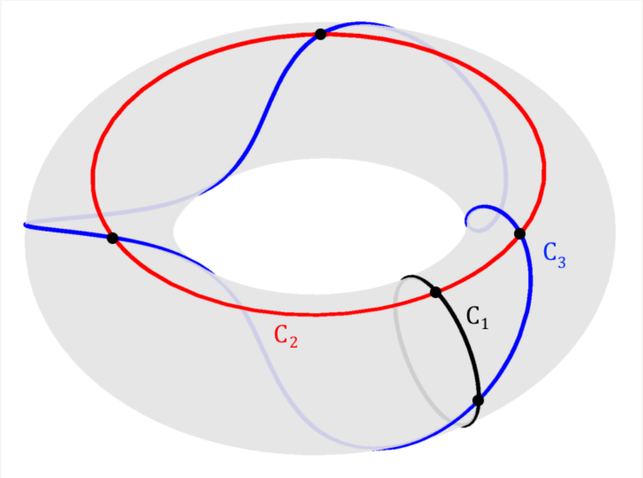
what we have to do: *- identify the fields, their conformal dimensions and their symmetries*
- determine (all) the correlation functions
(this will be achieved if we have determined the OPEs and the thre-point couplings

the Hilbert space of the CFT:

The field (operator) content

- We want to know how to write $\mathcal{H} = \bigoplus O(n) \otimes (\mathrm{Vir}, \overline{\mathrm{Vir}})$
- This can be extracted from the trace $Z = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}$
- Using conformal mappings, this trace can be re-expressed as the torus partition function Cardy 1988
- Z can be calculated using "Coulomb gas" as well as algebraic techniques : particular care has to be taken of non-contractible loops (all loops have weight n irrespective of their topology)
  Branching rules from Brauer to (affine) Temperley-Lieb algebras

  DiFrancesco, Saleur, Zuber 1992; Read Saleur 2007



$q = e^{2i\pi\tau}$, τ the torus modular parameter

- The result should have the form

$$Z = \sum_{h,\bar{h}} \text{degeneracy} \times q^{h-c/24} \bar{q}^{\bar{h}-c/24}$$

The degeneracies should be integer for n integer and in general correspond to (the dimensions of) the irreducible representations of the symmetry

E.g. the order parameter comes with [1] the vector representation

Set $h_{rs} = \dfrac{[(x+1)r - xs]^2 - 1}{4x(x+1)}$

- It turns out there is much more structure

Kac theorem - Virasoro Verma modules are reducible when $r, s \in \mathbb{N}^*$

$$\boxed{Z_{O(n)} = \sum_{s \in 2\mathbb{N}+1} \chi^D_{\langle 1,s\rangle} + \sum_{r \in \frac{1}{2}\mathbb{N}^*} \sum_{s \in \frac{1}{r}\mathbb{Z}} \left(E_{r,s} + \delta_{r,1}\delta_{s\in 2\mathbb{Z}+1}\right) \chi^N_{(r,s)}}$$

$\chi^D_{\langle r,s\rangle},\ r, s \in \mathbb{N}^*$: characters of irreducible diagonal representations $K_{h_{rs}} \otimes \overline{K}_{h_{r,s}}$

$$\chi^D_{\langle r,s\rangle} = q^{h_{rs} - \frac{c}{24}} \frac{1 - q^{rs}}{P(q)} \times \text{h.c.}$$

$\chi^N_{(r,s)}$: characters of non-diagonal potentially reducible representations $V_{h_{rs}} \otimes \overline{V}_{h_{r,-s}}$

$$\chi^N_{(r,s)} = \frac{q^{h_{rs} - \frac{c}{24}}}{P(q)} \times \frac{\bar{q}^{h_{r,-s} - \frac{c}{24}}}{P(\bar{q})}$$

For $r, s \in \mathbb{N}^*$ we have non-vanishing zero norm square states

- As for the degeneracies

(r,s)	$E_{r,s}$
$(\frac{1}{2},0)$	n
$(1,0)$	$\frac{1}{2}(n+2)(n-1)$
$(1,1)$	$\frac{1}{2}n(n-1)$
$(\frac{3}{2},0)$	$\frac{1}{3}n(n^2-1)$
$(\frac{3}{2},\frac{2}{3})$	$\frac{1}{3}n(n^2-4)$
$(2,0)$	$\frac{1}{4}n\left(n^3-3n+2\right)$
$(2,\frac{1}{2})$	$\frac{1}{4}\left(n^4-5n^2+4\right)$
$(2,1)$	$\frac{1}{4}(n-2)n(n+1)^2$
$(2,\frac{3}{2})$	$\frac{1}{4}\left(n^4-5n^2+4\right)$
$(3,0)$	$\frac{1}{6}\left(n^6-6n^4+n^3+11n^2-n-6\right)$

