Asymmetric Transport in Topological Insulators

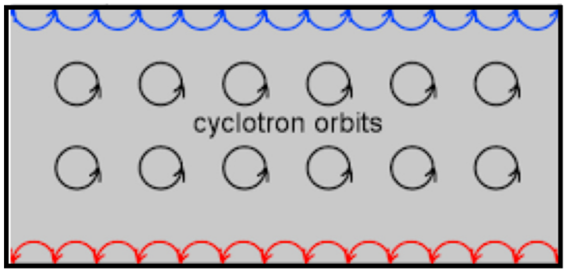
Guillaume Bal

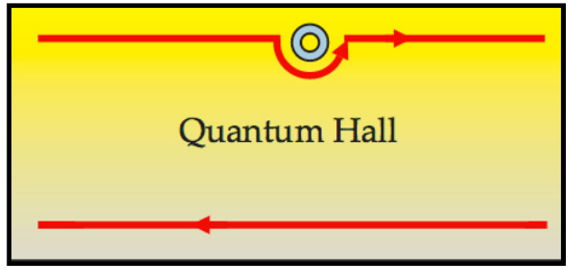
Computational and Applied Mathematics

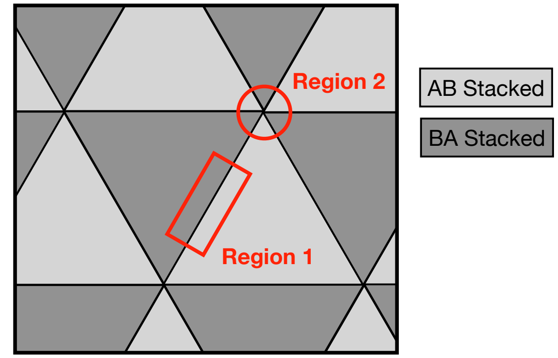
University of Chicago

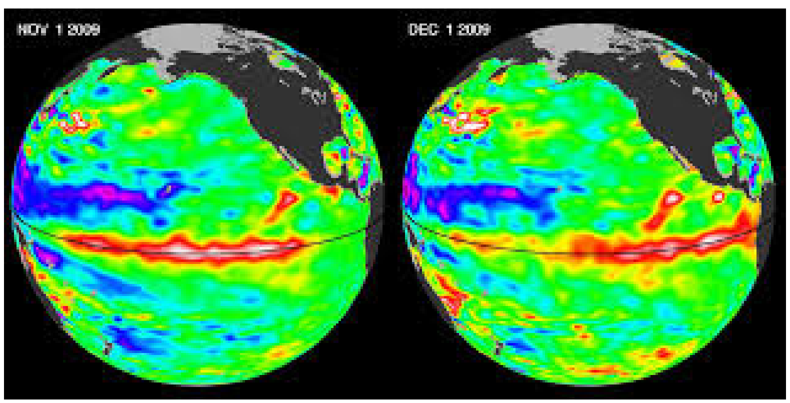
guillaumebal@uchicago.edu

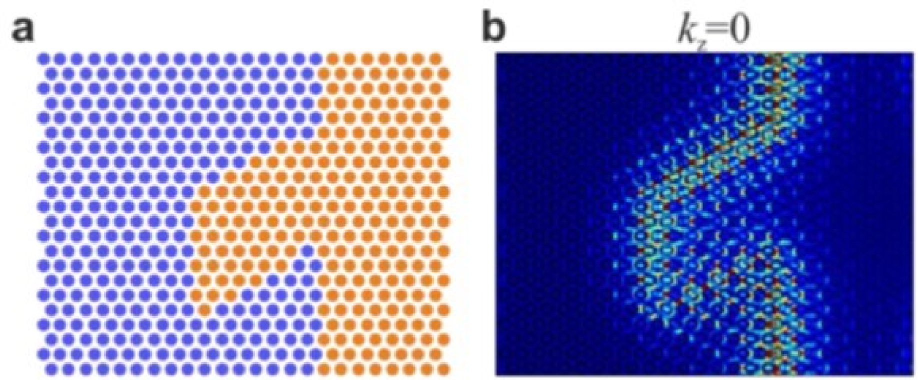
Robust Asymmetric Transport in TIs







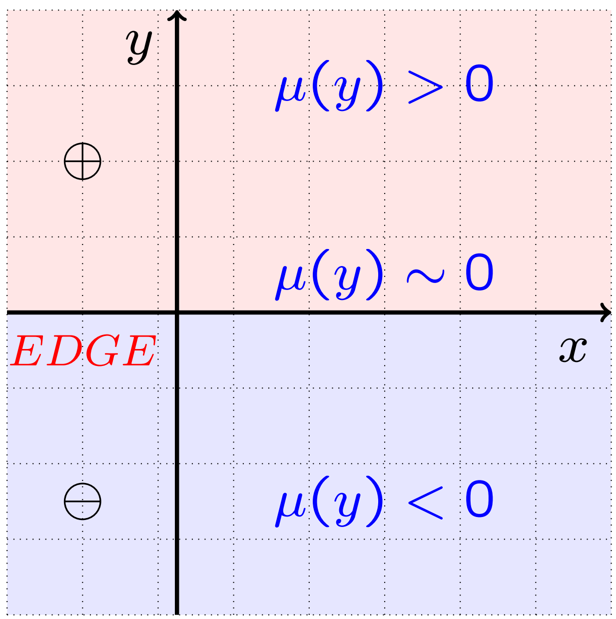


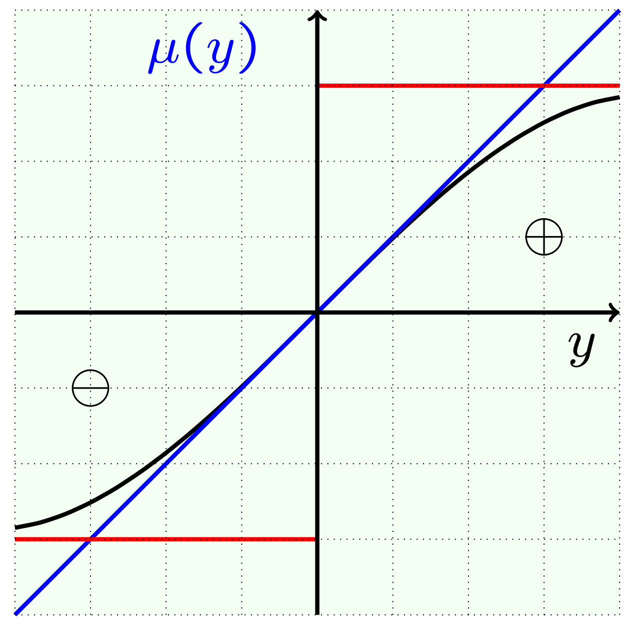


- **Quantized asymmetric transport** at different scales.
- IQHE,Twister biilayer Graphene, Atmospherics waves, Photonics.

