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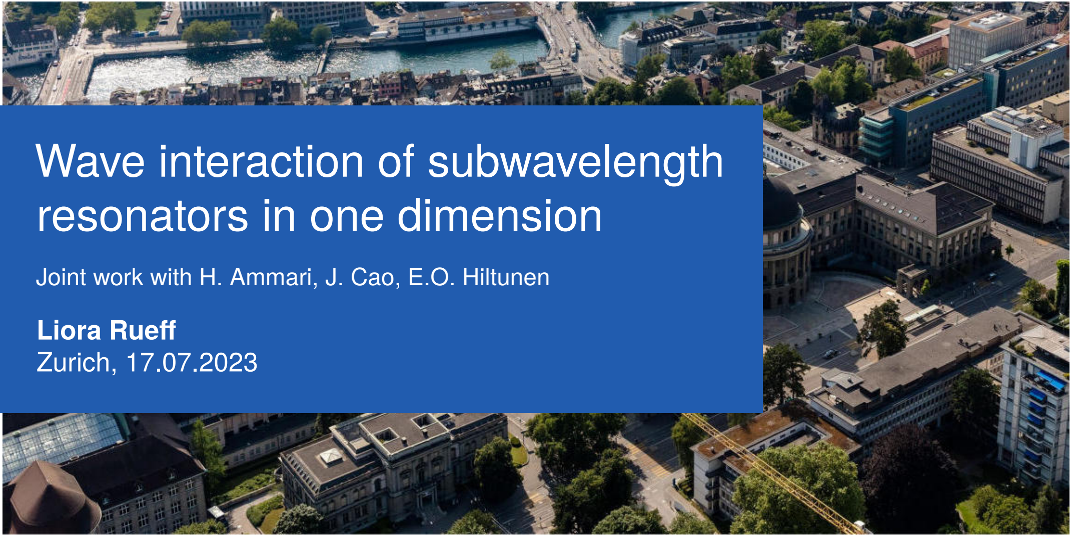

Wave interaction of subwavelength resonators in one dimension

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Outline

1. Motivation

2. Problem Formulation

3. Numerical Solution and Approximation

4. Conclusion & Outlook

Outline

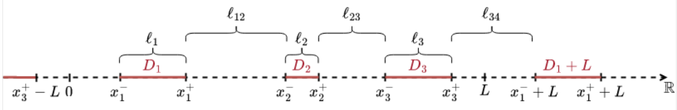
Motivation

- Goal: Focus, trap, guide, manipulate and control waves at subwavelength scales.
- Why 1D?
  - Explicit calculations are possible;
  - Only neighboring resonators interact with each other;
  - Analogies with quantum mechanical phenomena (tight-binding approximation for quantum systems) $\Rightarrow$ connects the field of high-contrast metamaterials to condensed-matter theory.
- Why time-modulated?
  - Formation of k-gaps;
  - Many wave operations such as signal amplification/compression, spacetime cloaking, ...
- Applications: Wireless communications, biomedical superresolution imaging, quantum computing.
- Tools: PDE model, capacitance matrix

Outline

Problem Formulation
Geometric Setup

- Subwavelength resonators: Objects exhibiting resonant phenomena in response to wavelengths much greater than their size. Subwavelength = size of resonators is much smaller than the operating wavelength.
- Unit cell: An interval $Y := (0, L)$ containing N resonators $D_i := (x_i^-, x_i^+)$, $\forall\, i = 1, \dots, N$, each of length ℓ_i and spacing $\ell_{i(i+1)}$ between D_i and D_{i+1}.
- Infinite system: Infinitely many contiguous unit cells covering $\mathbb{R}$, the regime taken up by the resonators is denoted by $D + L\mathbb{Z} := \{x + kL : x \in D, k \in \mathbb{Z}\}$, where $D := \bigcup_{i=1}^{N} D_i$.



