Large-scale geometry and quantization of Hall conductance

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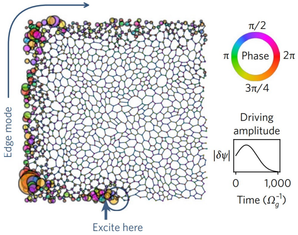
Outline

- Quantum Hall effect is large-scale spectral-geometric phenomenon exhibited by electrons coupled to gauge field.
- "Topological?" Does not care about small-scale holes, bumps, lattice vs continuum...
- **Today:** General quantization of conductance via coarse locality principle/index theory on any sample geometry.
- Is "topology" really needed to quantize σ_{Hall}?
- BEC for bounded sample will also be discussed.

Based on arXiv:2307.xxxxx with M. Ludewig (Regensburg)

Exhibit 1: Amorphous phenomenon

Small-scale structure and homogeneity unimportant:



N. Mitchell et al, Nature Phys. (2018)

Exhibit 2: Geometric curvatures?

Precision of quantization ($\sim 10^{-9}$) far exceeds flatness of laboratory sample, or uniformity of magnetic field.

Are 2D Interfaces Really Flat?

Zhihui Cheng,* Huairuo Zhang, Son T. Le, Hattan Abuzaid, Guoqing Li, Linyou Cao, Albert V. Davydov, Aaron D. Franklin,* and Curt A. Richter*

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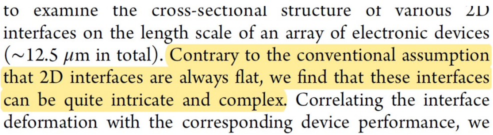


to examine the cross-sectional structure of various 2D interfaces on the length scale of an array of electronic devices (~12.5 μm in total). Contrary to the conventional assumption that 2D interfaces are always flat, we find that these interfaces can be quite intricate and complex. Correlating the interface deformation with the corresponding device performance, we

- How to explain experimental quantization of conductance in (very) non-Euclidean geometry?
- Can we justify “geometry-free” effective topological field theory?

I. Traces of commutators

Trace-class ideal

A Hilbert space operator S is trace class, if for a(ny) O.N.B. $\{e_i\}_i$,

$$\sum_i \langle e_i, \sqrt{S^* S} e_i \rangle < \infty. \qquad \text{(sum singular values)}$$

$$\rightsquigarrow \operatorname{Tr}(S) := \sum_i \langle e_i, S e_i \rangle \ \in \mathbb{C}.$$

Lidskii: ST and TS trace class $\Rightarrow \operatorname{Tr}[S, T] = 0$.

Examples:

- Smooth integral kernel operator on $L^2(M_{\mathrm{cpt}})$.
- Operator on $L^2(\mathbb{R})$ with Schwartz class integral kernel.
- Rapid decay kernels $\rightsquigarrow$ *locally* trace class.

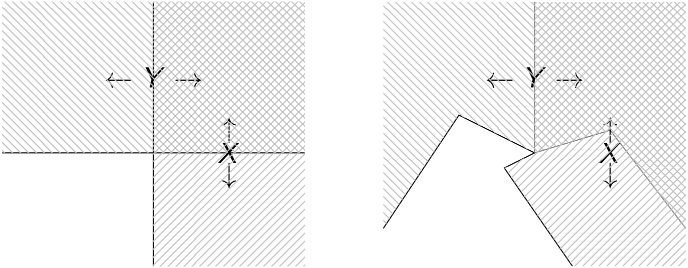
Trace and quantum theory

- Bounded operators B (observables) are continuously dual to trace class operators (states):
$$\langle B\rangle_\rho = \mathrm{Tr}(\rho B).$$
- Locality structure: metric measure space M, subsets $A \subset M$ act as multiplication-by-χ_A on $L^2(M)$.
- Laplacians, gauge fields, unitary gauge transformations etc.
- Local Hamiltonian H gives energy spectrum.
Fermi energy: Dirac-sea vacuum/∞-fermion ground state.
- Fermi projection P not trace class, yet it has "renormalizable observables".

Main result (L+T '23)

$M =$ metric space. For any projection $P = P^* = P^2$ on $L^2(M;\mu)$ with rapid-decay kernel, and any "coarsely transverse" half-spaces $X, Y \subset M$,

$$\boxed{2\pi i \cdot \mathrm{Tr}[PXP, PYP] \ \in \ \mathbb{Z}}$$



"Physics proof": Quantum Hall effect

Maths: Coarse pairing of P with partition.

Commutator-trace is a "small-scale filter"

Write $P_X = PXP$ and $P_Y = PYP$.