Topological phase transition: domain walls

- **Insulating phases** (typically) described by **mass term** $\mu \neq 0$.
- **Transition** (typically) modeled by **Domain Wall** $\mu(y)$.
- **Asymmetric transport** observed near interface $\mu^{-1}(0)$.





- **Interface Hamiltonian** H_μ modeling transition between bulk insulators.

Asymmetric transport and Interface Conductivity

- Let $P(x)$ model density on right of $x = x_0$.
- **Observable:** $\langle P \rangle = \langle \psi(t) | P | \psi(t) \rangle$, $i\partial_t \psi = H\psi$.
- **Rate of change**:

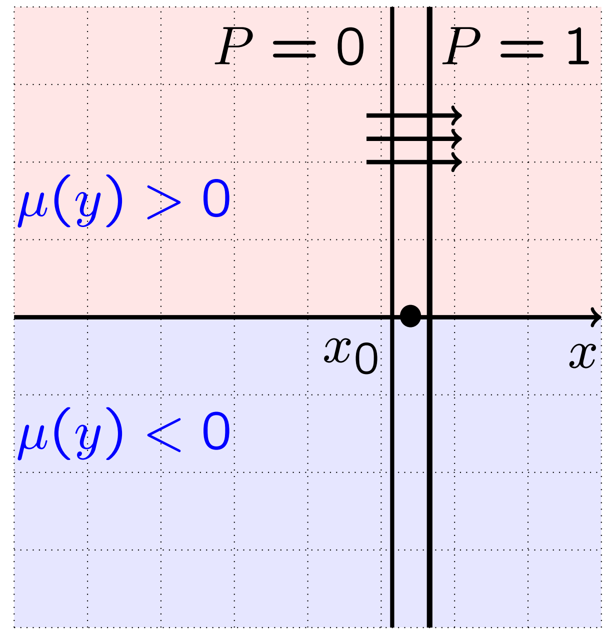
$$\frac{d}{dt}\langle P \rangle = \text{Tr } i[H,P]\psi(t)\psi^*(t).$$

- Models **current** across line $x = x_0$.

- Density $\varphi'(E) \geq 0$ with $\int \varphi'(E)dE = 1$ supported within **bulk gap**.

- We define **interface conductivity** as:

$$\boxed{\sigma_I = \text{Tr } i[H,P]\varphi'(H).}$$



Topological invariants for asymmetric transport

- **Asymmetric transport** modeled by $\sigma_I = \text{Tr}\ i[H,P]\varphi'(H)$.
- **Objectives:**

1. Identify classes of **Interface Hamiltonians** H.

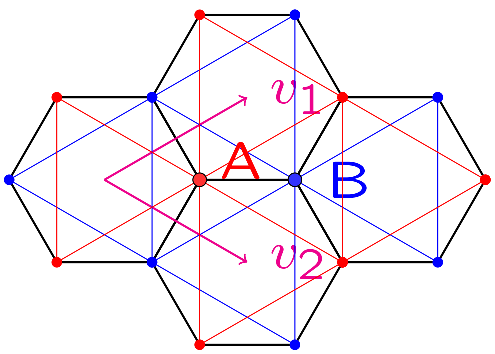
2. Introduce **Topological invariants** via two different **Fredholm operators**: $T(H) = \text{P}U(H)\text{P}$ and $F(H)$.

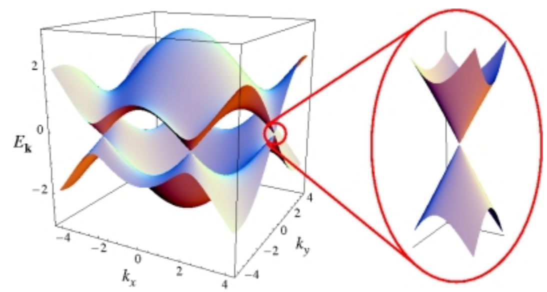
3. Prove **Topological Charge:** $2\pi\sigma_I = \text{Index}T = \text{Index}F \in \mathbb{Z}$.
   This is a form of **bulk-edge** correspondence.

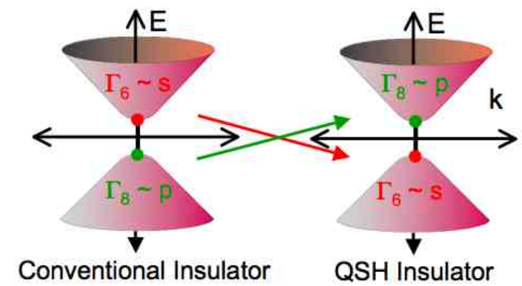
4. **Computation of invariants:** Index$F$ by *winding number/Chern number/topological degree* formulas; index$T$ by *spectral flow*.

Genericity of PDE models / Dirac operators

Bloch decomposition of microscopic problems (e.g., Schrödinger/Maxwell equation with periodic coefficients or tight-binding problems) provide:







Low energy models near Dirac points (*generic* in honeycomb structures [Fefferman-Weinstein 2012]) are Dirac equations :

$$H = D_x\sigma_1 + D_y\sigma_2 + m(x,y)\sigma_3 = \begin{pmatrix} m(x,y) & D_x - iD_y \\ D_x + iD_y & -m(x,y) \end{pmatrix}$$

with $D_x = -i\partial_x$, $D_y = -i\partial_y$ and $m(x,y)$ a mass term.

