Adiabatic Lindbladian Evolution with Small Dissipators*

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* CMP '22

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Quantum Setup

- State (density matrix): $\mathscr{H}$, Hilbert space

$$\rho \in \mathscr{T}(\mathscr{H}) \quad \text{s.t.} \quad \rho = \rho^* \geq 0\,, \mathrm{tr}\,\rho = 1$$

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- Evolution of states: $\qquad H = H^* \in \mathscr{B}(\mathscr{H})$ Hamiltonian

$$\begin{cases} i\dot{\rho} = [H, \rho] \\ \rho\,|_{t=0} = \rho_0 \in \mathscr{T}(\mathscr{H}) \end{cases} \quad \Rightarrow \quad \rho(t) = e^{-itH}\rho_0 e^{itH} \in \mathscr{T}(\mathscr{H})$$

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- Open Quantum Systems: Effect of Environment

  $\rightsquigarrow$ Approx. evolution eq. for $\rho \in \mathscr{T}(\mathscr{H})$

Lindbladian Evolution

- Markovian approximation of Quantum Dynamics:

$\dot{\rho} = \mathscr{L}(\rho),\ \rho(0) = \rho_0$ a state: $\rho_0 \geq 0,\ \ \mathrm{Tr}\rho_0 = 1.$

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$$\mathscr{L}(\rho) = -i[H,\rho] + \underbrace{\sum_j \Gamma_j \rho \Gamma_j^* - \frac{1}{2}\{\Gamma_j^*\Gamma_j, \rho\}}_{\text{dissipator } \mathscr{D}(\rho)}$$

$$H = H^* \in \mathscr{B}(\mathscr{H}),\ \ \Gamma_j \in \mathscr{B}(\mathscr{H})$$

s.t. $0 \in \sigma(\mathscr{L})$ & $\rho(t) = e^{t\mathscr{L}}\rho_0$ is a state

Quantum Engineering/Control

- Time dep. operators:

$$[0,1] \ni t \mapsto H(t) = H(t)^* \in \mathscr{B}(\mathscr{H})$$ smooth

$$[0,1] \ni t \mapsto \Gamma_j(t) \in \mathscr{B}(\mathscr{H})$$ (const. is OK)

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- Small dissipator: Coupling $0 \leq g \to 0$

$$\mathscr{L}_t^{[g]}(\,\cdot\,) = -i[H(t), \cdot\,] + g \sum_j \Gamma_j(t) \cdot \Gamma_j^*(t) - \frac{1}{2}\{\Gamma_j^*(t)\Gamma_j(t), \cdot\,\}$$

$$\equiv \mathscr{L}_t^0(\,\cdot\,) + g\mathscr{L}_t^1(\,\cdot\,)$$

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- Adiabatic regime: Time scale $1/\epsilon \to \infty$

$$\begin{cases} \epsilon\dot{\rho} = \mathscr{L}_t^{[g]}(\rho), & t \in [0,1] \\ \rho\,|_{t=0} = \rho_0 \in \mathscr{T}(\mathscr{H}) \end{cases}$$ as $(\epsilon, g) \to (0,0)$

Time dep. Lindbladian Evol.

- Two-param. Evolution op.: as $(\epsilon, g) \to (0,0)$

$$\begin{cases} \epsilon \partial_t \mathscr{U}(t,s) = (\mathscr{L}_t^0 + g\mathscr{L}_t^1)(\mathscr{U}(t,s)), \\ \mathscr{U}(s,s) = \mathbb{I}\,,\ 0 \le s \le t \le 1 \end{cases}$$

s.t. $\rho(t) = \mathscr{U}(t,0)(\rho_0)$

$\mathscr{U}(t,s) \in \mathscr{B}(\mathscr{B}(\mathscr{H}))$, contraction on $\mathscr{T}(\mathscr{H})$