- As for the degeneracies

they correspond to glueings of $O(n)$ representations into larger blocks

(r,s)	$E_{r,s}$	
$(\frac{1}{2},0)$	n	$\Lambda_{(\frac{1}{2},0)} = [1]$
$(1,0)$	$\frac{1}{2}(n+2)(n-1)$	$\Lambda_{(1,0)} = [2]$
$(1,1)$	$\frac{1}{2}n(n-1)$	$\Lambda_{(1,1)} = [11]$
$(\frac{3}{2},0)$	$\frac{1}{3}n(n^2-1)$	$\Lambda_{(\frac{3}{2},0)} = [3] + [111]$
$(\frac{3}{2},\frac{2}{3})$	$\frac{1}{3}n(n^2-4)$	$\Lambda_{(\frac{3}{2},\frac{2}{3})} = \Lambda_{(\frac{3}{2},\frac{4}{3})} = [21]$
$(2,0)$	$\frac{1}{4}n\left(n^3-3n+2\right)$	$\Lambda_{(2,0)} = [4] + [22] + [211] + [2] + []$
$(2,\frac{1}{2})$	$\frac{1}{4}\left(n^4-5n^2+4\right)$	$\Lambda_{(2,\frac{1}{2})} = \Lambda_{(2,\frac{3}{2})} = [31] + [211] + [11]$
$(2,1)$	$\frac{1}{4}(n-2)n(n+1)^2$	$\Lambda_{(2,1)} = [31] + [22] + [1111] + [2]$
$(2,\frac{3}{2})$	$\frac{1}{4}\left(n^4-5n^2+4\right)$	$\Lambda_{(\frac{5}{2},0)} = [5] + [32] + 2[311] + [221] + [11111] + [3] + 2[21] + [111] + [1]$
$(3,0)$	$\frac{1}{6}\left(n^6-6n^4+n^3+11n^2-n-6\right)$	$\Lambda_{(\frac{5}{2},\frac{2}{5})} = [41] + [32] + [311] + [221] + [2111] + [3] + 2[21] + [111] + [1]$

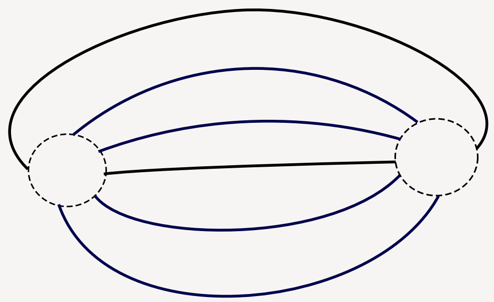
Exact expressions for the $E_{r,s}$ and $\Lambda_{(r,s)}$ are now known

- The $O(n)$ symmetry is global, not LR factorized so this is not a WZW model

- The symmetry is however enhanced (to a non-invertible topological symmetry)

Jacobsen Saleur 2023

- The $\Phi^N_{(r,0)} = \phi_{r,0} \otimes \phi_{r,0}$ are the $2r$ watermelon operators

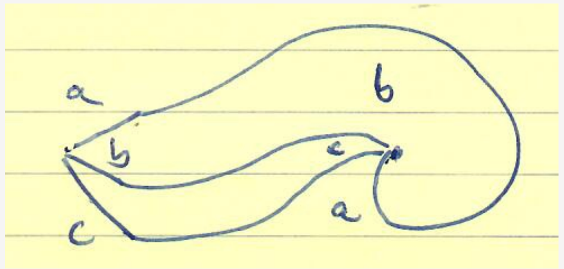


(associated in particular with the $[2r]$ representations)

Duplantier Saleur

- The $\Phi^N_{(r,s)} = \phi_{r,s} \otimes \phi_{r,-s}$ are the $2r$ watermelon operators where an elementary cyclic permutation of the $2r$ lines around an extremity gains a phase $e^{i\pi s}$ $(2rs = 0 \text{ mod } 2)$.

- Not every $O(n)$ Young diagram gives rise to a different primary field
  This is because in 2D not all tensors can be realized without crossings
  For instance, $\square\square\square \equiv \begin{array}{c}\square\\\square\\\square\end{array}$



- Recall the model with crossings flow anyway to the same CFT as the model without

Virasoro representations and "ghosts"

- The appearance of $\chi^D_{\langle 1,s\rangle}$ shows that $\Phi^D_{\langle 12\rangle} = \phi_{12} \otimes \overline{\phi}_{12}$ is degenerate, ie it has a descendent at level two that vanishes indeed

  Feigin Fuchs, Kac

  ϕ_{12} having conformal weight h_{12}, we know that it has a zero (Virasoro) norm descendent at level 2:

  $$\langle \phi_{12} A^\dagger_{12} | A_{12} \phi_{12} \rangle = 0 \qquad \left(\text{where } A_{12} \equiv L_{-2} - \frac{3}{2+4h_{12}} L^2_{-1}\right)$$