Examples of Hamiltonians H

Examples of (unperturbed) Hamiltonians in different applications:

$$H_D = D_x\sigma_1 + D_y\sigma_2 + m(y)\sigma_3 = \begin{pmatrix} m(y) & D_x - iD_y \\ D_x + iD_y & -m(y) \end{pmatrix}$$

$$H_p = (\frac{1}{2m}(D_x^2 + D_y^2) - \mu(y))\sigma_1 + \tfrac{1}{2}\{c(y), D_y\}\sigma_2 + c_0 D_x \sigma_3$$

$$H_d = (\frac{1}{2m}(D_x^2 + D_y^2) - \mu(y))\sigma_1 + c_0(D_y^2 - D_x^2)\sigma_2 + \tfrac{1}{2}D_x\{c(y), D_y\}\sigma_3$$

$$H_F = \begin{pmatrix} 1 + D\cdot\sigma & \varepsilon B^*(y) & O \\ \varepsilon B(y) & D\cdot\sigma & \varepsilon B^*(y) \\ O & \varepsilon B(y) & -1 + D\cdot\sigma \end{pmatrix} \qquad H_W = \begin{pmatrix} 0 & D_x & D_y \\ D_x & 0 & if(y) \\ D_y & -if(y) & 0 \end{pmatrix}$$

H_D: **Dirac** operator in electronics and photonics; H_p and H_d: BdG p-wave and d-wave superconductor Hamiltonians; H_F: 3-replica model in graphene-based Floquet TI (and bilayer graphene); H_W: Atmospheric Fluid-wave Hamiltonian.

Classes of Hamiltonians

We consider Hamiltonians in Weyl form $H = \mathrm{Op}^w a$ on $\mathbb{R}^d$, where

$$(\mathrm{Op}^w a)\ f(x) \ = \ \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi}\ a(\frac{x+y}{2},\xi)\ f(y)\ d\xi dy.$$

• In $d = 2$ for (differential) Hamiltonians with *bounded* domain walls:

$$a \in S^m \equiv S^m_{1,0}, \qquad \text{i.e.,} \qquad |\partial^\alpha_\xi \partial^\beta_x a|(x,\xi) \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}.$$

[$H1$] Assume $m > 0$, $a \in S^m$, *Hermitian*, **elliptic** (s.v. $\ge C_1\langle\xi\rangle^m - C_2$).

[$H2$] Assume H **insulating** for $|y| \ge L$, i.e., $H = H_\pm$ for $\pm y > L$ with $H_\pm$ gapped : $\mathrm{spec}(H_\pm) \cap (E_-, E_+) = \emptyset$.

[$H3$] Assume $\varphi' \in C^\infty_c(\mathbb{R})$ supported inside that *gap* (E_-, E_+).

• $d \ge 2$. Let $x = (x'_k, x''_k)$ for $x'_k \in \mathbb{R}^k$, $X = (x,\xi)$, $w_k(X) = \sqrt{1 + |x'_k|^2 + |\xi|^2}$.

$$a_k \in S^m_k, \qquad \text{i.e.,} \qquad \langle x\rangle^{|\alpha|}\langle\xi\rangle^{|\beta|}|\partial^\alpha_x \partial^\beta_\xi a_k(X)| \le C_{\alpha,\beta} w^m_k(X).$$

[$H1$] Symbol $a_k \in ES^m_k$: Hermitian, **elliptic** (s.v. $\ge C_1 w^m_k(X) - C_2$).

S^m_k : k confining unbounded domain walls. (Also chiral when $d + k$ even.)

Topological classification by domain walls: 2D

• **Topology** of operator tested by **domain walls** leading to transverse **asymmetric transport**. In $\mathbb{R}^2$:

• Bare (graphene) 2D Dirac $H_0 = D_x\sigma_1 + D_y\sigma_2 = \begin{pmatrix} 0 & D_x - iD_y \\ D_x + iD_y & 0 \end{pmatrix}$.

• Construct **confined** $H_1 = H_0 + y\sigma_3$ (or *start from* H_1 **confined** in y).

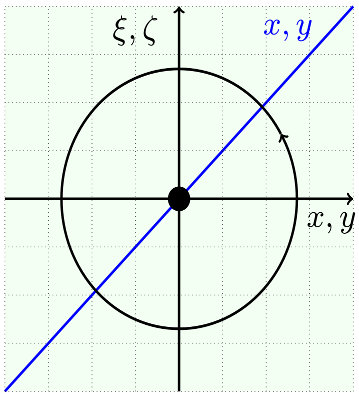
• Further **confine** in x and define the **Fredholm operator** $F = H_1 - ix$.

• For Dirac, $\underline{\text{Index}\, F := \dim \text{Ker} F - \dim \text{Ker} F^*} = -1 = \text{Index}\, F + V$.

• With $a(x, y, \xi, \zeta) = \xi\sigma_1 + \zeta\sigma_2 + y\sigma_3 - ix$ **symbol** of F:

$$\text{Index} F = \frac{1}{24\pi^2} \int_{\mathbb{S}^3} \text{tr}\ (a^{-1}da)^{\wedge 3} = -1.$$

This is the Fedosov-Hörmander **index** formula.



Topological classification by domain walls: nD

• Using Clifford algebras generalizing $\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij}$, we may construct an arbitrary number of domain walls.

• Start: H_k **confined** in k axes. Construct $H_{d-1}=\gamma_0\otimes H_k+\mu\cdot\gamma\otimes I_{n_k}$.

• Construct **Fredholm operator** $F=H_{d-1}-i\mu(x_d)$.

• **Theorem.** Let $H_k=\mathrm{Op}^w a_k$ for $a_k\in ES^m_k$ (elliptic symbols). Then $F=\mathrm{Op}^w a$ with

$$\mathrm{Index}F=-\frac{(d-1)!}{(2\pi i)^d(2d-1)!}\int_{\mathbb{S}^{2d-1}_R}\mathrm{tr}\,(a^{-1}da)^{\wedge(2d-1)}.$$

Fedosov-Hörmander formula: **Topological Charge** associated to H_k.

[B. JMP 23 Topological charge conservation for continuous insulators].

Main Results

- H_k, H_{d-1} as above (confined in $d-1$ directions) and $F = H_{d-1} - i\mu(x_d)$.

- **Theorem (stability):** $\sigma_I = \mathrm{Tr}\, i[H_{d-1}, P]\varphi'(H_{d-1})$ is well-defined *edge conductivity*. $2\pi\sigma_I \in \mathbb{Z}$ *stable* w.r.t. class- and ellipticity-preserving perturbations including $H_k \to H_k + V$ and $D \to hD$.

- Proof based on $2\pi\sigma_I = \mathrm{Index} T$, $T = \mathrm{P} e^{2\pi i \varphi(H_{d-1})} \mathrm{P}_{|\mathrm{RanP}}$ Fredholm.

- **Theorem (TCC / BEC)**: $\boxed{\mathrm{Index}\, F = 2\pi\sigma_I}$.

