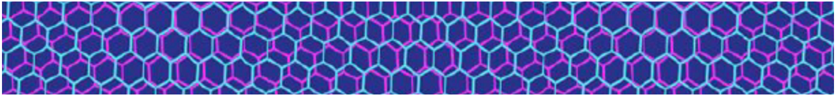
Dirac points for TBG with in-plane magnetic field

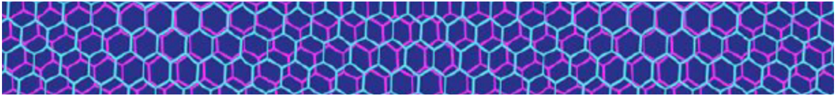
Mathematical Aspects of Condensed Matter Physics, ETH, Zürich

Maciej Zworski

July 18, 2023



Joint work with Simon Becker



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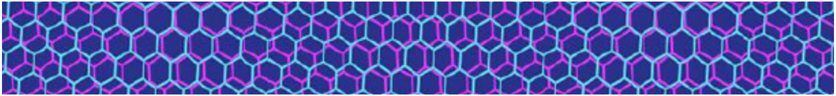
with contributions by Patrick Ledwith



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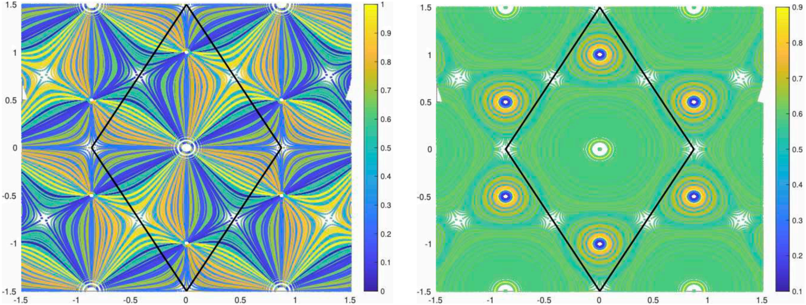

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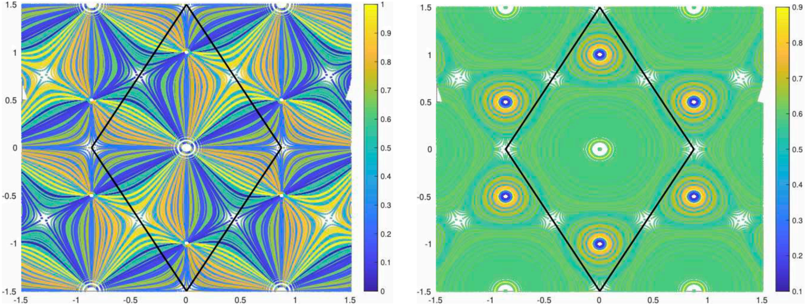

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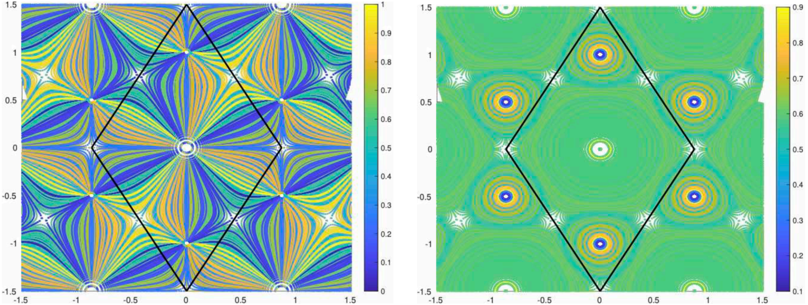

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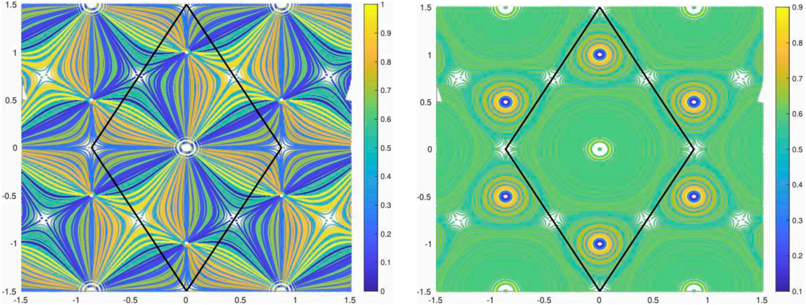