Problem Formulation

Material Parameters

- Time-dependency: Periodic in x with period L and in t with period $T := 2\pi/\Omega$, given by

$$\kappa(x,t) = \begin{cases} \kappa_0, & x \notin D, \\ \kappa_\mathrm{r}\kappa_i(t), & x \in D_i, \end{cases} \quad \frac{1}{\kappa_i(t)} = \sum_{n=-M}^{M} k_{i,n} \mathrm{e}^{\mathrm{i}n\Omega t},$$

$$\rho(x,t) = \begin{cases} \rho_0, & x \notin D, \\ \rho_\mathrm{r}\rho_i(t), & x \in D_i, \end{cases} \quad \frac{1}{\rho_i(t)} = \sum_{n=-M}^{M} r_{i,n} \mathrm{e}^{\mathrm{i}n\Omega t}.$$

- High contrast assumption: $\delta := \rho_\mathrm{r}/\rho_0 \ll 1$.
- Wave speed: $v_0 := \sqrt{\kappa_0/\rho_0}$ outside D and $v_\mathrm{r} := \sqrt{\kappa_\mathrm{r}/\rho_\mathrm{r}}$ inside D.
- Difficulty: Folding of resonant frequencies into the first Brillouin zone in time. $\Rightarrow$ Only consider resonant frequencies corresponding to eigenmodes essentially supported in the subwavelength regime. $\Rightarrow$ subwavelength quasifrequencies

Problem Formulation
Goal

- Goal: For $\Omega = O(\delta^{1/2})$ find $\omega = O(\delta^{1/2})$ s.t.

$$\begin{cases} \left(\dfrac{\partial}{\partial t} \dfrac{1}{\kappa(x,t)} \dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial x} \dfrac{1}{\rho(x,t)} \dfrac{\partial}{\partial x} \right) u(x,t) = 0, \quad x \in \mathbb{R},\, t \in \mathbb{R}, \\ u(x,t)\mathrm{e}^{-\mathrm{i}\omega t} \;\text{ is } T\text{−periodic}, \\ u(x,t)\mathrm{e}^{-\mathrm{i}\alpha x} \;\text{ is } L\text{−periodic}, \end{cases}$$

has a non-trivial solution $u(x,t)$.

Problem Formulation
Governing Equations

- Fourier expansion + Floquet-Bloch in time domain + superposition of Bloch waves: $u(x,t) = \mathrm{e}^{\mathrm{i}\omega t} \sum_{n=-\infty}^{\infty} \int_{-\pi/L}^{\pi/L} \hat{v}_n(x,\alpha)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}\alpha\,\mathrm{e}^{\mathrm{i}n\Omega t}$, where α is the momentum.
- Coupled Helmholtz equations: Find $v_n(x,\alpha) := \hat{v}_n(x,\alpha)\mathrm{e}^{\mathrm{i}\alpha x}$ s.t.

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} v_n + \dfrac{\rho_0(\omega+n\Omega)^2}{\kappa_0} v_n = 0 & \text{in } (0,L)\setminus D, \\ \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} v_{i,n}^* + \dfrac{\rho_\mathrm{r}(\omega+n\Omega)^2}{\kappa_\mathrm{r}} v_{i,n}^{**} = 0 & \text{in } D_i, \\ v_n|_-\left(x_i^\pm\right) = v_n|_+\left(x_i^\pm\right) & \text{for all } 1 \le i \le N, \\ \left.\dfrac{\mathrm{d}v_{i,n}^*}{\mathrm{d}x}\right|_\pm \left(x_i^\mp\right) = \delta \left.\dfrac{\mathrm{d}v_n}{\mathrm{d}x}\right|_\mp \left(x_i^\mp\right) & \text{for all } 1 \le i \le N, \end{cases}$$

where

$$v_{i,n}^*(x,\alpha) = \sum_{m=-\infty}^{\infty} r_{i,m} v_{n-m}(x,\alpha), \quad v_{i,n}^{**}(x,\alpha) = \sum_{m=-\infty}^{\infty} k_{i,m} \frac{\omega+(n-m)\Omega}{\omega+n\Omega} v_{n-m}(x,\alpha).$$