[B. JMP 23 Topological charge conservation for continuous insulators]

[B. CPDE 22 Topological invariants for interface modes]

[Quinn B. 21 Approximations of Top. inv. for interface Hamiltonians]

Related mathematical works

• IQHE [Avron, Seiler, Simon 80s',90s']. Bulk invariant as Index of pairs of projections Index(P, UPU^*) applied to *magnetic Schrödinger* equations.

• [Germinet et al. 05'] Asymmetric transport (with σ_I as interface invariant) for magnetic Schrödinger; [Quinn B. 22'] for magnetic Dirac.

• [Graf. et al. 00s'] Generalization of Asymmetric Transport to *discrete* Hamiltonians. σ_I associated to half-space Hamiltonians and Bulk-Edge correspondence (BEC).

• [Bellissard et al. 80s',90s'] *Non-commutative geometry* techniques applied to Bulk Invariant in IQHE.

• [Kellendonk, Prodan, Schulz-Baldes 00' 10'] Extension to *general discrete* Hamiltonians and BEC. • K-theoretic approaches for *general continuous* operators [Bourne, Carrie, Kaufmann, Kellendonk, Lorie, Thiang 10s', 20s'].

• (Magnetic) Schrödinger/photonic operators; periodic small scale structure [Fefferman-Weinstein 12-13'] [Drouot-Fefferman-Weinstein 19'] [Ablowitz et al. 13']; BEC [Drouot 19' & 21'].

• Topological Charge Conservation, Green's functions [Essin-Gurarie], [Volovik].

Functional calculus

We consider Hamiltonians in Weyl form $H = \mathrm{Op}^w a$

$$(\mathrm{Op}^w a)f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi} a(\frac{x+y}{2},\xi)f(y)d\xi dy.$$

In $d = 2$ for Hamiltonians with *bounded* domain walls, consider:

$$a \in S^m \equiv S^m_{1,0}, \qquad \text{i.e.,} \qquad |\partial_\xi^\alpha \partial_x^\beta a|(x,\xi) \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}.$$

•$[H1]$ Assume $m > 0$, $a \in S^m$, Hermitian, **elliptic** (s.v. $\ge C_1\langle\xi\rangle^m - C_2$).

•$[H2]$ Assume H **insulating** for $|y| \ge L$, i.e., $H = H_\pm$ for $\pm y > L$ with $H_\pm$ gapped : $\mathrm{spec}(H_\pm) \cap (E_-, E_+) = \emptyset$.

•$[H3]$ Assume $\varphi' \in C_c^\infty(\mathbb{R})$ supported inside that *gap* (E_-, E_+).

Then: (i) H self-adjoint. Functional calculus (Helffer-Sjöstrand formula):

$$f(H) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar\partial \tilde f(z)(z-H)^{-1} d^2z.$$

(ii) $i[H,P] \in \mathrm{Op}^w(\langle x\rangle^{-\infty}\langle\xi,\zeta\rangle^{m-1})$ while $\varphi'(H) \in \mathrm{Op}^w(\langle y,\xi,\zeta\rangle)^{-\infty}$.
Thus $i[H,P]\varphi'(H)$ is **trace-class** by composition calculus.

Fredholm operator in Toeplitz form

- By cyclicity of trace, for $\phi \in C_c^\infty(\mathbb{R})$,

$$\mathrm{Tr}[H^n, P]\phi(H) = \mathrm{Tr}[H, P]nH^{n-1}\phi(H)$$

so that (essentially) by density

$$2\pi\sigma_I = \mathrm{Tr}2\pi i[H, P]\varphi'(H) = \mathrm{Tr}[U(H), P]U^*(H), \quad U(H) = e^{i2\pi\varphi(H)}.$$

- For (modified) $\mathrm{P}^2 = \mathrm{P}$, then the Calderón-Fedosov formula implies

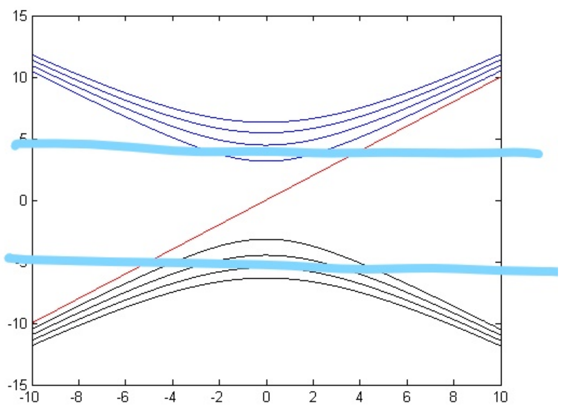
$$T := \mathrm{P}U(H)\mathrm{P}_{|\mathrm{RanP}} \quad \text{Fredholm} : \quad \mathrm{Index}\mathrm{P}U(H)\mathrm{P} = \mathrm{Tr}[U(H), \mathrm{P}]U^*(H).$$

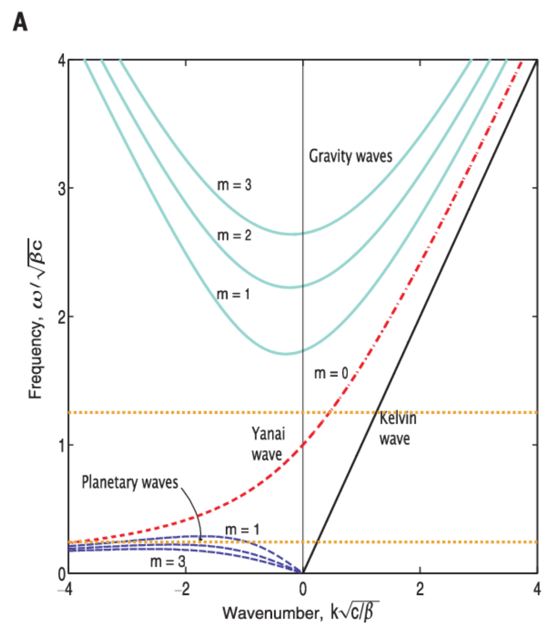
Thus,

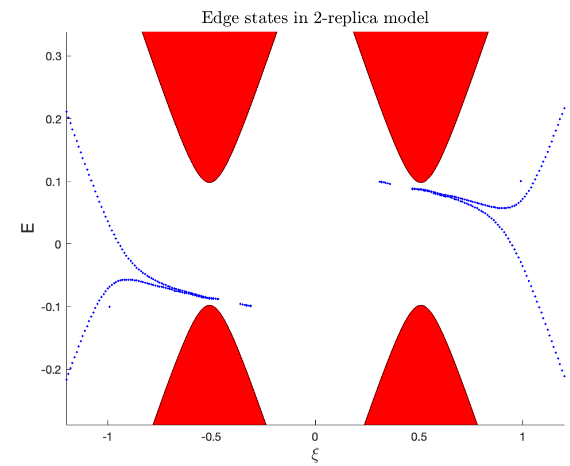
$$\boxed{2\pi\sigma_I(H) = \mathrm{Index}\mathrm{P}U(H)\mathrm{P} \in \mathbb{Z}.}$$

This implies stability w.r.t. P; P, φ'; $H \to H + V$; $\xi \to h\xi$; $x \to hx$.
The index remains hard to compute (essentially by spectral flow).

