Exact Dirac-Bogoliubov-de Gennes Dynamics for Inhomogeneous Quantum Liquids*

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Dirac-Bogoliubov-de Gennes (DBdG) equations

Problem: Given smooth functions $v(x)$ and $K(x)$, consider

$$\begin{pmatrix} v(x)\partial_x + \partial_t & \Delta(x) \\ \Delta(x) & v(x)\partial_x - \partial_t \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$\Delta(x) \equiv v(x)\partial_x \log \sqrt{K(x)}$$

for $u_\pm = u_\pm(x,t)$ with given initial conditions.

Questions:

- What is the general solution?
- What is the effect of $\Delta(x) \neq 0$?
- What is the behavior as $t \to \infty$?

Applications of DBdG-type equations

◇ [Andreev, Sov. Phys. JETP (1964)]:

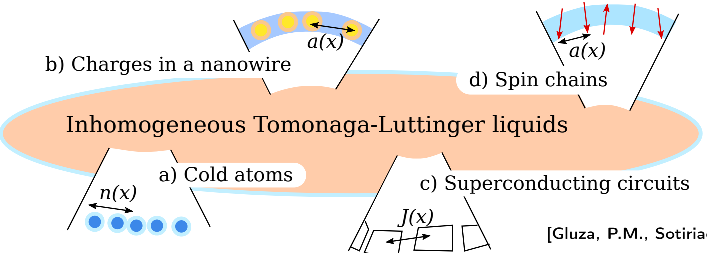
Interfaces between normal metals and superconductors

◇ [Takayama, Lin-Liu, Maki, PRB (1980)]:

Continuum description of Su-Schrieffer-Heeger model

◇ [**P.M.**, arXiv:2208.14467]:

Dynamics in inhomogeneous Tomonaga-Luttinger liquids (TLLs)



[Gluza, P.M., Sotiriadis, JPA (2022)]

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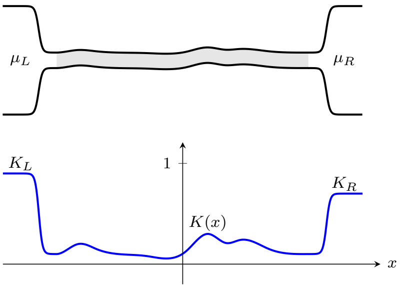
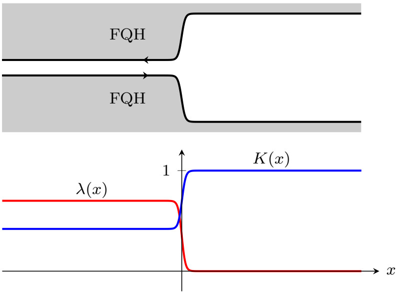
Quantum wires



Fractional quantum Hall (FQH) edges



Some previous works on inhomogeneous TLLs

◇ [Maslov, Stone], [Safi, Schulz], [Ponomarenko] {PRB (1995)}:

Quantum wires

◇ [Stringari, PRL (1996)], . . . , [Citro et al., New J. Phys. (2008)]:

Effective descriptions of trapped ultra-cold atoms in equilibrium

◇ [Brun, Dubail, SciPost (2018)], [Bastianello, Dubail, Stéphan, JPA (2020)], [Gluza, **P.M.**, Sotiriadis, JPA (2022)], [Ruggiero, Calabrese, Giamarchi, Foini, SciPost (2022)]:

Inhomogeneous TLLs out of equilibrium

Outline

- Tomonaga-Luttinger liquids (TLLs)
- DBdG equations from TLL theory
- Solving the DBdG equations

Tomonaga-Luttinger liquids (TLLs)

TLL theory / Free compactified bosons

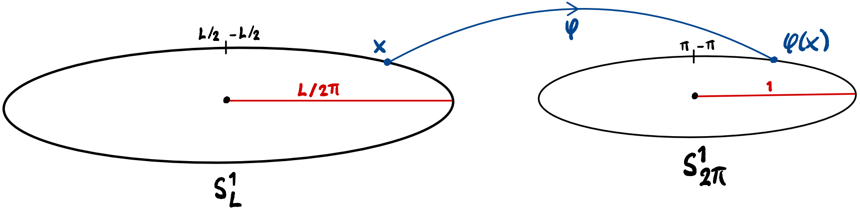
Given $v > 0$ and $K > 0$. Consider the action functional

$$S = \frac{R^2}{8\pi} \int_{\mathbb{R}\times S^1_L} \mathrm{d}^2 x \, (\partial^\mu \varphi)(\partial_\mu \varphi)$$

for fields $\varphi : S^1_L \to S^1_{2\pi}$ with compactification radius R satisfying

$$K = \frac{R^2}{4}$$

and metric $(h_{\mu\nu}) = \mathrm{diag}(1, -1)$ in coordinates $(x^0, x^1) = (vt, x)$.