Outline

Lemma (Exterior Solution) [FCA23, Lemma 2.1]

The following exponential Ansatz solves the Helmholtz equation in $\mathbb{R}\backslash D$:

$$v_n(x) = \alpha_i^n e^{ik^n x} + \beta_i^n e^{-ik^n x}, \quad \forall x \in \left(x_i^+, x_{i+1}^-\right),$$

for all $i = 1, \ldots, N-1$. The coefficients $(\alpha_i^n, \beta_i^n)_{i=1}^N \subset \mathbb{R}^2$ can be determined in terms of the boundary values v through

$$\begin{bmatrix} \alpha_i^n \\ \beta_i^n \end{bmatrix} = \frac{-1}{2i \sin\left(k^n \ell_{i(i+1)}\right)} \begin{bmatrix} e^{ik^n x_{i+1}^-} & -e^{-ik^n x_i^+} \\ -e^{ik^n x_{i+1}^-} & e^{ik^n x_i^+} \end{bmatrix} \begin{bmatrix} v_n(x_i^+) \\ v_n(x_{i+1}^-) \end{bmatrix},$$

for all $i = 1, \ldots, N$ and for all $n \in \mathbb{Z}$.

To do: determine $(\alpha_i^n, \beta_i^n)_{i=1}^N \subset \mathbb{C}^2, \forall n \in \mathbb{Z}$, *i.e.* determine the boundary values of v_n.

Numerical Solution and Approximation

Interior Solution

Lemma (Interior Solution) [ACHR23, Lemma 3.3]

For each resonator D_i, for $i = 1, \dots, N$, the interior problem can be written as an infinitely-dimensional system of ODEs $\Delta \mathbf{v}_i + C_i \mathbf{v}_i = \mathbf{0}$ with the unknown $\mathbf{v}_i(x, \alpha) := [v_n(x, \alpha)]_{n \in \mathbb{Z}} \in \mathbb{C}^\infty$ for all $x \in D_i$, for fixed α. Let $\{\tilde{\lambda}_n^i\}_{n \in \mathbb{Z}}$ be the set of all eigenvalues of C_i with corresponding eigenvectors $\{\mathbf{f}^{n,i}\}_{n \in \mathbb{Z}}$. Using the square-roots $\pm \lambda_n^i$ of the eigenvalues $\tilde{\lambda}_n^i$, the solution to the interior problem over D_i takes the form

$$\mathbf{v}_i = \sum_{n=-\infty}^{\infty} \left(a_i^n \mathrm{e}^{\mathrm{i} \lambda_n^i x} + b_i^n \mathrm{e}^{-\mathrm{i} \lambda_n^i x} \right) \mathbf{f}^{n,i}, \quad \forall\, x \in \left(x_i^-, x_i^+ \right),$$

for coefficients $\{(a_i^n, b_i^n)\}_{n \in \mathbb{Z}} \subset \mathbb{C}^2$.

To do: determine $\{(a_i^n, b_i^n)\}_{n \in \mathbb{Z}} \subset \mathbb{C}^2, \forall\, i = 1, \dots, N$.

Truncation: choose $K \in \mathbb{N}$ and truncate the solution, $\mathbf{v}_i = \sum_{n=-K}^{K} \left(a_i^n \mathrm{e}^{\mathrm{i} \lambda_j^i x} + b_i^n \mathrm{e}^{-\mathrm{i} \lambda_j^i x} \right) \mathbf{f}^{n,i}, \forall\, x \in D_i$.