Edge conductivity and spectral flow illustrations







When spectral decomposition of H available, Index$PU(H)P = SF(H)$.

• Left: Dirac model for $m(y) = -y$ with $SF = 1$. • Middle: Geophysical fluid model with $f(y) = f$ with $SF = 2$. • Right: gated twisted bilayer graphene with $SF = -2$ close to $E = 0$ (finite spectral gap).

Derivation of $2\pi\sigma_I = \mathrm{Index} F \quad (i)$

- Deform symbol $a \to a(y,\xi,\zeta)$ and compute

$$2\pi\sigma_I = \mathrm{Tr}_y \int_{\mathbb{R}^2} 2\pi i [H,P](x,x')\varphi'(H)(x',x)dx'dx = \mathrm{Tr}_y \int_{\mathbb{R}} \partial_\xi \hat{H}\varphi'(\hat{H})d\xi$$

- Use invariance of σ_I w.r.t. $\zeta \to h\zeta$, $Y = (y,\zeta)$, and define

$$\partial_\xi \hat{H}_h = -\mathrm{Op}_h^w(\partial_\xi \sigma_z), \quad \varphi'(\hat{H}_h) = \mathrm{Op}_h^w \varsigma, \quad (z-\hat{H}_h)^{-1} = \mathrm{Op}_h^w r_z$$

to obtain (using $\sharp_h$ for **semiclassical** (Moyal) symbol product):

$$2\pi\sigma_I = \frac{-1}{2\pi h}\int_{\mathbb{R}^3} \mathrm{tr}\ \partial_\xi \sigma_z \sharp_h \varsigma\ dYd\xi = \frac{1}{2\pi^2 h}\int_{\mathbb{R}^3\times\mathbb{C}} \bar{\partial}\tilde{\varphi}'(z)\ \mathrm{tr}\partial_\xi\sigma_z\sharp_h r_z\ dYd\xi d^2z.$$

- With *semiclassical expansion* in h using $\sigma_z \sharp_h r_z = I$, the $O(h^0)$ term is

$$2\pi\sigma_I = \frac{i}{4\pi^2}\int_{\mathbb{R}^3\times\mathbb{C}} \bar{\partial}\tilde{\varphi}'(z)\ \mathrm{tr}\tau\ dYd\xi d^2z, \quad \tau = \partial_\xi\sigma_z\{\sigma_z^{-1},\sigma_z\}\sigma_z^{-1} - \{\partial_\xi\sigma_z,\sigma_z^{-1}\}.$$

$2\pi\sigma_I = \mathrm{Index} F \quad (ii)$

$$2\pi\sigma_I = \frac{i}{4\pi^2}\int_{\mathbb{R}^3\times\mathbb{C}} \bar\partial\tilde\varphi'(z)\ \mathrm{tr}\,\tau\ dY d\xi d^2 z, \quad \tau = \partial_\xi\sigma_z\{\sigma_z^{-1},\sigma_z\}\sigma_z^{-1} - \{\partial_\xi\sigma_z,\sigma_z^{-1}\}.$$

- Use $z\to\sigma^{-1}(z)$ defined and *analytic* for $|(\xi, Y=(y,\zeta))|\geq R$, write Poisson brackets in divergence form, and use Stokes theorem to get

$$2\pi\sigma_I = \int_{\mathbb{R}} \varphi'(\lambda) I(\lambda) d\lambda,\ \ I(\lambda) = \frac{1}{8\pi^2}\int_{|(Y,\xi)|\leq R} \mathrm{tr}\left[\sigma_z^{-1}\partial_\xi\sigma_z\{\sigma_z^{-1},\sigma_z\}\right]_{\lambda-0i}^{\lambda+0i} dY d\xi.$$

- Compute

$$\sigma_z^{-1}\partial_\xi\sigma_z\{\sigma_z^{-1},\sigma_z\}dY d\xi = \frac{1}{3}(\sigma_z^{-1}d\sigma_z)^{\wedge 3}$$

- Use $I(\lambda)$ independent of λ, closedness $d\Big(\mathrm{tr}(\sigma_z^{-1}d\sigma_z)^{\wedge 3}\Big)=0$, Stokes:

$$2\pi\sigma_I = \frac{1}{24\pi^2}\int_{\mathbb{S}^3_R} \mathrm{tr}\,(\sigma_z^{-1}d\sigma_z)^{\wedge 3}.$$

$\mathbb{S}^3_R$ is three-sphere in $(x\equiv\omega,\xi,y,\zeta)$ variables.
This is the explicit Fedosov-Hörmander formula for Index$F$. □

Arbitrary dimension with infinite domain walls

- Let $x=(x'_k,x''_k)$ for $x'_k\in\mathbb{R}^k$, $X=(x,\xi)$, and for $H_k=\mathrm{Op}^w a_k$, define

$$w_k(X)=\langle x'_k,\xi\rangle,\qquad \langle x\rangle^{|\alpha|}\langle\xi\rangle^{|\beta|}|\partial_x^\alpha\partial_\xi^\beta a_k(X)|\le C_{\alpha,\beta}w_k^m(X)\quad(a_k\in S_k^m).$$

Symbol $a_k\in ES_k^m$ assumed elliptic (singular values $\ge C_1 w_k^m(X)-C_2$).