TLL theory in Hamiltonian framework

Hamiltonian

$$H_{v,K} = \frac{1}{2\pi} \int_{S^1_L} \mathrm{d}x : \Big(\frac{v}{K} [\pi \Pi(x)]^2 + vK [\partial_x \varphi(x)]^2 \Big) :$$

with bosonic field $\varphi(x)$ and conjugate $\Pi(x)$ for $x \in S^1_L$ satisfying

$$[\partial_x \varphi(x), \Pi(y)] = \mathrm{i}\delta'(x-y).$$

Diagonalizable by simple Bogoliubov transformation in terms of bosonic creation and annihilation operators after expanding in plane waves:

$$H_{v,K} = \frac{\pi v}{L} \big(a_0^2 + \bar{a}_0^2 \big) + \frac{\pi v}{L} \sum_{n \neq 0} :(a_{-n} a_n + \bar{a}_{-n} \bar{a}_n):$$

with $a_n = a^\dagger_{-n}$ and $\bar{a}_n = \bar{a}^\dagger_{-n}$ ($n \in \mathbb{Z}$) for right/left movers satisfying

$$[a_n, a_m] = n\delta_{n+m,0} = [\bar{a}_n, \bar{a}_m], \qquad [a_n, \bar{a}_m] = 0.$$

Inhomogeneous TLL

Hamiltonian

$$H_{v(\cdot),K(\cdot)} = \frac{1}{2\pi}\int_{S^1_L} \mathrm{d}x : \left(\frac{v(x)}{K(x)} [\pi\Pi(x)]^2 + v(x)K(x)[\partial_x \varphi(x)]^2 \right):$$

with inhomogeneous periodic $v(x) > 0$ and $K(x) > 0$ on the circle S^1_L.
Not diagonalizable by simple Bogoliubov transformation for $K(x) \neq K$.

For inhomogeneous periodic $v(x)$ and $K(x) = K$ constant:

[Dubail, Stéphan, Viti, Calabrese, SciPost Phys. (2017)], [Dubail, Stéphan, Calabrese, SciPost Phys. (2017)]
[Gawedzki, Langmann, P.M., JSP (2018)], [Langmann, P.M., PRL (2019)], [P.M., AHP (2021)]

Corresponding action functional

$$S_{R(\cdot)} = \frac{1}{8\pi}\int_{\mathbb{R}\times S^1_L} \mathrm{d}^2x \sqrt{-h} R(x)^2 (\partial^\mu \varphi)(\partial_\mu \varphi)$$

with inhomogeneous compactification radius $R(x) = 2\sqrt{K(x)}$ and
metric $(h_{\mu\nu}) = \mathrm{diag}(v(x)^2/v^2, -1)$ in coordinates $(x^0, x^1) = (vt, x)$.

Inhomogeneous TLL

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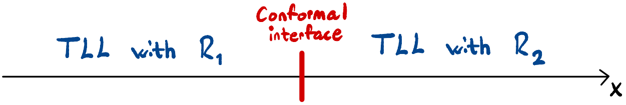
Related special case: Conformal interfaces

[Bachas, Brunner, JHEP (2008)]:



DBdG equations from TLL theory

PDE approach

Instead of diagonalizing $H_{v(\cdot),K(\cdot)}$ rewrite it as

$$H_{v(\cdot),K(\cdot)} = \int_{-L/2}^{L/2} \mathrm{d}x \, \pi v(x) \, \colon\!\Big(\widetilde{\rho}_+(x)^2 + \widetilde{\rho}_-(x)^2\Big)\!\colon$$

with right/left-moving densities

$$\widetilde{\rho}_\pm(x) \equiv \frac{1}{2\pi\sqrt{K(x)}}\Big[\pi\Pi(x) \mp K(x)\partial_x\varphi(x)\Big].$$

Result: $\widetilde{\rho}_\pm(x)$ satisfy

$$[\widetilde{\rho}_\pm(x), \widetilde{\rho}_\pm(y)] = \mp\frac{\mathrm{i}}{2\pi}\delta'(x-y),$$
$$[\widetilde{\rho}_+(x), \widetilde{\rho}_-(y)] = \frac{\mathrm{i}}{2\pi}\Lambda(x)\delta(x-y)$$

with $\Lambda(x) \equiv \partial_x \log\sqrt{K(x)}$ coupling right/left movers.