Numerical Solution and Approximation
Dirichlet-to-Neumann Map

Definition (Dirichlet-to-Neumann Map) [FCA23, Definition 2.1]

For any $k^n \in \mathbb{C}$, for fixed $n \in \mathbb{Z}$, which is not of the form $m\pi/\ell_{i(i+1)}$ for some $m \in \mathbb{Z}\backslash\{0\}$ and $1 \leq i \leq N-1$, the *Dirichlet-to-Neumann map* with wave number $k^n := (\omega + n\Omega)/v_0$ is the linear operator $\mathcal{T}^{k^n,\alpha} : \mathbb{C}^{2N,\alpha} \to \mathbb{C}^{2N,\alpha}$ defined by

$$\mathcal{T}^{k^n,\alpha}[(v_i^\pm)_{1\leq i\leq N}] := \left(\pm \frac{\mathrm{d}v_n}{\mathrm{d}x}(x_i^\pm)\right)_{1\leq i\leq N},$$

where v_n is the unique solution to the exterior Helmholtz equation and $(v_i^\pm)_{i=1}^N \subset \mathbb{C}^{2N,\alpha}$ is a sequence of quasi-periodic boundary data defined s.t. $v_{i+N}^\pm = \mathrm{e}^{\mathrm{i}\alpha L} v_i^\pm$.

The Dirichlet-to-Neumann map can be expressed explicitly through a matrix-vector multiplication, where we denote the matrix by $\mathcal{T}^{k^n,\alpha} \in \mathbb{C}^{2N\times 2N}$.

Transmission condition: $\left.\frac{\mathrm{d}v_{i,n}^*}{\mathrm{d}x}\right|_\pm (x_i^\mp) = \delta \left.\frac{\mathrm{d}v_n}{\mathrm{d}x}\right|_\mp (x_i^\mp) \Rightarrow \pm\frac{\mathrm{d}}{\mathrm{d}x} v_{i,n}^*(x_i^\pm, \alpha) = \delta \mathcal{T}^{k^n,\alpha}[v_n]_i^\pm$

Lemma (Transmission Condition) [ACHR23, Theorem 3.4]

The subwavelength quasifrequencies ω are approximately satisfying, as $\delta \to 0$, the following truncated system of non-linear equations:

$$\sum_{j=-K}^{K} \left(\mathcal{G}^{n,j} - \delta \mathcal{T}^{k^n,\alpha} \times \mathcal{V}^{n,j} \right) \mathbf{w}_j = \mathbf{0}, \ \forall -K \leq n \leq K, \quad \mathbf{w}_j := \begin{bmatrix} a_i^j \\ b_i^j \end{bmatrix}_{1 \leq i \leq N} \in \mathbb{C}^{2N}, \ \forall -K \leq j \leq K,$$

and the matrices $\mathcal{G}^{n,j} = \mathcal{G}^{n,j}(\omega)$ and $\mathcal{V}^{n,j} = \mathcal{V}^{n,j}(\omega)$ are given by

$$\mathcal{G}^{n,j} := \operatorname{diag}\left(\sum_{m=-M}^{M} r_{i,m} f_{K+1-n+m}^{j,i} \begin{bmatrix} -\mathrm{i}\lambda_j^i \mathrm{e}^{\mathrm{i}\lambda_j^i x_i^-} & \mathrm{i}\lambda_j^i \mathrm{e}^{-\mathrm{i}\lambda_j^i x_i^-} \\ \mathrm{i}\lambda_j^i \mathrm{e}^{\mathrm{i}\lambda_j^i x_i^+} & -\mathrm{i}\lambda_j^i \mathrm{e}^{-\mathrm{i}\lambda_j^i x_i^+} \end{bmatrix} \right)_{1 \leq i \leq N} \in \mathbb{C}^{2N \times 2N},$$

$$\mathcal{V}^{n,j} := \operatorname{diag}\left(f_{K+1-n}^{j,i} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\lambda_j^i x_i^-} & \mathrm{e}^{-\mathrm{i}\lambda_j^i x_i^-} \\ \mathrm{e}^{\mathrm{i}\lambda_j^i x_i^+} & \mathrm{e}^{-\mathrm{i}\lambda_j^i x_i^+} \end{bmatrix} \right)_{1 \leq i \leq N} \in \mathbb{C}^{2N \times 2N}.$$

Theorem [ACHR23, Theorem 3.4]