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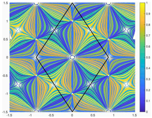

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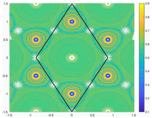

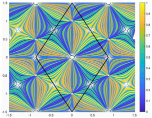

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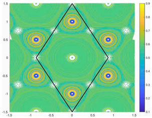

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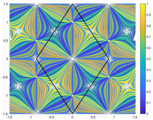

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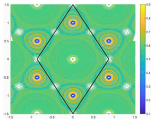

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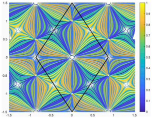

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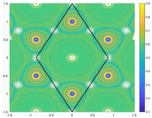

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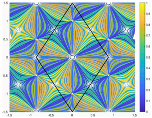

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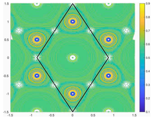

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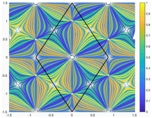

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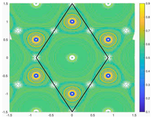

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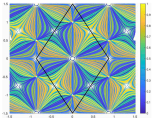

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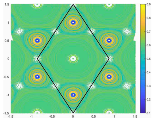

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- Qualitative agreement of the chiral model with the BM model

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Mathematical derivation:
Cancès–Garrigue–Gontier, Watson–Kong–MacDonald–Luskin '22

Bands: eigenvalues $H_k(\alpha_1, \alpha_0)$ obtained by $2D_{\bar z} \to 2D_{\bar z} + k$.

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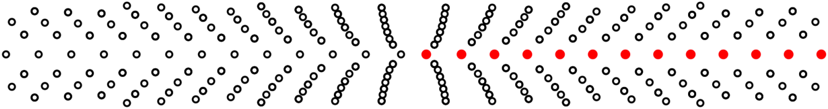

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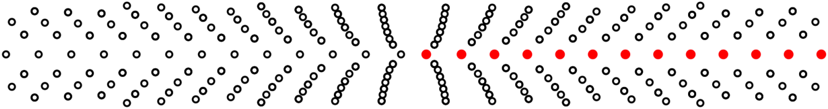

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Watson–Luskin '21: Existence of the first real α

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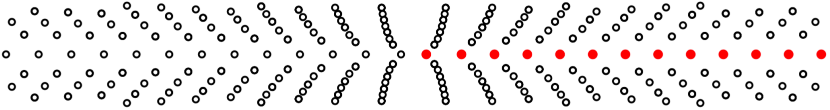

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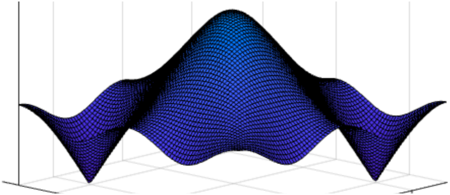
Tarnopolsky–Kruchkov–Vishwanath '19:
symmetry protected states fixed at $\pm K$ $_{\omega K \equiv K \mod \Lambda^*}$



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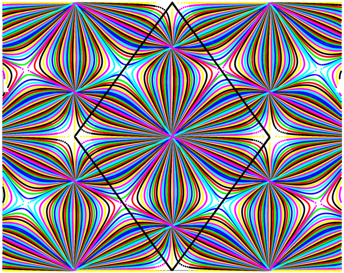

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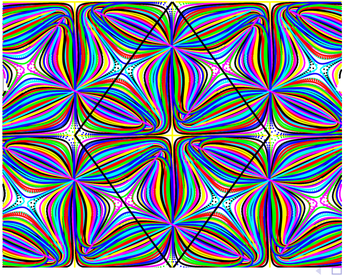