  Using this kind of property is the essence of the BPZ strategy to determine four-point functions and solve the theory

  Belavin, Polyakov, Zamolodchikov

  In the $O(n)$ CFT a (very) few four-point functions (essentially, the energy) can indeed be determined this way

- But these are the exception, not the rule

  The theory being non-unitary, the Virasoro norm is not positive definite and there are (infinitely many) null states which are not vanishing

Logarithmic CFT:

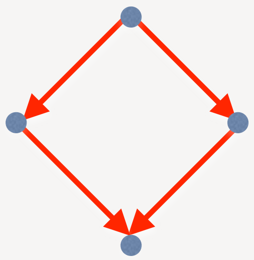
Generic logarithmic structure

For all the other ϕ_{rs} in the theory (r, s integer, $r > 1$) the null descendents are indeed non zero

and involved in rank two Jordan blocks $\left(Z_{O(n)} = \sum_{s \in 2\mathbb{N}+1} \chi_{\langle 1,s \rangle} + \sum_{r \in \frac{1}{2}\mathbb{N}^*} \sum_{s \in \frac{1}{r}\mathbb{Z}} (E_{r,s} + \delta_{r,1}\delta_{s \in 2\mathbb{Z}+1})\, \chi^N_{(r,s)} \right)$

- $$\begin{tikzcd}
 & \Psi_{(r,s)} \arrow[dl, "A^\dagger"'] \arrow[dr, "\bar{A}^\dagger"] \arrow[dd, "(L_0 - h_{r,-s})"] & \\
\Phi^N_{(r,s)} = \phi_{r,s} \otimes \bar{\phi}_{r,-s} \arrow[dr, "A"'] & & \bar{\Phi}^N_{(r,s)} = \phi_{r,-s} \otimes \bar{\phi}_{r,s} \arrow[dl, "\bar{A}"] \\
 & A_{r,s}\Phi^N_{(r,s)} = \bar{A}_{r,s}\bar{\Phi}^N_{(r,s)} &
\end{tikzcd}$$

Loewy diagram



"One-way" Virasoro actions

Top and bottom fields have $h = \bar{h} = h_{r,-s} = h_{r,s} + rs$
$A, \bar{A}$ are combinations of Vir ($\overline{\text{Vir}}$) producing null states

- A simple example: (see also Gorbenko Zan 2020)

$$\bar{\partial} J^a \neq 0$$
$$\partial \bar{J}^a \neq 0$$

but both have zero norms square

Exact solution via the bootstrap (focus on 4 point functions)

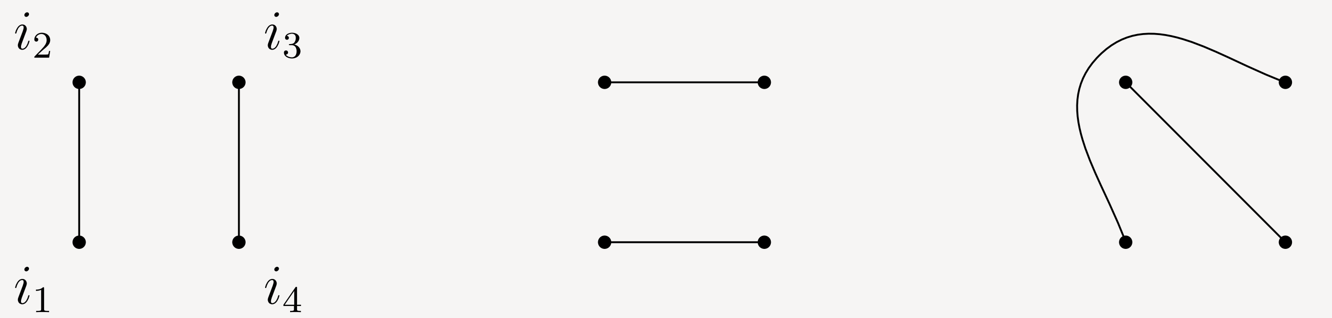
- The order operator $\vec{S}$ transforms in [1] and creates an extra open line in the lattice model
- Its dimension is well known to be $h_{1/2,0} = \frac{(x-1)(x+3)}{16x(x+1)}$. Nienhuis 1980's
- From $[1]^{\otimes 2} = [] \oplus [11] \oplus [2]$ we get the tensor structure

$$\langle V^{i_1}_{(\frac{1}{2},0)} V^{i_2}_{(\frac{1}{2},0)} V^{i_3}_{(\frac{1}{2},0)} V^{i_4}_{(\frac{1}{2},0)} \rangle = T^{O(n)}_{[]} A^{(s)}_{[]} + T^{O(n)}_{[11]} A^{(s)}_{[11]} + T^{O(n)}_{[2]} A^{(s)}_{[2]}$$

$$\begin{aligned}
T^{O(n)}_{[]} &= \delta_{i_1 i_2}\delta_{i_3 i_4} \ , \\
T^{O(n)}_{[11]} &= \delta_{i_1 i_4}\delta_{i_2 i_3} - \delta_{i_1 i_3}\delta_{i_2 i_4} \ , \\
T^{O(n)}_{[2]} &= \delta_{i_1 i_3}\delta_{i_2 i_4} + \delta_{i_1 i_4}\delta_{i_2 i_3} - \frac{2}{n}\delta_{i_1 i_2}\delta_{i_3 i_4}
\end{aligned}$$