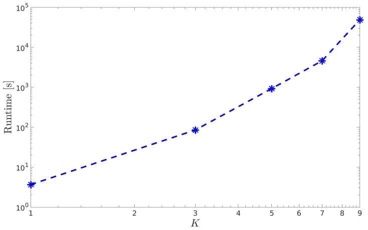
The subwavelength quasifrequencies ω are approximately satisfying $\mathcal{A}(\omega,\delta)[\mathbf{w}_j]_{j=K}^{-K} = \mathbf{0}$, where $\mathcal{A}(\omega,\delta) \in \mathbb{C}^{2N(2K+1)\times 2N(2K+1)}$ and $\mathbf{w}_j \in \mathbb{C}^{2N}$ are given by:

$$\mathcal{A}(\omega,\delta) := \begin{bmatrix} \mathcal{G}^{K,K} - \delta \mathcal{T}^{k^K,\alpha} \times \mathcal{V}^{K,K} & \cdots & \mathcal{G}^{K,-K} - \delta \mathcal{T}^{k^K,\alpha} \times \mathcal{V}^{K,-K} \\ \vdots & & \vdots \\ \mathcal{G}^{0,K} - \delta \mathcal{T}^{k^0,\alpha} \times \mathcal{V}^{0,K} & \cdots & \mathcal{G}^{0,-K} - \delta \mathcal{T}^{k^0,\alpha} \times \mathcal{V}^{0,-K} \\ \vdots & & \vdots \\ \mathcal{G}^{-K,K} - \delta \mathcal{T}^{k^{-K},\alpha} \times \mathcal{V}^{-K,K} & \cdots & \mathcal{G}^{-K,-K} - \delta \mathcal{T}^{k^{-K},\alpha} \times \mathcal{V}^{-K,-K} \end{bmatrix}, \quad \mathbf{w}_j := \begin{bmatrix} a_i^j \\ b_i^j \end{bmatrix}_{1\leq i\leq N}.$$

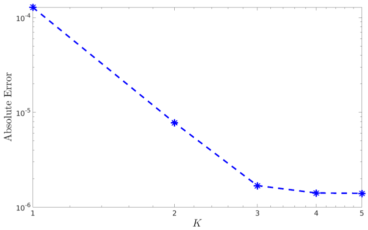
Use Muller's method to find ω for which $\mathcal{A}(\omega,\delta)$ is not invertible.

Numerical Solution and Approximation
Problems

⚠ Run-time increases with increasing N and K, K must be sufficiently large for sufficient accuracy.



The run-time depends algebraically on K.



With increasing K, the absolute error decreases.

💡 We introduce the Capacitance matrix!

Numerical Solution and Approximation

Capacitance Matrix Approximation

Lemma [AH21, Lemma 4.1]

As $\delta \to 0$, the functions $v_{i,n}^*(x, \alpha)$ are approximately constant inside the resonator:

$$v_{i,n}^*(x, \alpha)\Big|_{(x_i^-, x_i^+)} = c_{i,n} + O(\delta^{(1-\gamma)/2}).$$

Define $c_i(t) = \mathrm{e}^{\mathrm{i}\omega t} \sum_{n=-\infty}^{\infty} c_{i,n} \mathrm{e}^{\mathrm{i}n\Omega t}$.

Definition [ACHR23]

For any smooth function $f : \mathbb{R} \to \mathbb{R}$, we define $I_{\partial D_j}[f]$ by $I_{\partial D_j}[f] := \frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{-}(x_j^-) - \frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{+}(x_j^+)$.

Numerical Solution and Approximation
Capacitance Matrix Approximation

Capacitance matrix: $C^\alpha := \left(C^\alpha_{ij}\right)_{1\leq i,j\leq N}$ (nearly tridiagonal) same as in the static case [FCA23].