Generically, $\mathrm{Tr}[P_X, P_Y] = 0$:

- P supported within X or Y; or X or Y compact.
- P is real.

$\mathrm{Tr}[P_X, P_Y] \neq 0$ requires:

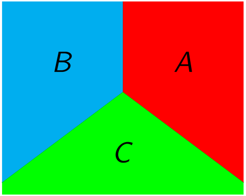
- P breaks time-reversal and orientation-reversal symmetry.
- P is supported on "all of M" and "delocalized" (e.g. Wannier sense, [L+T, JMP '22]).
- $P_X P_Y$ and $P_Y P_X$ *not* trace class. So

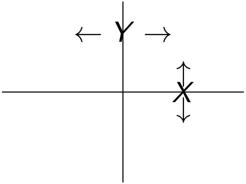
$$\mathrm{Tr}[P_X, P_Y] = \text{``}\infty - \infty\text{''} = ??$$

Hall conductance

$$[P_X, P_Y] = P\big[[X, P], [Y, P]\big]$$

- Current $Y^c \to Y$ in response to electric potential $X^c \to X$. (e.g. Elgart–Schlein '03).
- Adiabatic curvature, Kubo formula. . .



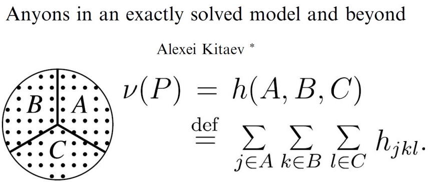


$$\mathrm{Tr}[P_X, P_Y] = 2 \cdot \underbrace{\mathrm{Tr}[P_A, P_B]}_{\text{Kitaev ``2-current''}}$$

- Response to a magnetic flux at intersection [Mitchell '18].

II: Coarse viewpoint

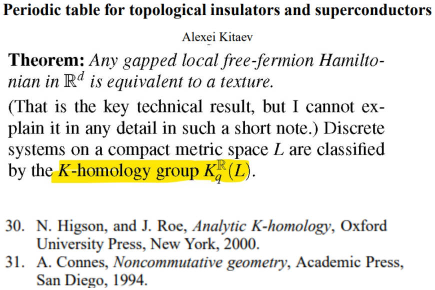
2000s: Coarse geometry was anticipated (A. Kitaev)



Anyons in an exactly solved model and beyond

Alexei Kitaev *

$$\nu(P) = h(A,B,C) \overset{\text{def}}{=} \sum_{j\in A}\sum_{k\in B}\sum_{l\in C} h_{jkl}.$$



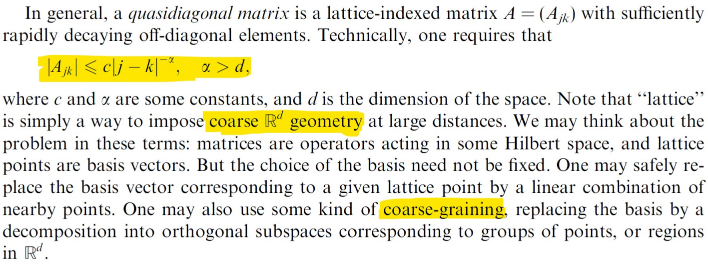
Periodic table for topological insulators and superconductors

Alexei Kitaev

Theorem: *Any gapped local free-fermion Hamiltonian in $\mathbb{R}^d$ is equivalent to a texture.*

(That is the key technical result, but I cannot explain it in any detail in such a short note.) Discrete systems on a compact metric space L are classified by the K-homology group $K_q^{\mathbb{R}}(L)$.

30. N. Higson, and J. Roe, *Analytic K-homology*, Oxford University Press, New York, 2000.
31. A. Connes, *Noncommutative geometry*, Academic Press, San Diego, 1994.

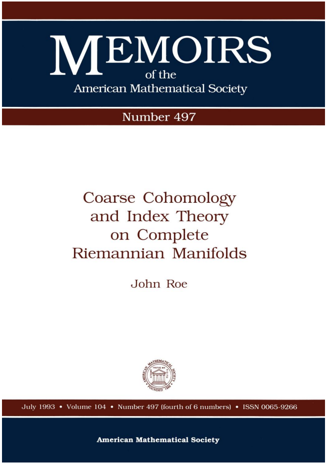


In general, a *quasidiagonal matrix* is a lattice-indexed matrix $A = (A_{jk})$ with sufficiently rapidly decaying off-diagonal elements. Technically, one requires that

$$|A_{jk}| \leqslant c|j-k|^{-\alpha}, \quad \alpha > d,$$

where c and α are some constants, and d is the dimension of the space. Note that "lattice" is simply a way to impose coarse $\mathbb{R}^d$ geometry at large distances. We may think about the problem in these terms: matrices are operators acting in some Hilbert space, and lattice points are basis vectors. But the choice of the basis need not be fixed. One may safely replace the basis vector corresponding to a given lattice point by a linear combination of nearby points. One may also use some kind of coarse-graining, replacing the basis by a decomposition into orthogonal subspaces corresponding to groups of points, or regions in $\mathbb{R}^d$.