Dirac-Bogoliubov-de Gennes (DBdG) equations

Heisenberg equation and commutation relations imply that $\widetilde{\rho}_{\pm}(x)$ and $\widetilde{j}_{\pm}(x) \equiv \pm v(x)\widetilde{\rho}_{\pm}(x)$ satisfy coupled continuity equations

$$\partial_t \widetilde{\rho}_{\pm} + \partial_x \widetilde{j}_{\pm} = \pm \Delta(x) \widetilde{\rho}_{\mp}$$

with $\Delta(x) \equiv v(x)\Lambda(x)$.

Result: $\widetilde{j}_{\pm}(x,t)$ satisfy the inhomogeneous DBdG equations

$$\begin{pmatrix} v(x)\partial_x + \partial_t & \Delta(x) \\ \Delta(x) & v(x)\partial_x - \partial_t \end{pmatrix} \begin{pmatrix} \widetilde{j}_{+}(x,t) \\ \widetilde{j}_{-}(x,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with a local gap $\Delta(x) = v(x)\partial_x \log \sqrt{K(x)}$. [P.M., arXiv:2208.14467]

Solving the DBdG equations

◇ Recall: $\widetilde{j}_\pm(x,t)$ satisfy

$$\begin{pmatrix} v(x)\partial_x + \partial_t & \Delta(x) \\ \Delta(x) & v(x)\partial_x - \partial_t \end{pmatrix} \begin{pmatrix} \widetilde{j}_+ \\ \widetilde{j}_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $\Delta(x) = v(x)\partial_x \log\sqrt{K(x)}$.

◇ [Magnus, Comm. Pure Appl. Math. (1954)]:

$$\frac{\mathrm{d}}{\mathrm{d}s}Y(s) = A(s)Y(s), \qquad Y(s_0) = Y_0.$$

◇ Recall: $\widetilde{j}_{\pm}(x,t)$ satisfy

$$\partial_x \begin{pmatrix} \widetilde{j}_+ \\ \widetilde{j}_- \end{pmatrix} + \begin{pmatrix} v(x)^{-1}\partial_t & \Lambda(x) \\ \Lambda(x) & -v(x)^{-1}\partial_t \end{pmatrix} \begin{pmatrix} \widetilde{j}_+ \\ \widetilde{j}_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $\Lambda(x) = \partial_x \log \sqrt{K(x)}$.

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Analogy with non-Hermitian (PT-symmetric) 2-level system

DBdG eqs. in frequency space ω for expectations in the infinite volume:

$$\partial_x \begin{pmatrix} \langle \hat{j}_+(x,\omega) \rangle \\ \langle \hat{j}_-(x,\omega) \rangle \end{pmatrix} = \mathrm{i}\mathsf{P}_\omega(x) \begin{pmatrix} \langle \hat{j}_+(x,\omega) \rangle \\ \langle \hat{j}_-(x,\omega) \rangle \end{pmatrix} + \frac{1}{v(x)}\sigma_3 \begin{pmatrix} \langle \widetilde{j}_+(x,0) \rangle \\ \langle \widetilde{j}_-(x,0) \rangle \end{pmatrix}$$

for $x \in \mathbb{R}$ with the $\mathfrak{sl}(2,\mathbb{C})$ matrix

$$\mathsf{P}_\omega(x) \equiv \frac{\omega}{v(x)}\sigma_3 + \mathrm{i}\Lambda(x)\sigma_1.$$

In general, $\mathsf{P}_\omega(x)\mathsf{P}_\omega(y) \neq \mathsf{P}_\omega(y)\mathsf{P}_\omega(x)$, so need spatial ordering $\overleftarrow{\mathcal{X}}$ ($\overrightarrow{\mathcal{X}}$) where positions decrease (increase) from left to right.