Theorem [ACHR23, Theorem 5.3]

The quasifrequencies in the subwavelength regime are, at leading order, given by the quasifrequencies of the system of ordinary differential equations

$$M^\alpha(t)\Psi(t) + \Psi''(t) = 0,$$

where $M^\alpha(t) = \frac{\delta\kappa_\mathrm{r}}{\rho_\mathrm{r}} W_1(t)C^\alpha W_2(t) + W_3(t)$ with W_1, W_2 and W_3 being diagonal matrices defined as

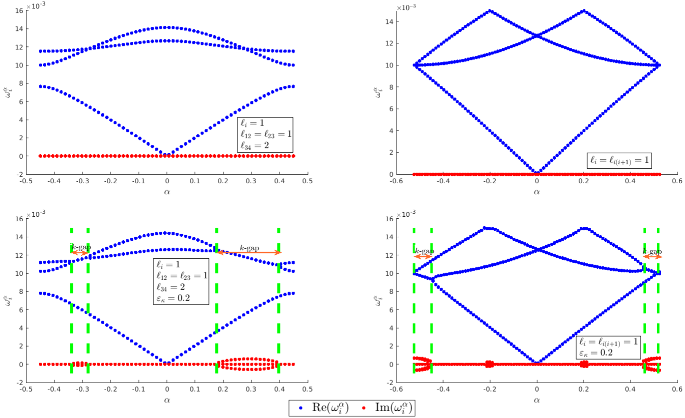
$$(W_1)_{ii} = \frac{\sqrt{\kappa_i}}{\ell_i}, \quad (W_2)_{ii} = \sqrt{\kappa_i}, \quad (W_3)_{ii} = \frac{\sqrt{\kappa_i}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\kappa_i'}{\kappa_i^{3/2}},$$

for $i = 1, \dots, N$, with

$$\Psi(t) = \left(\frac{c_i(t)}{\sqrt{\kappa_i(t)}}\right)_{i=1,\dots,N}.$$

Numerical Solution and Approximation

Numerical Simulations



Observations:

- $$\kappa_i(t) = \frac{1}{1+\varepsilon_{\kappa,i}\cos\left(\Omega t+\phi_{\kappa,i}\right)}$$
- k-gaps: undesirable α for which wave propagation is uncontrollable.
- ρ does not affect band structure at leading order.

Outline

Conclusion & Outlook

- Solve the coupled Helmholtz equations exactly up to numerical errors.
- Capacitance matrix approximation to the subwavelength quasifrequencies in one dimension for a quasi-periodic, time-modulated problem.
- Time-modulating ρ does not affect the subwavelength quasifrequencies at leading order.
- Time-modulating κ leads to the formation of k-gaps.
- Next step: Formulate the scattering problem in the dilute regime and let $N \to \infty$ while the resonators have fixed size. Derive an approximation for $N = 1$. Obtain an effective medium theory.

References

[ACHR23] Ammari Habib, Cao Jinghao, Hiltunen Erik Orvehed, and Rueff Liora, *Transmission properties of time-dependent one-dimensional metamaterials*, 2023, arXiv: 2301.09983 [math.AP].

[AH21] Ammari Habib and Hiltunen Erik Orvehed, "Time-dependent high-contrast subwavelength resonators", in: *J. Comput. Phys.* 445 (2021), p. 110594, doi: https://doi.org/10.1016/j.jcp.2021.110594.

[FCA23] Feppon Florian, Cheng Zijian, and Ammari Habib, "Subwavelength Resonances in One-Dimensional High-Contrast Acoustic Media", in: *SIAM Journal on Applied Mathematics* 83.2 (2023), pp. 625–665, doi: 10.1137/22M1503841, eprint: https://doi.org/10.1137/22M1503841.

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Additional Material

Consider the solution $V_i^\alpha : \mathbb{R} \to \mathbb{R}$ of the following problem:

$$\begin{cases} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} V_i^\alpha = 0, & (0,L)\backslash D, \\ V_i^\alpha(x) = \delta_{ij}, & x \in D_j, \\ V_i^\alpha(x+mL) = \mathrm{e}^{\mathrm{i}\alpha mL} V_i^\alpha(x), & m \in \mathbb{Z}. \end{cases}$$