Coarse cohomology, geometry, and index (J. Roe)



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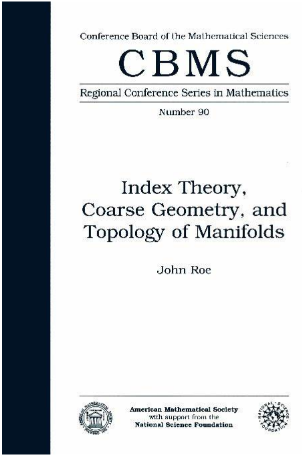
Coarse Cohomology
and Index Theory
on Complete
Riemannian Manifolds

John Roe

July 1993 • Volume 104 • Number 497 (fourth of 6 numbers) • ISSN 0065-9266

American Mathematical Society



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Index Theory,
Coarse Geometry, and
Topology of Manifolds

John Roe

American Mathematical Society
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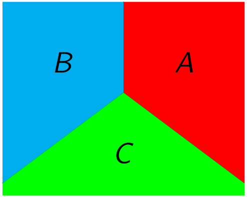
Finite propagation method, Dirac's unit speed of propagation.

Coarse ingredients:

$$\text{2-partition}: \qquad B_r(A) \cap B_r(B) \cap B_r(C) \quad \text{bounded} \ \ \forall r > 0.$$



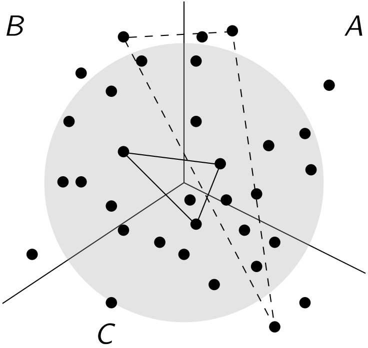
Dually: algebra $\mathscr{B}_{\text{fin}}(M)$ of operators L on $L^2(M)$ satisfying:

- finite propagation: $\exists r > 0$ such that $ALB = 0$ whenever $\text{dist}(A, B) > r$.
- locally trace class: AL and LA trace class whenever A bounded.

Partition-projection duality pairing

Coarse partitions pair with projections in $\mathscr{B}_{\mathrm{fin}}(M)$.

$$\langle A, B, C; P\rangle := \mathrm{Tr}(\underbrace{APBPCP}_{\text{trace class}} + \text{antisymm}) = \ldots = \mathrm{Tr}[P_A, P_B].$$

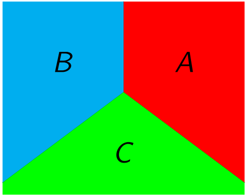


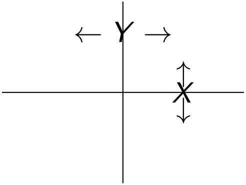
Oriented sum of “loop amplitudes”; large loops suppressed.

Quantization of pairing

"Coarse cobordism invariance" argument gives

$$2 \cdot [P_A, P_B] = [P_X, P_Y] \qquad \text{up to traceless term.}$$





- $P_X - P_X^2 = PXPX^cP$ is supported near $X \cap X^c$.
- So $(P_X - P_X^2)(P_Y - P_Y^2)$ is supported near intersection point, thus trace class.
- Conditions of abstract quantization theorem (next slide) hold.

Abstract functional analytic quantizations

- [L+T'23]: If projections P and X, Y satisfy

$$[P_X, P_Y] \quad \text{and} \quad (P_X - P_X^2)(P_Y - P_Y^2) \qquad \text{trace class},$$

then:

$$2\pi i \cdot \mathrm{Tr}[P_X, P_Y] \in \mathbb{Z}.$$

Compare

- If projection P is trace class, then $\mathrm{Tr}(P) \in \mathbb{Z}$.
- If unitary U and projection X have $X - UXU^{-1}$ trace class[1],

$$\mathrm{Tr}(X - UXU^{-1}) \in \mathbb{Z}.$$

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[1] Effros '89, Avron–Seiler–Simon '93

Proof of abstract quantization theorem

Holomorphic map $z \mapsto e^{2\pi i z} - 1$ has poles at $z = 0, 1$.