Note: Expectations $\langle \cdot \rangle$ w.r.t. arbitrary state in the infinite-volume limit $L \to \infty$. Assumed system prepared in an initial state for $t < 0$ and evolving for $t > 0$ with initial data $\langle \widetilde{j}_\pm(x, t=0) \rangle$. Fourier transforms: $\hat{j}_\pm(x,\omega) = \int_0^\infty \mathrm{d}t\, \widetilde{j}_\pm(x,t) \mathrm{e}^{\mathrm{i}\omega t}$.

Green's functions

Result: Given $\langle \widetilde{j}_\pm(x,0) \rangle$ and assuming $\lim_{|x|\to\infty} \langle \widetilde{j}_\pm(x,t) \rangle = 0$, then

$$\begin{pmatrix} \langle \widetilde{j}_+(x,t) \rangle \\ \langle \widetilde{j}_-(x,t) \rangle \end{pmatrix} = \int_{\mathbb{R}} \mathrm{d}y \, G(x,y;t) \frac{1}{v(y)} \begin{pmatrix} \langle \widetilde{j}_+(y,0) \rangle \\ \langle \widetilde{j}_-(y,0) \rangle \end{pmatrix}$$

using $G(x,y;t) = \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \hat{G}(x,y;\omega) \mathrm{e}^{-\mathrm{i}\omega t}$ with

$$\hat{G}(x,y;\omega) = \hat{G}_+(x,y;\omega) \frac{\sigma_0 + \sigma_3}{2} + \hat{G}_-(x,y;\omega) \frac{\sigma_0 - \sigma_3}{2},$$

$$\hat{G}_\pm(x,y;\omega) = \pm\theta(\pm[x-y]) \overset{\leftrightarrows}{\mathcal{X}} \mathrm{e}^{\mathrm{i} \int_y^x \mathrm{d}s \, \mathsf{P}_\omega(s)} \sigma_3.$$

Special case: If $K(x) = K$, then $\hat{G}_\pm(x,y;\omega)$ equal

$$\hat{G}^0_\pm(x,y;\omega) = \pm\theta(\pm[x-y]) \mathrm{e}^{\mathrm{i}\omega\tau_{x,y}\sigma_3} \sigma_3, \qquad \tau_{x,y} = \int_y^x \mathrm{d}s \, \frac{1}{v(s)}.$$

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$$\hat{G}(x,y;\omega) = \hat{G}_+(x,y;\omega) \frac{\sigma_0 + \sigma_3}{2} + \hat{G}_-(x,y;\omega) \frac{\sigma_0 - \sigma_3}{2},$$
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Magnus expansion

Result:

$$\overleftrightarrow{\mathcal{X}} \mathrm{e}^{\mathrm{i}\int_y^x \mathrm{d}s\, \mathsf{P}_\omega(s)} = \exp\left[\sum_{n=1}^{\infty} \Omega_\omega^n(x,y;x)\right] \mathrm{e}^{\mathrm{i}\omega\tau_{x,y}\sigma_3}$$

with

$$\Omega_\omega^1(x,y;a) = \mathrm{i}\int_y^x \mathrm{d}s\, \mathsf{P}_\omega^1(s;a), \qquad \mathsf{P}_\omega^1(s;a) \equiv \mathrm{i}\Lambda(s)\begin{pmatrix} 0 & \mathrm{e}^{-2\mathrm{i}\omega\tau_{s,a}} \\ \mathrm{e}^{2\mathrm{i}\omega\tau_{s,a}} & 0 \end{pmatrix},$$

$$\Omega_\omega^2(x,y;a) = -\mathrm{i}\int_y^x \mathrm{d}s_1 \int_y^{s_1} \mathrm{d}s_2\, \Lambda(s_1)\Lambda(s_2)\sin(2\omega\tau_{s_1,s_2})\sigma_3,$$

and

$$\Omega_\omega^n(x,y;a) = \mathrm{i}\sum_{k=1}^{n-1} \frac{B_k}{k!} \sum_{\substack{m_1\geq 1,\ldots,m_k\geq 1 \\ m_1+\ldots+m_k=n-1}} \int_y^x \mathrm{d}s \prod_{j=1}^{k} \mathrm{ad}_{\Omega_\omega^{m_j}(s,y;a)}\, \mathsf{P}_\omega^1(s;a)$$

for $n \geq 3$ consist of similar nested spatial integrals of $\mathfrak{sl}(2,\mathbb{C})$-valued functions that vanish at $\omega = 0$. (Bernoulli numbers B_k with $B_1 = -1/2$)