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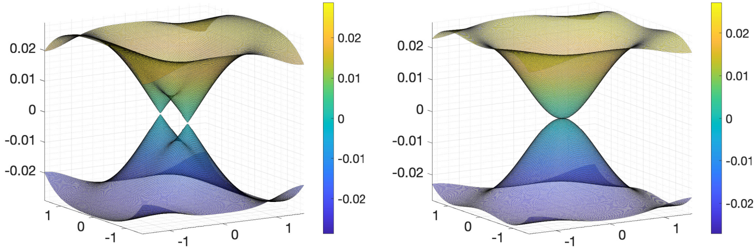

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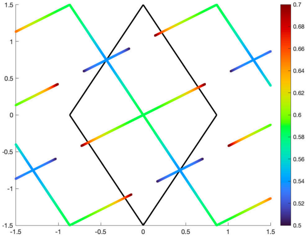

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Becker–Humbert–Z '22: If $\alpha \in \mathcal{A}$ is simple then the unique zero has to appear at the stacking point $z_S := -z(K) = \sqrt{3}/i$.

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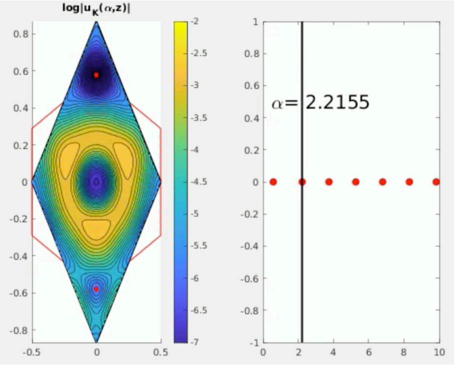

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In-plane magnetic field as a perturbation

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Magic angle $\underline{\alpha}$	0.585	2.221	3.751	5.276	6.794
$\lvert g_0(\underline{\alpha}) \rvert \simeq$	7e-02	5 e-04	7 e-04	2 e-05	3 e-05

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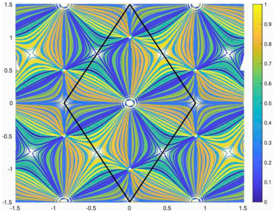
More Grushin problems + symmetries + spectral characterization:



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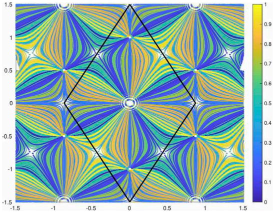

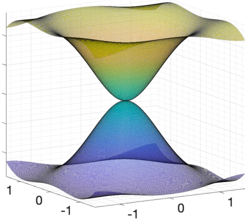

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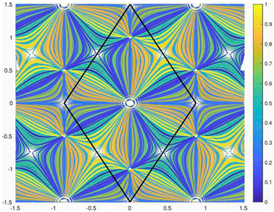
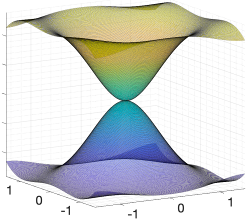
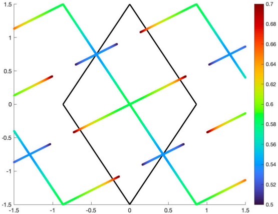

Relevance to the full Bistritzer–MacDonald model?

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$$H_B(\alpha_1, \alpha_0) := \begin{pmatrix} \alpha_0 C & D(\alpha_1)^* \\ D_B(\alpha_1) & \alpha_0 C \end{pmatrix} : H^1(\mathbb{C}; \mathbb{C}^4) \to L^2(\mathbb{C}; \mathbb{C}^4)$$

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$E_{\pm 1}(k,\alpha,t\alpha) = \textbf{?}, \quad t \to 0+$

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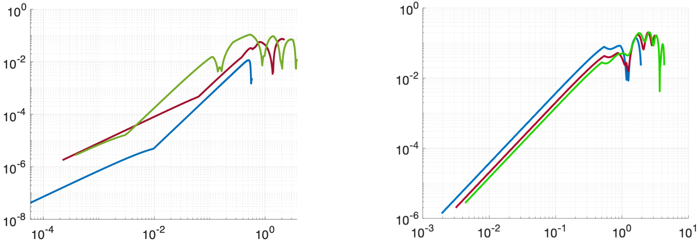

$\alpha_1 \simeq \{0.586, 2.221, 3.751\}$ $\quad$ $\alpha_1 \in \{1.121 + 1.57i, 1.312 + 2.862i, 1.438 + 4.11i\}$