- So the following is trace class:

$$(e^{2\pi i P_X} - 1)(e^{2\pi i P_Y} - 1) = \psi(P_X) \cdot \overbrace{P_X(1 - P_X)}^{P_X - P_X^2} \cdot \overbrace{P_Y(1 - P_Y)}^{P_Y - P_Y^2} \cdot \psi(P_Y)$$

- Kitaev's observation (2000), proved by Elgart–Frass (2023):

$$\det(e^{2\pi i P_X} e^{2\pi i P_Y} e^{-2\pi i P_X} e^{-2\pi i P_Y}) = 1.$$

- By Pincus–Helton–Howe '73,

$$1 = \det(e^{2\pi i P_X} e^{2\pi i P_Y} e^{-2\pi i P_X} e^{-2\pi i P_Y}) = \exp\left((2\pi i)^2 \mathrm{Tr}[P_X, P_Y]\right).$$

Thus $2\pi i \cdot \mathrm{Tr}[P_X, P_Y] \in \mathbb{Z}$.

For NCG experts:

Note: *"No topology" was needed for quantization...*

Partition $\rightsquigarrow$ coarse 2-cocycle $\rightsquigarrow$ cyclic 2-cocycle on $\mathscr{B}_{\mathrm{fin}}(M)$,

$$(L_0, L_1, L_2) \mapsto \mathrm{Tr}(AL_0BL_1CL_2 + \mathrm{antisymm})$$

Formula descends to coarse cohomology class of partition and algebraic K_0-theory class of P.

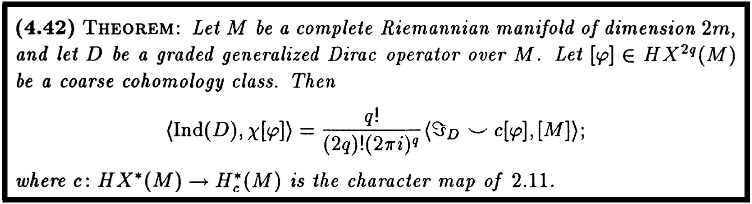
$$\begin{aligned} HX^2(M) \times K_0(\mathscr{B}_{\mathrm{fin}}(M)) &\to \mathbb{Z} \subset \mathbb{C} \\ \Big((A, B, C)\,,\ P\Big) \quad &\mapsto 4\pi i \cdot \mathrm{Tr}[P_A, P_B]. \end{aligned}$$

→ Additivity in P, functorial in M (coarse-metric category), etc.

III. Coarse index, briefly

Roe's coarse index

- Roe was trying to generalize Atiyah–Singer index theory to non-compact manifolds M.
- Constructed abstract index $\mathrm{Ind}(\not{D}) \in K_0(\mathscr{B}_{\mathrm{fin}}(M))$, and proved:



(4.42) THEOREM: *Let M be a complete Riemannian manifold of dimension $2m$, and let D be a graded generalized Dirac operator over M. Let $[\varphi] \in HX^{2q}(M)$ be a coarse cohomology class. Then*

$$\langle \mathrm{Ind}(D), \chi[\varphi] \rangle = \frac{q!}{(2q)!(2\pi i)^q} \langle \mathfrak{I}_D \smile c[\varphi], [M] \rangle;$$

where $c\colon HX^*(M) \to H_c^*(M)$ is the character map of 2.11.

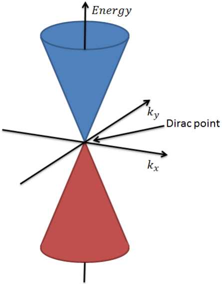
- Demonstrates non-trivial pairing with projections representing Dirac index.

2D Dirac coarse index

Massless Dirac operator on Euclidean $\mathbb{R}^2$ is

$$D = \begin{pmatrix} 0 & -i\partial_x - \partial_y \\ -i\partial_x + \partial_y & 0 \end{pmatrix}$$

- (Massless) Gapping out of Dirac point is obstructed by "Index(D)".



Method 1: Atiyah–Singer families index bundle over momentum space ("T-duality").