- It can be reinterpreted in terms of diagrams (recall $\vec{S}$ creates a line)

$$\langle V^{i_1}_{(\frac{1}{2},0)} V^{i_2}_{(\frac{1}{2},0)} V^{i_3}_{(\frac{1}{2},0)} V^{i_4}_{(\frac{1}{2},0)} \rangle = \delta_{i_1 i_2}\delta_{i_3 i_4} C_1 + \delta_{i_2 i_3}\delta_{i_1 i_4} C_2 + \delta_{i_1 i_3}\delta_{i_2 i_4} C_3$$



$\delta_{i_1 i_2}\delta_{i_3 i_4} C_1$ $\delta_{i_2 i_3}\delta_{i_1 i_4} C_2$ $\delta_{i_1 i_3}\delta_{i_2 i_4} C_3$

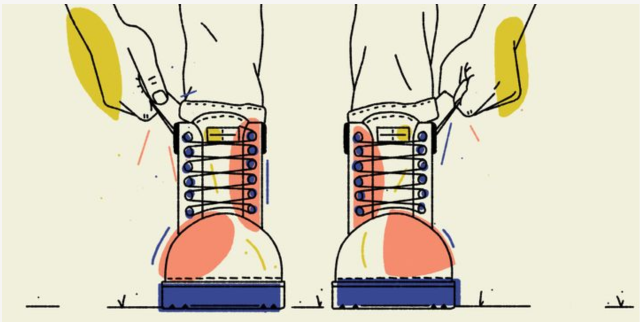
with

$$C_1 = A^{(s)}_{[]} - \frac{2}{n} A^{(s)}_{[2]} \quad , \quad C_2 = A^{(s)}_{[2]} + A^{(s)}_{[11]} \quad , \quad C_3 = A^{(s)}_{[2]} - A^{(s)}_{[11]}$$

- The bootstrap program

Regge, Mandelstamm, Polyakov, BPZ, El-Showk, Rychkov,...

Ferrara, Gatto, Parisi

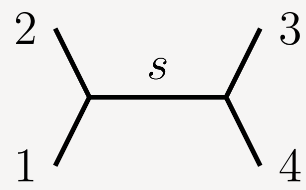


different values of the anharmonic ratio

Expand the four point function onto conformal blocks

$$\sum_{(\Delta,\bar{\Delta})\in\mathcal{S}^{(k)}} \mathcal{A}^{(k)}_{\Delta,\bar{\Delta}} \mathcal{F}^{(k)}_{\Delta}(\{z_i\}) \mathcal{F}^{(k)}_{\bar{\Delta}}(\{\bar{z}_i\}), \quad k \in \{s,t,u\}$$

channel	limit
s	$z_1 \to z_2$
t	$z_1 \to z_4$
u	$z_1 \to z_3$



$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

The unknowns are the values of Δ, Δ i.e. the spectrum.
The $\mathcal{F}^{(k)}_{\Delta}$ are determined from general principles as functions of c, Δ and the external weights h

Zamolodchikov

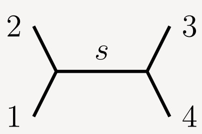
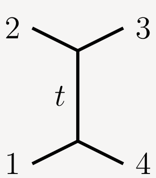
If the spectrum is known and identical for several channels consistency conditions can be written, e.g.

$$\mathcal{F}^{(t)}_{\Delta}(z) = \mathcal{F}^{(s)}_{\Delta}(1-z)$$

$$\sum_{(\Delta,\bar{\Delta})\in\mathcal{S}} \mathcal{A}_{\Delta,\bar{\Delta}} \left(\mathcal{F}^{(s)}_{\Delta}(\{z_i\}) \mathcal{F}^{(s)}_{\bar{\Delta}}(\{\bar{z}_i\}) - \mathcal{F}^{(t)}_{\Delta}(\{z_i\}) \mathcal{F}^{(t)}_{\bar{\Delta}}(\{\bar{z}_i\}) \right) = 0$$