Late-time asymptotics

If $\omega = 0$, then $\mathsf{P}_0(x) = \mathsf{P}_0^1(x;\cdot) = \mathrm{i}\Lambda(x)\sigma_1$ for different x commute.
$\implies$ Only non-zero contribution in the Magnus expansion is

$$\exp\left[-\int_y^x \mathrm{d}s\,\Lambda(s)\sigma_1\right] = \begin{pmatrix} \frac{\sqrt{\frac{K(y)}{K(x)}}+\sqrt{\frac{K(x)}{K(y)}}}{2} & \frac{\sqrt{\frac{K(y)}{K(x)}}-\sqrt{\frac{K(x)}{K(y)}}}{2} \\ \frac{\sqrt{\frac{K(y)}{K(x)}}-\sqrt{\frac{K(x)}{K(y)}}}{2} & \frac{\sqrt{\frac{K(y)}{K(x)}}+\sqrt{\frac{K(x)}{K(y)}}}{2} \end{pmatrix} \equiv \mathsf{T}(x,y)$$

since $\Lambda(x) = \partial_x \log\big(\sqrt{K(x)}\,\big)$.

Result: Leading $t \gg 1$ contribution to $G(x,y;t)$ is $\mathsf{T}(x,y)G^0(x,y;t)$.

Corollary: For the current $\jmath = \sqrt{K(x)}\big(\widetilde{j}_+ + \widetilde{j}_-\big)$,

$$\langle \jmath(x,t)\rangle = \int_{\mathbb{R}} \mathrm{d}y\,\frac{\delta(\tau_{x,y}-t)-\delta(\tau_{x,y}+t)}{2}\langle\rho(y,0)\rangle + \int_{\mathbb{R}} \mathrm{d}y\,\frac{\delta(\tau_{x,y}-t)+\delta(\tau_{x,y}+t)}{2v(y)}\langle\jmath(y,0)\rangle + o(t^{-1})$$

when $t \gg 1$ for all $K(x)$.

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Transfer matrix

Consider a subsystem on a finite interval $[y, x]$ with $\langle \widetilde{j}_{\pm}(\cdot, 0) \rangle = 0$ inside and currents instead incident at y and x.

Result: The transfer matrix $\mathsf{T}(\omega)$ between $(\hat{j}_+(y,\omega), \hat{j}_-(y,\omega))^T$ and $(\hat{j}_+(x,\omega), \hat{j}_-(x,\omega))^T$ for $x > y$ is

$$\mathsf{T}(\omega) = \begin{pmatrix} \mathsf{T}_{++}(\omega) & \mathsf{T}_{+-}(\omega) \\ \mathsf{T}_{-+}(\omega) & \mathsf{T}_{--}(\omega) \end{pmatrix} = \overleftarrow{\mathcal{X}} \mathrm{e}^{\mathrm{i} \int_y^x \mathrm{d}s \, \mathsf{P}_\omega(s)}.$$

Simplifies for $\omega = 0$:

$$\mathsf{T}(\omega = 0) = \begin{pmatrix} \frac{\sqrt{\frac{K(y)}{K(x)}} + \sqrt{\frac{K(x)}{K(y)}}}{2} & \frac{\sqrt{\frac{K(y)}{K(x)}} - \sqrt{\frac{K(x)}{K(y)}}}{2} \\ \frac{\sqrt{\frac{K(y)}{K(x)}} - \sqrt{\frac{K(x)}{K(y)}}}{2} & \frac{\sqrt{\frac{K(y)}{K(x)}} + \sqrt{\frac{K(x)}{K(y)}}}{2} \end{pmatrix} = \mathsf{T}(x, y).$$

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Scattering matrix

Result: The scattering matrix is

$$\mathsf{S}(\omega) = \begin{pmatrix} T(\omega) & R(\omega) \\ \widetilde{R}(\omega) & T(\omega) \end{pmatrix}$$

with the transmission and reflection amplitudes $\left(|T(\omega)|^2 + |R(\omega)|^2 = 1\right)$

$$T(\omega) = \frac{1}{\mathsf{T}_{--}(\omega)}, \qquad R(\omega) = \frac{\mathsf{T}_{+-}(\omega)}{\mathsf{T}_{--}(\omega)}, \qquad \widetilde{R}(\omega) = -\overline{R(\omega)}\,\frac{T(\omega)}{\overline{T(\omega)}}.$$

Again, simplifies for $\omega = 0$:

$$T(\omega = 0) = \frac{2\sqrt{K(y)K(x)}}{K(y) + K(x)}, \qquad R(\omega = 0) = \frac{K(y) - K(x)}{K(y) + K(x)}.$$

Generalizes results for conformal interfaces and yields simple proof of independence on intermediate values of $K(\cdot)$ for quantum wires.