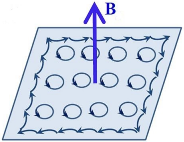
Method 2: Twist by gauge field: get ∞-degenerate zero modes $\leftrightarrow$ Landau levels.

$$[P_{\mathrm{Landau}}] = \mathrm{Index}(D).$$

Landau level = Dirac Index

$$D_b^2 = \begin{pmatrix} 0 & -i\partial_x - (\partial_y - ibx) \\ -i\partial_x + (\partial_y - ibx) & 0 \end{pmatrix}^2$$

$$= \begin{pmatrix} H_{\mathrm{Landau}} - b & 0 \\ 0 & H_{\mathrm{Landau}} + b \end{pmatrix}$$



Landau Spectrum : $\bullet_b$ $\bullet_{3b}$ $\bullet_{5b}$ $\bullet_{7b}$ $\ldots$

- Landau level spectral projection ∼ coarse Dirac index.
- Geometry affects Landau spectrum:
  Helical geometry on $\mathbb{R}^2$, no gaps[2]!
- These projections are not finite propagation. . . quantization??

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[2] Kubota+L+T, CMP '21- '22

IV. Rapid decrease operators

Algebra of rapid decrease operators

- Choose any tiling $\mathcal{T}$ of M, and define seminorms for each $\nu \geq 0$,

$$||L||_\nu := \sup_{V \in \mathcal{T}} \sum_{W \in \mathcal{T}} ||VLW||_{\mathrm{Tr}}(1 + d(V, W))^\nu < \infty.$$

- Finiteness of seminorms determines Fréchet algebra $\mathscr{B}(M)$, whose local traces decay rapidly from diagonal.

For subsets $Z \subset M$, there are ideals $\mathscr{B}(M; Z) \subset \mathscr{B}(M)$ defined by rapid decrease of local traces away from Z.

Properties of $\mathscr{B}(M)$

We prove:

- **Trace class.** $\mathscr{B}(M; K)$ in trace class, if K is bounded and M has polynomial growth.
- **Localization.** If Z_1, Z_2 are polynomially excisive, meaning that $\exists \mu$ such that

  $$B_r(Z_1) \cap B_r(Z_2) \subset B_{r^\mu}(Z_1 \cap Z_2) \qquad \forall r > 0,$$

  then

  $$\mathscr{B}(M; Z_0) \cdot \mathscr{B}(M; Z_1) \subset \mathscr{B}(M; Z_1 \cap Z_2).$$

Universal quantization of conductance

May now adapt "algebraic" proof from finite-prop. case:

- If X, Y are coarsely transverse and polynomially-excisive, then

  $$\mathrm{Tr}[P_X, P_Y], \qquad P = P^2 = P^* \in \mathscr{B}(M)$$

  makes sense, quantized to $\frac{1}{2\pi i} \cdot \mathbb{Z}$.
- Continuous, thus constant in P as it is deformed within space of projections in topological algebra $\mathscr{B}(M)$.
- Coarse cobordism invariant[3] w.r.t. choice of X, Y.
- Applies to P_{Landau}, and other P_{Fermi} with rapid decrease integral kernels.
  (Generalizable to mobility gap?)

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[3] For edge-following states, see [L+T, ATMP '22]

V. Finite size?

Trivial theory for finite size M?!

- Define bulk $K \subset M$ to be region at distance $> r$ from ∂M, where $r =$ propagation of P.

  $$\sigma_{\mathrm{bulk}}(P) = \mathrm{Tr}\big((KAK)P(KBK)P(KCK)P + \mathrm{antisymm}\big)$$

  is not quantized, because PKP is not a projection.
- Similarly, K^c gives boundary contribution $\sigma_{\mathrm{boundary}}(P)$.

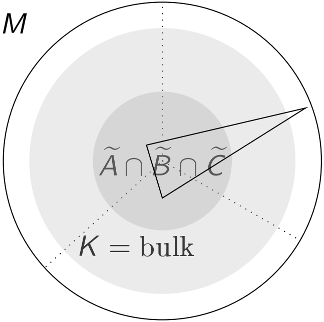
  $$0 = \sigma_{\mathrm{total}} = \sigma_{\mathrm{bulk}}(P) + \sigma_{\mathrm{boundary}}(P),$$

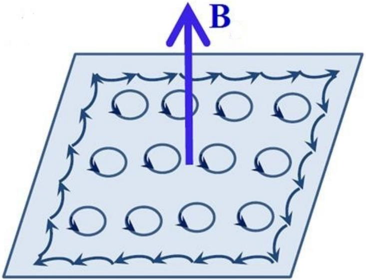
  up to cross-terms like

  $$(KAK)P(K^cBK^c)P(KCK)P.$$

Finite size M

Cross-terms vanish if $\mathrm{diam}(K) \gg r$, then $\sigma_{\mathrm{bulk}}(P) = \sigma_{\mathrm{boundary}}(P)$.





Precision of $\sigma_{\mathrm{bulk}}(P) \approx \mathbb{Z}$ depends on how well P approximates unbounded model, decay rate of P, volume and growth rate of M...

End/Discussion