Seemingly quadratic at real magic angles and linear at complex magic angles

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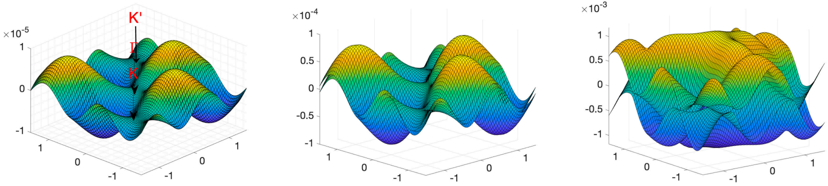
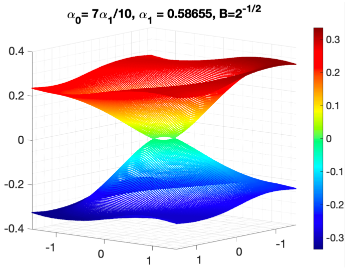
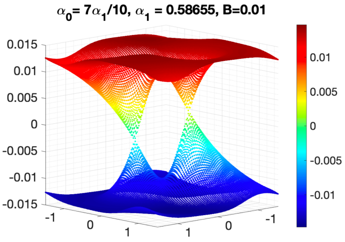


$t\alpha = 10^{-3}, 10^{-2}, 10^{-1}$

Relevance to the full Bistritzer–MacDonald model?

Adding in-plane magnetic field: Dirac points not at zero energy





$$\theta \mapsto B = e^{i\theta}/\sqrt{2} \qquad\qquad 0 < B < 1/\sqrt{2}$$

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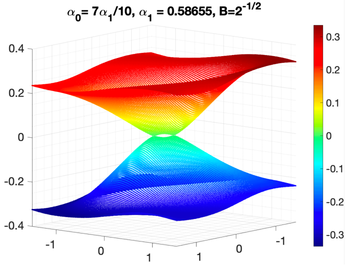

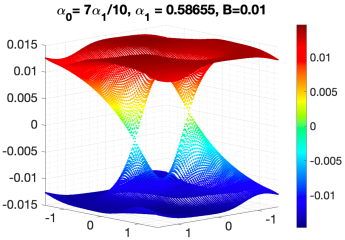

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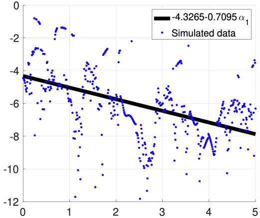
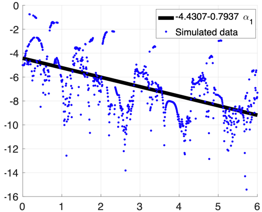
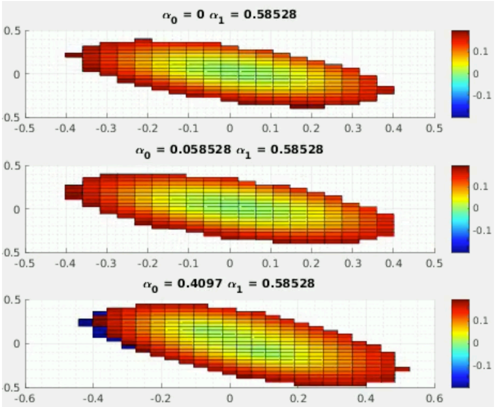
Approximate Dirac tips splitting for $B = 1/\sqrt{2}$, $\frac{1}{2}(1+i)$ and $\alpha_0 = 0.7\alpha_1$: $e^{-c\alpha_1}$?

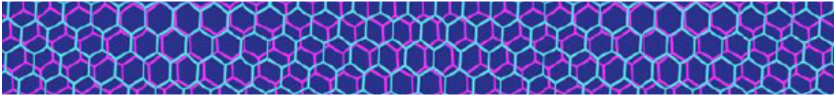




Relevance to the full Bistritzer–MacDonald model?

Comparison of Dirac points for chiral, weakly interacting, and full BM Hamiltonian with in-plane field $B = 0.5(1 + i)$; many features persist...





Thanks for your attention!