[Bachas, Brunner, JHEP (2008)], [Maslov, Stone, PRB (1995)], [Safi, Schulz, PRB (1995)]

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Summary

- Showed that the dynamics of inhomogeneous TLLs are described by inhomogeneous DBdG equations.
- Obtained general solution of the DBdG equations.
- Derived explicit results at late time or at stationarity that generalize known results in the literature.
- Used results to study coupled FQH edges, quantum wires, and quantum quenches.
- Results applicable whenever DBdG-type equations appear and approach directly generalizable to other algebras than $\mathfrak{sl}(2, \mathbb{C})$.
- Interesting to extend to heat transport and correlation functions.

Thank you for your attention!

Appendices

Remark 1: Vector and axial currents

The PDEs are equivalent to existence of vector and axial current with

$$\rho(x) = \Pi(x), \qquad \jmath(x) = v(x)K(x)\rho_5(x),$$
$$\rho_5(x) = -\partial_x\varphi(x)/\pi, \qquad \jmath_5(x) = \frac{v(x)}{K(x)}\rho(x),$$

satisfying

$$\partial_t\rho + \partial_x\jmath = 0, \qquad \partial_t\jmath + v(x)K(x)\partial_x\big[v(x)K(x)^{-1}\rho\big] = 0,$$
$$\partial_t\rho_5 + \partial_x\jmath_5 = 0, \qquad \partial_t\jmath_5 + v(x)K(x)^{-1}\partial_x\big[v(x)K(x)\rho_5\big] = 0,$$

In terms of quantities for right/left movers:

$$\rho = \sqrt{K(x)}\big(\widetilde{\rho}_+ + \widetilde{\rho}_-\big), \qquad \jmath = \sqrt{K(x)}\big(\widetilde{j}_+ + \widetilde{j}_-\big).$$

Remark 2: Coupled U(1) current algebras

Define

$$a_n \equiv \int_{S^1_L} \mathrm{d}x\, \widetilde{\rho}_+(x)\mathrm{e}^{-2\pi \mathrm{i} n x/L}, \qquad \bar{a}_n \equiv \int_{S^1_L} \mathrm{d}x\, \widetilde{\rho}_-(x)\mathrm{e}^{2\pi \mathrm{i} n x/L}.$$

Obtain coupled U(1) current algebras:

$$[a_n, a_m] = n\delta_{n+m,0} = [\bar{a}_n, \bar{a}_m], \qquad [a_n, \bar{a}_m] = \frac{\mathrm{i}}{2\pi}\Lambda_{n-m},$$

where $\Lambda_n \equiv \int_{S^1_L} \mathrm{d}x\, \Lambda(x)\mathrm{e}^{-2\pi \mathrm{i} n x/L}$.

$\Longrightarrow$ Infinitely many coupled quantum harmonic oscillators.

Special case: If $K(x) = K$, then $\Lambda_n = 0$ and the algebras decouple.

Application: Transport in quantum wire

Consider a quantum quench turning off a smooth chemical-potential profile $\mu(x)$ at $t = 0$. Suppose there is some finite $\ell > 0$ so that

$$\mu(x), K(x), v(x) = \begin{cases} \mu_L, K_L, v_L & \text{for } x < -\ell, \\ \mu_R, K_R, v_R & \text{for } x > +\ell. \end{cases}$$

Due to universality of $\frac{v(x)}{K(x)}\langle\rho\rangle$ and equilibrium before the quench:

$$\langle\rho(y,0)\rangle = \frac{K(y)}{\pi v(y)}\mu(y), \quad \langle\jmath(y,0)\rangle = 0.$$

Inserted into the $t \gg 1$ expression for $\jmath$:

$$\lim_{t\to\infty}\langle\jmath(x,t)\rangle = \frac{\mu_+ - \mu_-}{2\pi}$$

with $\mu_+ = K_L\mu_L$ and $\mu_- = K_R\mu_R$.