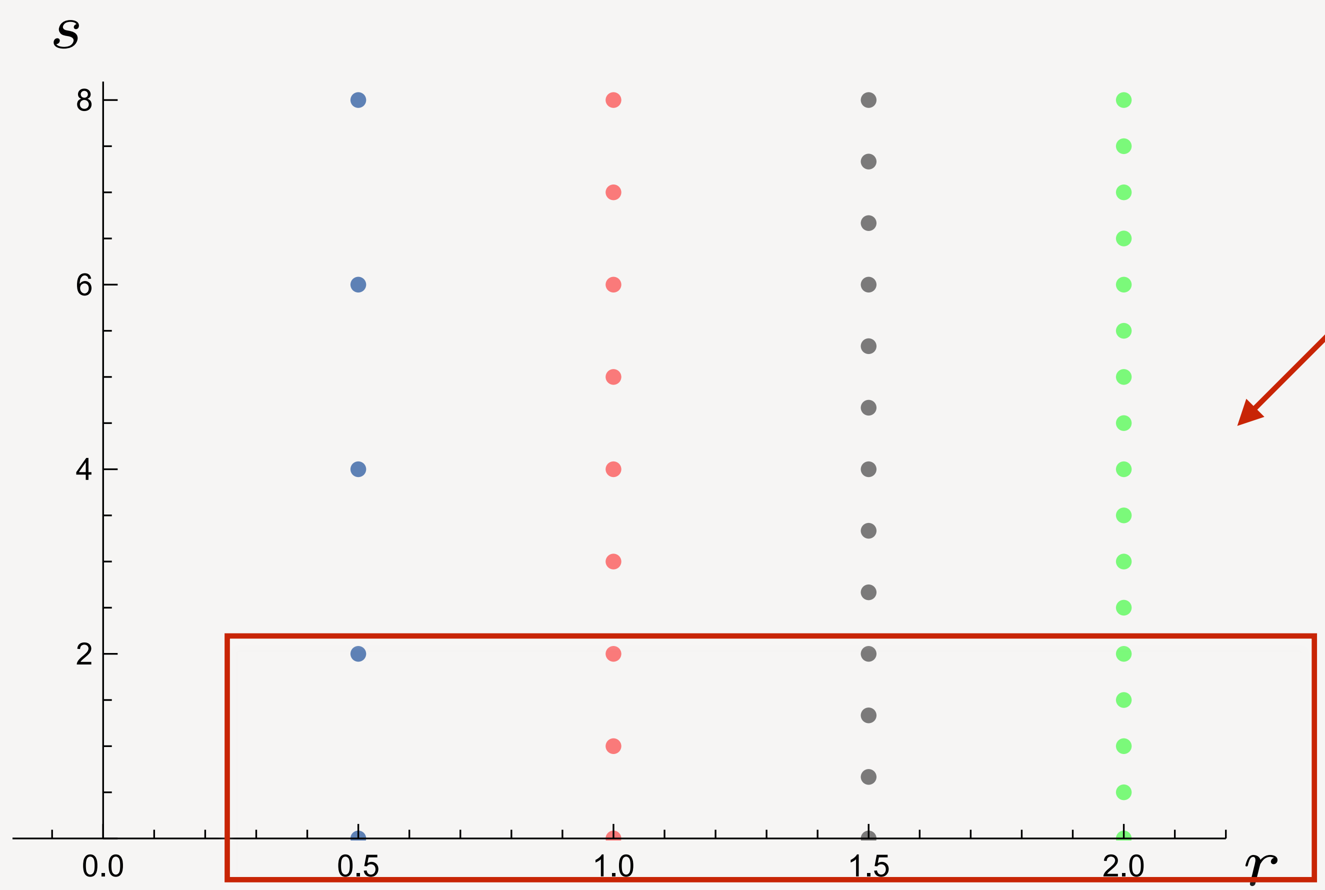
Solving numerically for a range of values of z determines the amplitudes

(of course this only works if the spectrum is consistent)

$\sum_{\Delta_s \in S} C_{12s} C_{s34}$  $= \sum_{\Delta_t \in S} C_{23t} C_{t41}$  .

- The spectrum is a priori (and in fact) very rich

$$\mathcal{S} = \left\{(r,s)^N; r \in \mathbb{N}^*/2, s \in \mathbb{Z}/r\right\} \cup \left\{\langle 1, 1+2\mathbb{N}\rangle^D\right\}$$



The non diagonal part has s values dense on the axis

- The fact that $\Phi^D_{(1,3)}$ is truly degenerate means the fusion rule

$$\phi_{13}\phi_{rs} = \phi_{r,s+2} + \phi_{rs} + \phi_{r,s-2}$$

is obeyed for all r, s

so matters somehow simplify and conformal blocks can be partly resummed
into interchiral conformal blocks (Gainutdinov Read Saleur 2018)

- However Φ_{21} is not degenerate (contrast with Liouville at $c < 1$) so only a numerical approach is possible

It can in fact be carried out to arbitrary accuracy, despite the large number of fields
Amazingly, the numbers can be fitted by (complicated) formulas, suggesting exact solvability

- We have also carried out a lattice bootstrap, measuring amplitudes of four-point functions directly on the lattice using transfer matrices (and sometimes Bethe-ansatz) The agreement is perfect.