- $F=\mathrm{Op}^w(a)$ Fredholm with index given by FH formula.
- $\sigma_I[H_{d-1}]$ defined similarly with $2\pi\sigma_I=\mathrm{Index}\,PU(H_{d-1})P$. Then:

$$2\pi\sigma_I=\frac{-ic_{d-1}}{(2\pi)^d}\int_{\mathbb{R}^{2d-1}\times\mathbb{R}}\varphi'(\lambda)\mathrm{tr}\,\sigma_z^{-1}\partial_\xi\sigma_z\{\sigma_z^{-1},\sigma_z\}_f^{d-1}\Big|_{\lambda-i0}^{\lambda+i0}dY\,d\xi\,d\lambda.$$

For *isotropic symbols* (i.e., $\langle X\rangle^{|\beta|}|\partial_X^\beta a_k(X)|\le C_{\alpha,\beta}w_k^m(X)$), σ_I invariant w.r.t. *rotations in variables* X: $\sigma_I=\frac{1}{(2d-2)!}\sum_{\rho\in\mathcal{S}_{2d-2}}(-1)^\rho\sigma_I(\rho(Y))$. So:

$$2\pi\sigma_I=\frac{(-1)^{d-1}}{(2\pi i)^d}\frac{(d-1)!}{(2d-1)!}\int_{\mathbb{R}^{2d-1}\times\mathbb{R}}\varphi'(\lambda)\mathrm{tr}\,(\sigma_z^{-1}d\sigma_z)^{2d-1}\Big|_{\lambda-i0}^{\lambda+i0}dY\,d\xi\,d\lambda.$$

We need to approximate $a_k\in ES_k^m$ by isotropic symbols. Rest as in $d=2$.

[B. JMP 23 Topological charge conservation for continuous insulators]

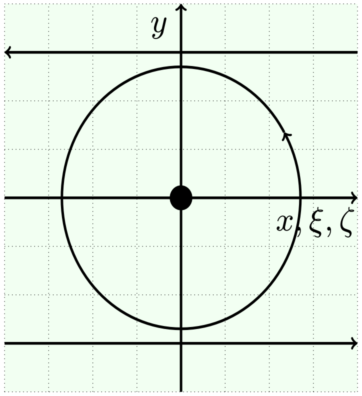
Bulk-Edge correspondence

- TCC is a **bulk-edge correspondence** in 2d (and a generalization to *high-order topological insulators* in higher dimensions).

- We can continuously deform the *topological* integral

$$2\pi\sigma_I = \frac{1}{24\pi^2}\int_{\mathbb{S}^3} \mathrm{tr}\,(a^{-1}da)^{\wedge 3}$$

$$= \frac{1}{24\pi^2}\int_{\{y=R\}\cup\{y=-R\}} \mathrm{tr}\,(a^{-1}da)^{\wedge 3}.$$



- Above integrals at $y = \pm R$ involves **bulk** quantities.

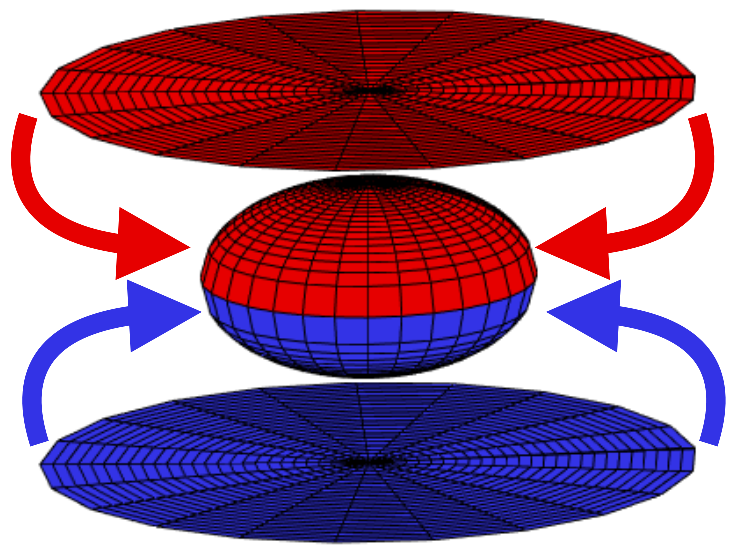
- Introduce the (imaginary frequency $\omega \equiv x$) Green's functions:

$$G^{N/S}(\omega,\xi,\zeta) = -a^{-1}(\omega, y = \pm R, \xi, \zeta) = (i\omega - \widehat{H}^{N/S}(\xi,\zeta))^{-1}.$$

- Defining the projectors $\Pi^{N/S}(\xi,\zeta) = \chi(\widehat{H}^{N/S}(\xi,\zeta) < 0)$, we have:

$$\int_{\mathbb{R}^3} \mathrm{tr}(G^\alpha d(G^\alpha)^{-1})^{\wedge 3} = 12 i\pi \int_{\mathbb{R}^2} \mathrm{tr}\Pi^\alpha d\Pi^\alpha \wedge d\Pi^\alpha, \qquad \alpha = N, S.$$

Bulk-difference invariant and correspondence



- Gluing two bulk quantities continuously by circle compactification generates invariant on sphere. Thus $2\pi\sigma_I =$*bulk-difference Chern invariant*:

$$\frac{i}{2\pi}\int_{\mathbb{S}^2} \mathrm{tr}\Pi d\Pi\wedge d\Pi = \frac{i}{2\pi}\int_{\mathbb{R}^2} \mathrm{tr}\Big(\Pi^S[\partial_1\Pi^S,\partial_2\Pi^S] - \Pi^N[\partial_1\Pi^N,\partial_2\Pi^N]\Big) d\xi.$$

- Explicitly integrates *curvature* of *connection* on *principal bundle*.
- Easier to define **relative** rather than *absolute* topological phases.
- Chern Bulk invariants *not* defined for many (such as Dirac) operators.

[B. CPDE 22 Topological invariants for interface modes]

Summary

- $H_k = \mathrm{Op}^w a_k$ for **elliptic symbols** $a_k \in ES_k^m$
- $H = H_{d-1}$ *confined* by **domain walls** in all variables but one
- **Physical observable** $\sigma_I = \mathrm{Tr} i[H,P]\varphi'(H)$ for **asymmetric transport**
- **Classification** by $F = H_{d-1} - ix_d$ **Fredholm operator**

$$
\begin{aligned}
2\pi\sigma_I &= \mathrm{Index}\, \mathrm{P}U(H)\mathrm{P} &&= \text{spectral flow} \\
&= \mathrm{Index}\, F &&= \text{bulk-difference Chern number} \\
&= \mathrm{Tr}\, T_+^* T_+ - \mathrm{Tr}\, T_-^* T_- &&= \text{asymmetric transport.}
\end{aligned}
$$

Requirement: Elliptic, confined, (partial differential) Hamiltonians.