The corresponding capacitance matrix is defined by

$$\begin{aligned} C_{ij}^\alpha &= \left.\frac{\mathrm{d}V_j^\alpha}{\mathrm{d}x}\right|_{-}(x_i^-) - \left.\frac{\mathrm{d}V_j^\alpha}{\mathrm{d}x}\right|_{+}(x_i^+) \\ &= -\frac{1}{\ell_{(j-1)j}}\delta_{i(j-1)} + \left(\frac{1}{\ell_{(j-1)j}} + \frac{1}{\ell_{j(j+1)}}\right)\delta_{ij} - \frac{1}{\ell_{j(j+1)}}\delta_{i(j+1)} \\ &\quad - \delta_{1j}\delta_{iN}\frac{\mathrm{e}^{-\mathrm{i}\alpha L}}{\ell_{N(N+1)}} - \delta_{1i}\delta_{jN}\frac{\mathrm{e}^{\mathrm{i}\alpha L}}{\ell_{N(N+1)}}, \end{aligned}$$

Additional Material

or equivalently by

$$C^\alpha = \begin{bmatrix} \frac{1}{\ell_{N(N+1)}} + \frac{1}{\ell_{12}} & -\frac{1}{\ell_{12}} & & & -\frac{e^{-i\alpha L}}{\ell_{N(N+1)}} \\ -\frac{1}{\ell_{12}} & \frac{1}{\ell_{12}} + \frac{1}{\ell_{23}} & -\frac{1}{\ell_{23}} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\frac{1}{\ell_{(N-1)N}} \\ -\frac{e^{i\alpha L}}{\ell_{N(N+1)}} & & & -\frac{1}{\ell_{(N-1)N}} & \frac{1}{\ell_{(N-1)N}} + \frac{1}{\ell_{N(N+1)}} \end{bmatrix}.$$

Additional Material

For fixed $n \in \mathbb{Z}$, the Dirichlet-to-Neumann map $\mathcal{T}^{k^n,\alpha}$ admits the following explicit matrix representation: for any $k^n \in \mathbb{C}\backslash\{m\pi/\ell_{i(i+1)} : m \in \mathbb{Z}\backslash\{0\}, 1 \leq i \leq N-1\}$, $f \equiv (f_i^{\pm})_{1\leq i\leq N}$, $\mathcal{T}^{k^n,\alpha}[f] \equiv (\mathcal{T}^{k^n,\alpha}[f]_i^{\pm})_{1\leq i \leq N}$ is given by

$$
\begin{bmatrix} \mathcal{T}^{k^n,\alpha}[f]_1^- \\ \mathcal{T}^{k^n,\alpha}[f]_1^+ \\ \vdots \\ \mathcal{T}^{k^n,\alpha}[f]_N^- \\ \mathcal{T}^{k^n,\alpha}[f]_N^+ \end{bmatrix} = \begin{bmatrix} -\frac{k^n \cos(k^n \ell_{N(N+1)})}{\sin(k^n \ell_{N(N+1)})} & & & & \frac{k^n}{\sin(k^n \ell_{N(N+1)})} e^{-\mathrm{i}\alpha L} \\ & A^{k^n}(\ell_{12}) & & & \\ & & \ddots & & \\ & & & A^{k^n}(\ell_{(N-1)N}) & \\ \frac{k^n}{\sin(k^n \ell_{N(N+1)})} e^{\mathrm{i}\alpha L} & & & & -\frac{k^n \cos(k^n \ell_{N(N+1)})}{\sin(k^n \ell_{N(N+1)})} \end{bmatrix} \begin{bmatrix} f_1^- \\ f_1^+ \\ \vdots \\ f_N^- \\ f_N^+ \end{bmatrix},
$$

where for any $\ell \in \mathbb{R}$, $A^{k^n}(\ell)$ denotes the 2×2 symmetric matrix

$$
A^{k^n}(\ell) := \begin{bmatrix} -\frac{k^n \cos(k^n \ell)}{\sin(k^n \ell)} & \frac{k^n}{\sin(k^n \ell)} \\ \frac{k^n}{\sin(k^n \ell)} & -\frac{k^n \cos(k^n \ell)}{\sin(k^n \ell)} \end{bmatrix}.
$$