- The same analysis can be carried out for all four-point functions. Some interesting interplay with $O(n)$ symmetry

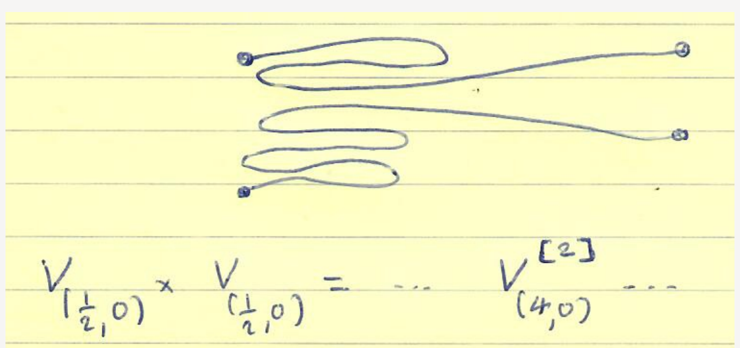
- Using the interchiral blocks define

$$\begin{aligned}
\mathcal{S}_{[]} &= \{(r,s)^N; r \in \mathbb{N}, s \in (2\mathbb{Z})/r \cap [-1,1)\} \cup \{\langle 1,1\rangle^D\} \\
\mathcal{S}_{[2]} &= \{(r,s)^N; r \in \mathbb{N}, s \in (2\mathbb{Z})/r \cap [-1,1)\} \\
\mathcal{S}_{[11]} &= \{(r,s)^N; r \in \mathbb{N}, s \in (2\mathbb{Z}+1)/r \cap [-1,1)\}
\end{aligned}$$

We have the OPE

$$V^{[1]}_{(1/2,0)^N} \times V^{[1]}_{(1/2,0)^N} = \sum_{k \in \mathcal{S}_{[]}} V^{[]}_k + \sum_{k \in \mathcal{S}_{[11]}} V^{[11]}_k + \sum_{k \in \mathcal{S}_{[2]}} V^{[2]}_k$$

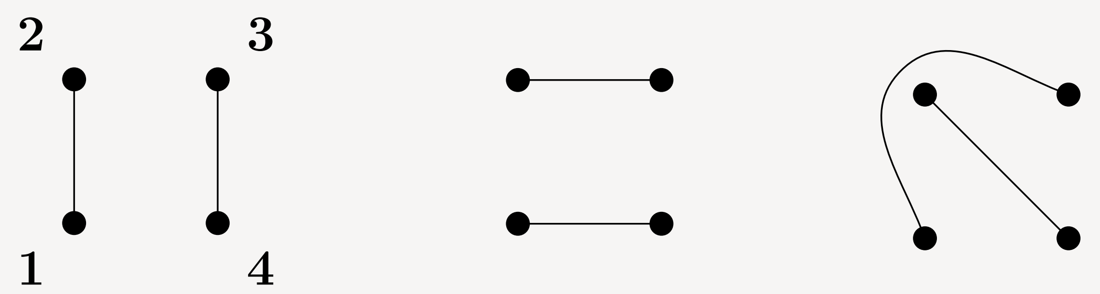
Generalized energy operators



Watermelon operators with even number of legs and winding phases compatible with the symmetry