Some applications and computations

Theory applies to:

- Dirac operators (TC=$\pm$1),

- BdG superconductors (TC=$\pm$1 (p-wave), TC=$\pm$2 (d-wave)),

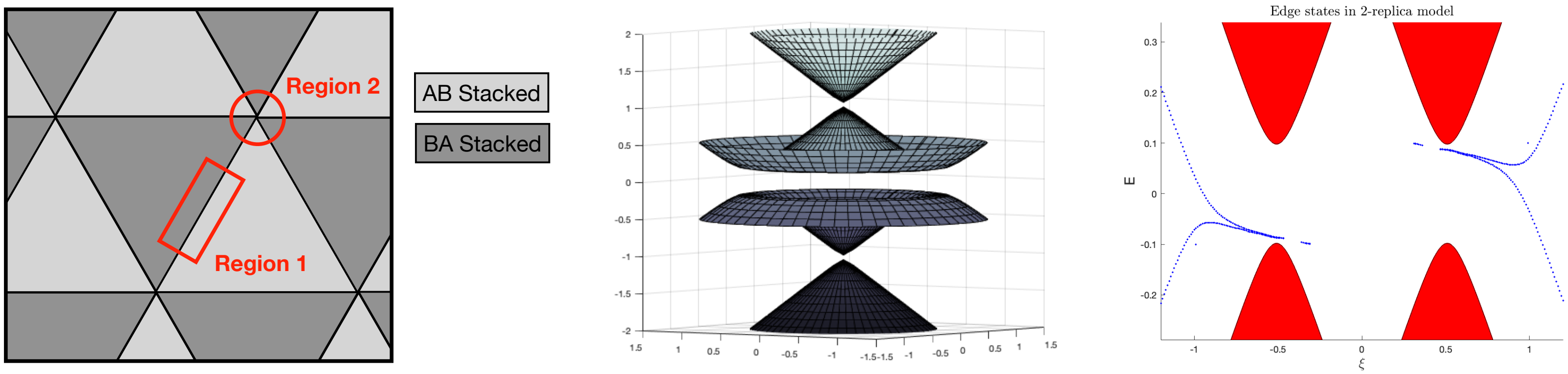
- Models of gated twisted bilayer graphene (TC=$\pm$2); see below

- Models of Floquet Topological insulators with TC $= -1 + 2n(n+1)$ with n number of replicas. Heuristically: -1 short-time; 3 longer-time; 11 even longer...

- (Regularized) Water Waves (TC=2); see below

- Dirac model of Higher-Order topological insulator; see below

gated twisted Bilayer Graphene



(a) (Relaxed) Moiré pattern in tBLG; (b) Bulk dispersion; (c) Edge dispersion.

- Model of gated twisted bilayer graphene in Region 1 (valley $\eta = \pm 1$)

$$H := \begin{pmatrix} \Omega + D_x\sigma_1 + \eta D_y\sigma_2 & \varepsilon V^*(y) \\ \varepsilon V(y) & -\Omega + D_x\sigma_1 + \eta D_y\sigma_2 \end{pmatrix} \quad \text{on} \quad L^2(\mathbb{R}^2; \mathbb{C}^4).$$

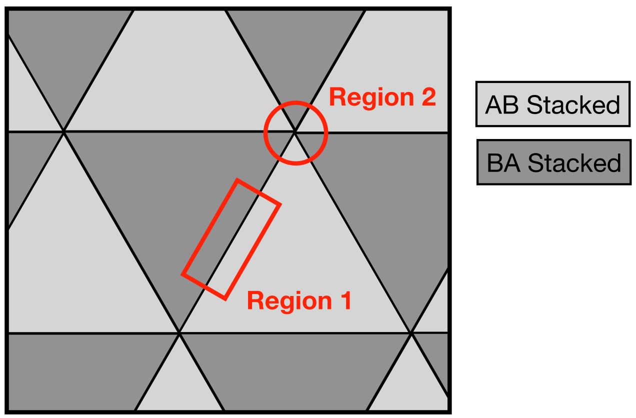
Domain wall $[0,1] \ni m(y)$, $V(y) = m(y)A + (1 - m(y))A^*$.

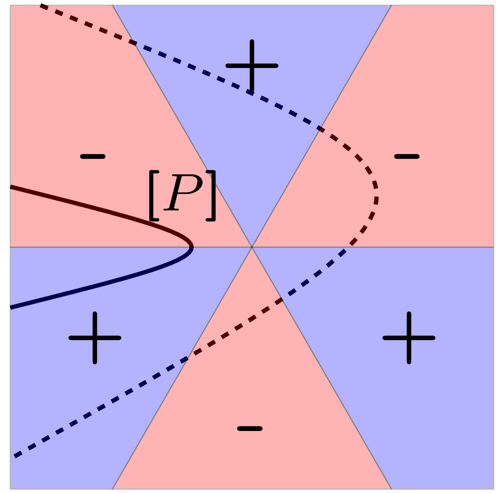
$\pm\Omega$ is voltage of top/bottom layer. ε is inter-layer coupling strength.

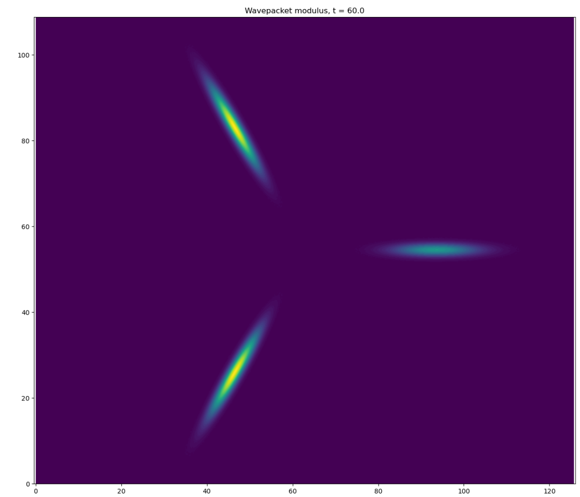
- TC $= -2\,\text{sign}(\eta\Omega)$. σ_I difficult to evaluate without TCC/BEC.