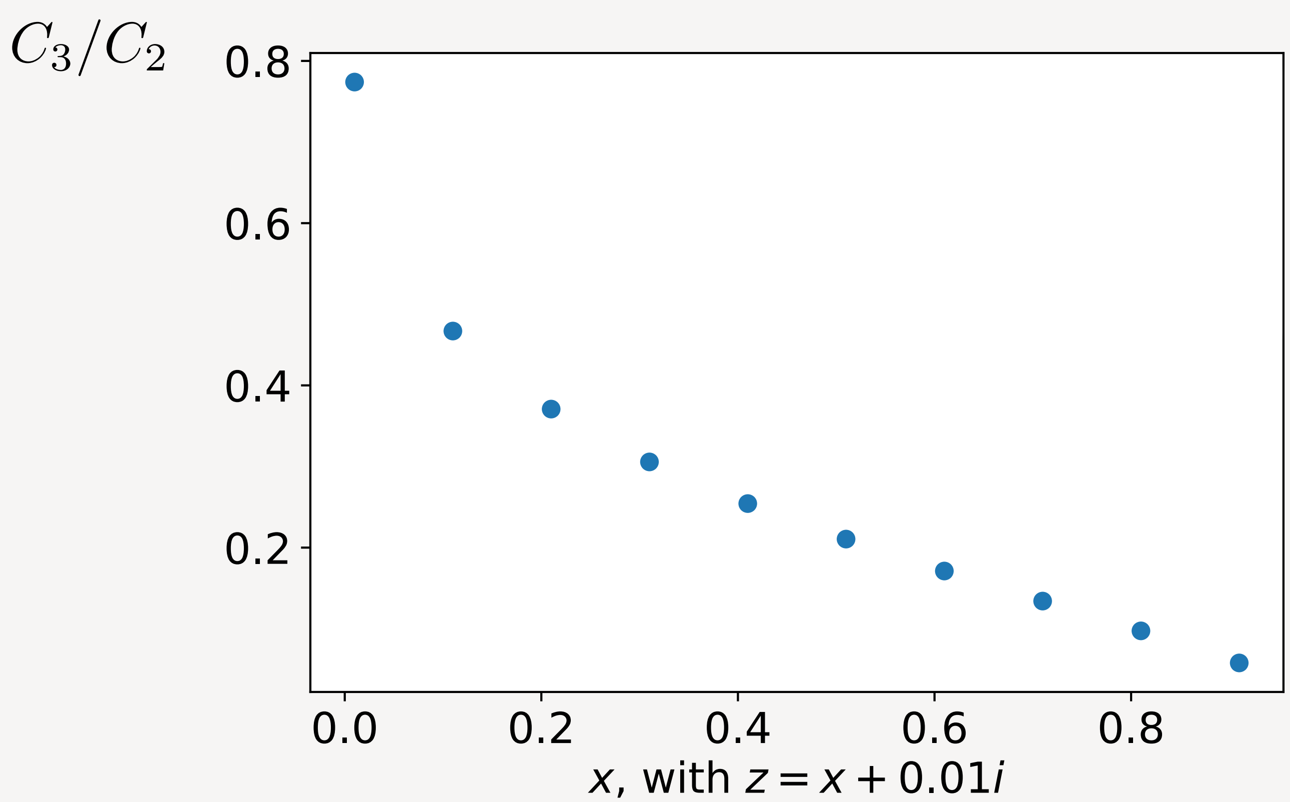
The CFT is not rational and not quasi-rational

Bootstrap for SAW ($n \to 0$)



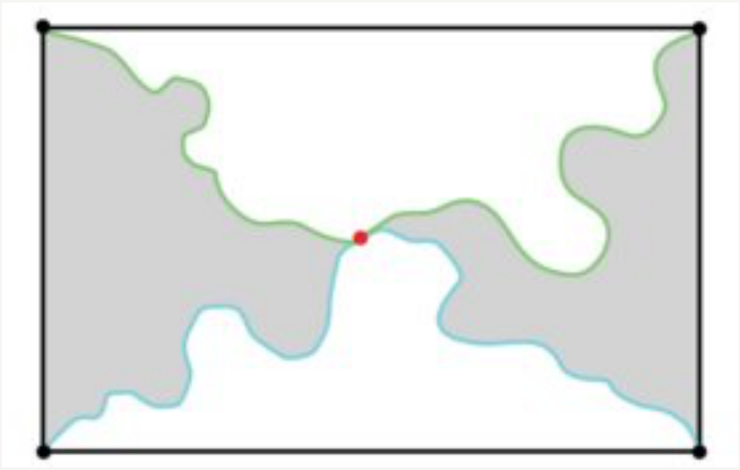
C_1 C_2 C_3

Applications in probability theory



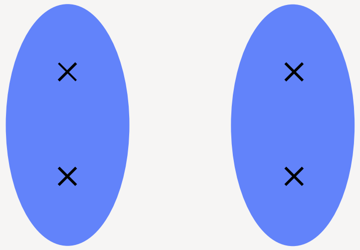
$$z = \frac{z_{12} z_{34}}{z_{13} z_{24}}$$

The same can be done for Q-state Potts

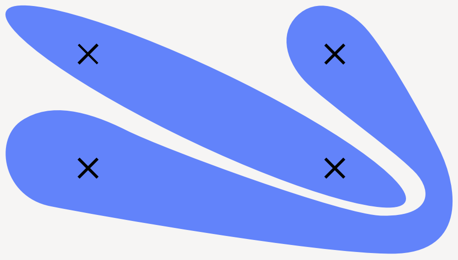


- Similar kind of glueings
- There no currents (S_Q is discrete)
- Now it is $\Phi_{\langle 21 \rangle^D}$ that is exactly degenerate
- The $\Phi^N_{(0,s)}$ are the $2s$ cluster boundaries (hull) operators
- In particular the following 4-point functions

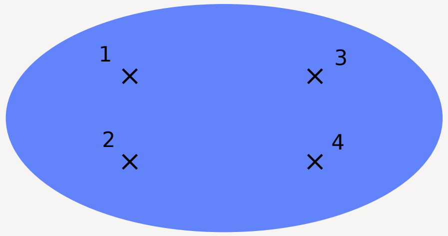
D_{abab}



D_{aabb}



D_{abba}



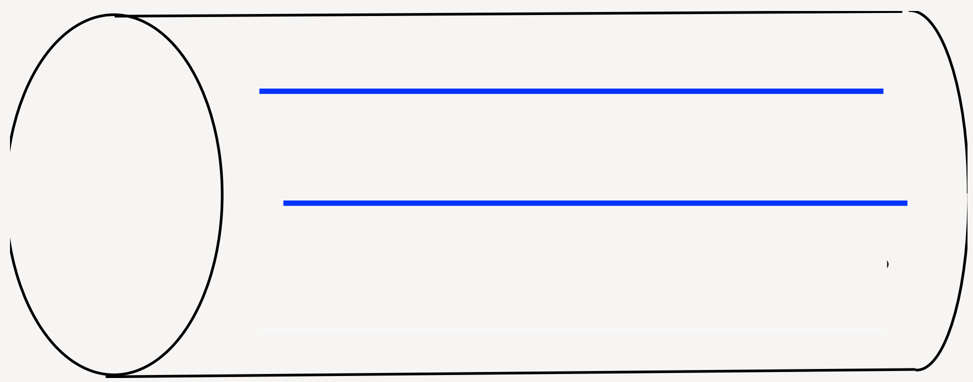
D_{aaaa}

can be determined exactly

A note on lattice techniques

- The spectrum and amplitudes can be determined by calculating C_2 (for instance) and tackling the inverse problem

Jacobsen Saleur



$$w_1 = ia,\ w_2 = -ia$$
$$w_3 = i(a+x)+l,\ w_4 = i(-a+x)+l$$

The C_2 etc are expanded on eigenvalues of the transfer matrix for a large set of w_i coordinates. By solving the inverse problem, we determine which of the eigenvalues actually contribute, and with which amplitude. We do this for a variety of sizes, and extrapolate to the continuum limit. Note: the number of eigenvalues is very large (in the thousands).

$$C_2 \propto \sum_{h,\bar{h}\in\mathcal{S}} C_{\sigma\sigma\Phi_{h,\bar{h}}} C_{\Phi_{h,\bar{h}}\sigma} \left(4\sin^2\frac{2\pi a}{L}\right)^{h+\bar{h}} (-1)^{h-\bar{h}} \xi^h \bar{\xi}^{\bar{h}} (1+O(\xi,\bar{\xi}))$$

with $\xi \equiv e^{-2\pi(l+ix)/L},\ \bar{\xi} \equiv e^{-2\pi(l-ix)/L}$

the usual contribution from transfer matrix eigenvalues : $\lambda^l e^{-iPx}$

$$\lambda = \exp\left[-2\frac{\pi}{L}(h+\bar{h})\right]$$

$$P = \frac{2\pi}{L}(h-\bar{h}) \in \mathbb{Z}$$

the amplitude corrected by logarithmic mapping

- Algebraic considerations (affine Temperley-Lieb algebra and the like) are crucial

(V. Jones, Ram, Martin, Graham Lehrer, Read Saleur, Jacobsen Saleur, Estienne Ikhlef, Morin-Duchesne Ikhlef)

- Action of Virasoro can be studied using discretizations of the L_n's

(Koo Saleur 1995, Vidal et al., Zini Wang)

Conclusions and summary

- Non-unitarity precludes these models to be WZW theories. There are “currents” but they are not purely chiral (e.g. the OPE $J^a(z)J^b(0)$ might contain some $\bar{z}$ dependency)
- In fact the currents do not seem to play much a role, and there is no qualitative difference between $O(n)$ and S_Q
- The symmetry is larger than $O(n)$ or S_Q. Non-invertible topological lines seem to play an important role.
- The spectrum is very rich, with a dense set of values of one Kac label. The theories are neither rational nor quais-rational
- One type of field (depending on $O(n)$ or S_Q) $\Phi^D_{\langle 1,r\rangle}$ or $\Phi^D_{\langle r,1\rangle}$ is truly degenerate but not both.
- There are many fields with integer Kac labels which are degenerate and involved in rank two Jordan blocks
- When x is a root of unity, Jordan blocks of higher rank (in fact, arbitrarily high) appear
- Four-point functions can be accurately determined using the bootstrap. They are regular as a function of x.
- There are strong indications (e.g. exact amplitude ratios) that these theories are analytically solvable

So this is almost (for physicists) the end of a story started more than 35 years ago

As for future directions

- For the more mathematically oriented: rigorous construction of these CFTs, relationship with SLE, categorical interpretation of $O(n)$ symmetry when $n \in \mathbb{C}$ (Binder Rychkov)

- It is not clear how these features will help solve more complicated models such as those describing plateau transitions: our understanding of non-compactedness in CFT is still insufficient

- And so is our understanding of the landscape of non-unitary CFTs (how many spectra lead to meaningful CFTs? how are the RG flows?)

- There are probably lessons about 3D to be learned from this

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Thank you!