[B. Cazeaux, Massatt, Quinn MMS 23]

Junction topology in twisted bilayer graphene







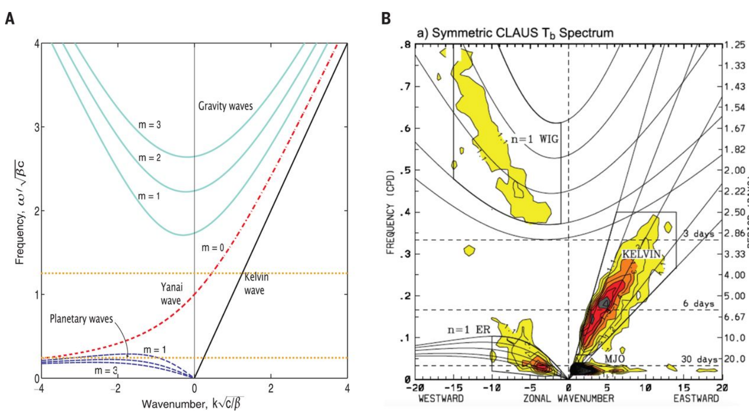
- Dirac operator $H = D \cdot \sigma + m(x,y)\sigma_3$ with $m \approx \Im(x+iy)^3$.
- Models propagation across junction in Region 2 of tBLG.
- $P(x,y)$ now with (smooth) jump across thick solid or dashed curves.
- Let $g(x,y)$ such that P jumps near $g^{-1}(0)$ and $F = H - ig(x,y)$ Fredholm.
- **Theorem:** $2\pi\sigma_I = 2\pi \text{Tr } i[H,P]\varphi'(H)$ equals TC $=$ Index F.
- **Corollary:** $1 =$ intersections solid curve $=$ intersections dashed curve $= 3 - 2$. Consistent with observed wavepacket propagation initially on middle left branch.

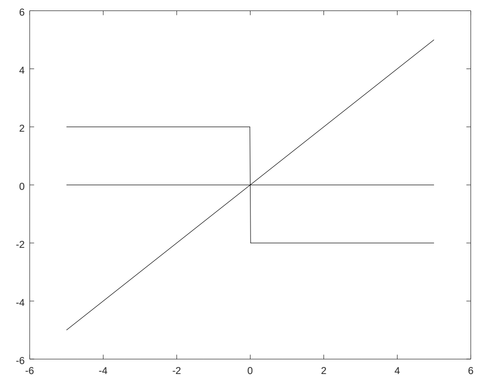
[B. Cazeaux Massatt Quinn MMS 23]

(Semi) Failure of Bulk-Edge correspondence

For Elliptic operators: $2\pi\sigma_I =$ Bulk-difference Chern number.

Does *not* always hold for the geophysical model (though mostly does):





$SF = 2 = Ch$ when $f(y) = y$ while $SF = 1 \neq Ch$ when $f(y) = \text{sign}(y)$.

$$\hat{H}(\xi,\zeta) = \begin{pmatrix} 0 & \xi & \zeta \\ \xi & 0 & if \\ \zeta & -if & 0 \end{pmatrix}, \quad E_\pm = \pm\sqrt{\xi^2+\zeta^2+f^2}, \quad E_0 = 0.$$

So Ellipticity condition [H1] important.

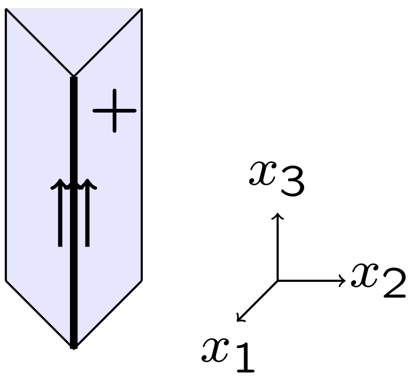
(Fix: make flat band smile/frown. Then TC$=2$.)

Higher-order Topological Insulators

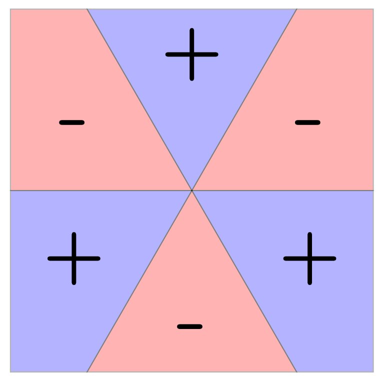
• Consider the 2×2 (bare) **Weyl Hamiltonian** $H_0 = D \cdot \sigma$ in 3D. Adding **two domain walls** (topological classification) leads to the 4×4

$$H_2 = \sigma_1 \otimes D \cdot \sigma + \sigma_2 \otimes I x_1 + \sigma_3 \otimes I x_2, \qquad F = H_2 - i x_3.$$

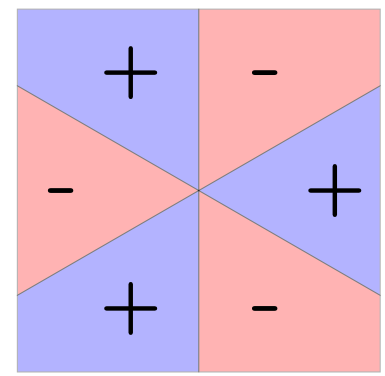
• Then for $P(x_3)$, TCC is $2\pi\sigma_I[H_2] = -\text{deg}(\xi_1, \xi_2, \xi_3) = \text{Index } F = -1$.



$m_1(x_1, x_2) =$



$m_2(x_1, x_2) =$



• Define $m_1(x_1, x_2) = \Im(x_1 + i x_2)^p$ and $m_2(x_1, x_2) = \Re(x_1 + i x_2)^p$ and

$$H_2 = \sigma_1 \otimes D \cdot \sigma + \sigma_2 \otimes I m_1(x_1, x_2) + \sigma_3 \otimes I m_2(x_1, x_2), \qquad F = H_2 - i x_3.$$

• $\underline{2\pi\sigma_I[H_2] = p}$. *Coaxial cable* with p protected modes along x_3-axis.

• Difficult to **topologize** with *bulk phases*. Simpler with domain walls.

[B. JMP 23 Topological charge conservation for continuous insulators]

References

- B. Topological perturbation perturbed edge states CMS 2019
- B. Continuous bulk and interface description of TIs JMP 2019
- B. Massatt, Multiscale invariants of Floquet TIs MMS 2022
- B. Topological invariants for interface modes CPDE 2022
- B. Topological Charge Conservation for continuous insulators JMP 2023
- Quinn B. Approximations of topological invariants arxiv2112:02686
- B. Cazeaux Massatt Quinn tBLG MMS 2023
- B. Becker Drouot Fermanian-Kammerer Lu Watson Edge state dyn. SIMA 2023
- B. Becker Drouot Magnetic slowdown edge states CPAM 2023
- B. Semiclassical propagation along curved domain walls, arXiv:2206.09439
- B. Chen Wang Long time asymptotics of mixed-type Kimura diffusions arXiv:2210.10037
- B. Hoskins Wang Asymmetric transport computations JCP 2023

Thank You !