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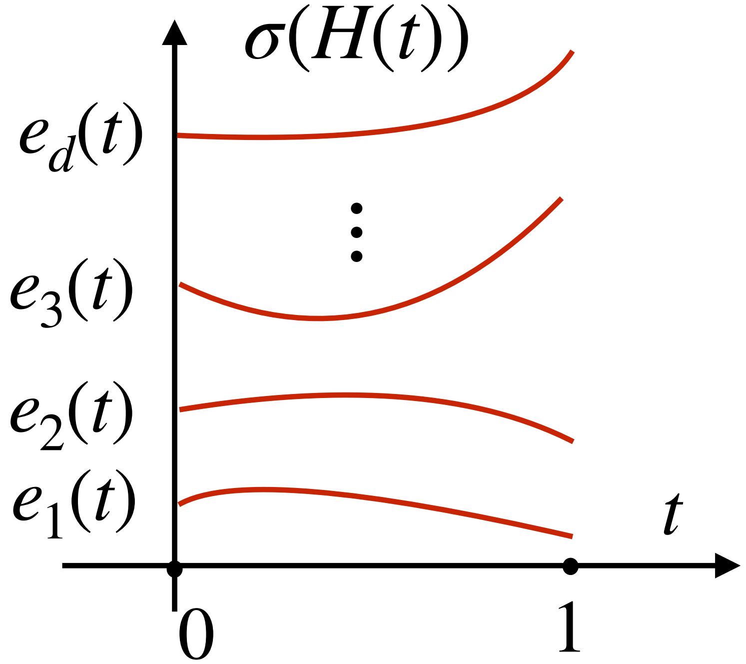
- Simplified Spectral Assumptions:

$$H(t) = \sum_{1 \le j \le d} e_j(t) P_j(t)$$

Unif. gap:

$P_j(t) = P_j^2(t) = P_j^*(t)$ spect. proj.



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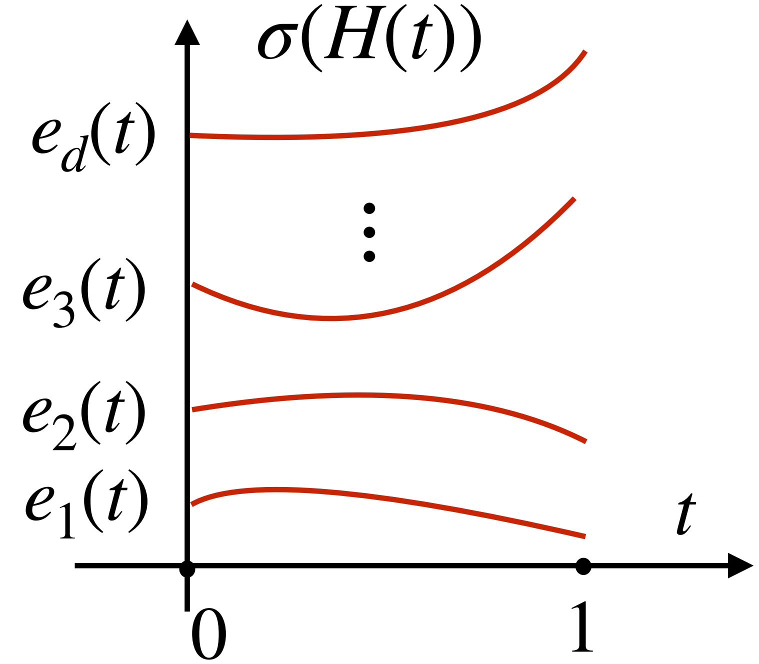

- Kato Operator:

$$\begin{cases} \partial_t W(t,s) = \sum_l P_l'(t) P_l(t) W(t,s), \\ W(s,s) = \mathbb{I},\ 0 \le s,t \le 1 \end{cases}$$

s.t. $W(t,0)P_l(0) = P_l(t)W(t,0)\ \forall l$

Transition probabilities

- Typical Question:

Let $\rho_j = P_j(0)\rho_j P_j(0)$ be a state and $\rho(t) = \mathcal{U}(t,0)(\rho_j)$

determine $\mathrm{tr}\,(P_k(t)\mathcal{U}(t,0)(\rho_j))$ as $(\epsilon, g) \to (0,0)$, for $k \neq j$

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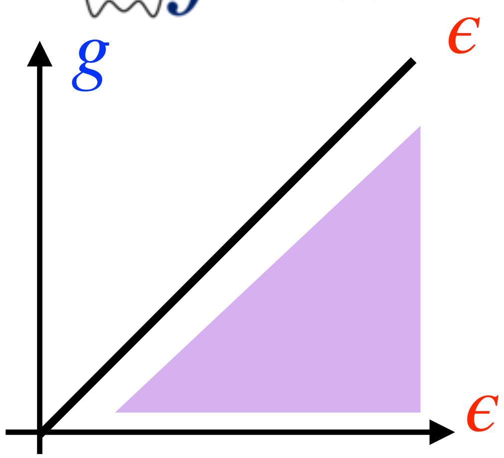
Kato '50

$$\tilde{\rho}_j(t) = W(t,0)\rho_j W(0,t) \equiv P_j(t)\tilde{\rho}_j(t)P_j(t)$$

Perturbative Regime

Thm: if $g \ll \epsilon \ll 1$ for $\rho_j = P_j(0)\rho_j P_j(0)$ a state



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Perturbative Regime

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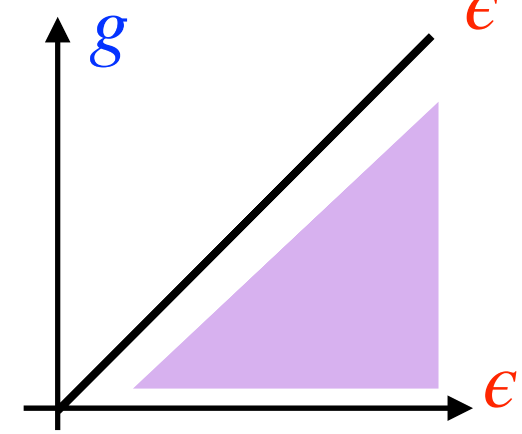

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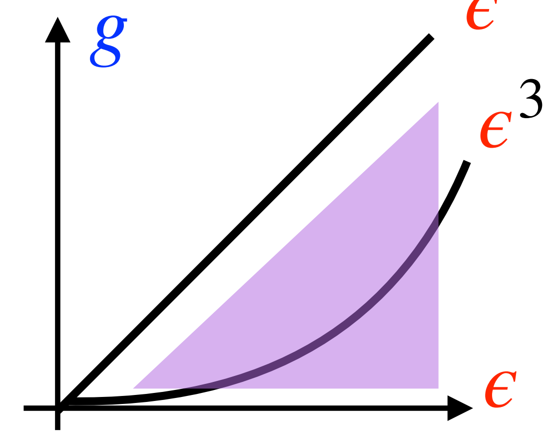

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- $g = \epsilon^3$: both terms are $O(\epsilon^2)$, with error $O(\epsilon^3)$

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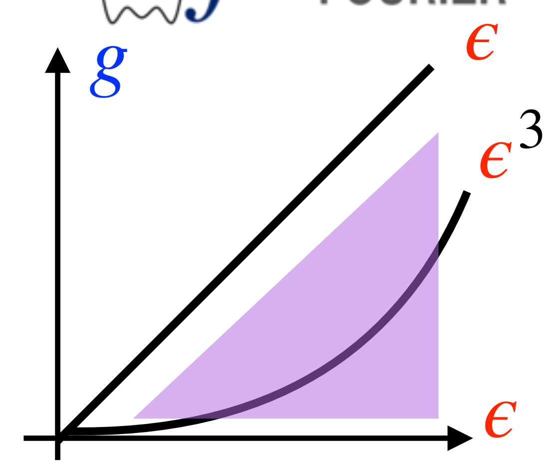

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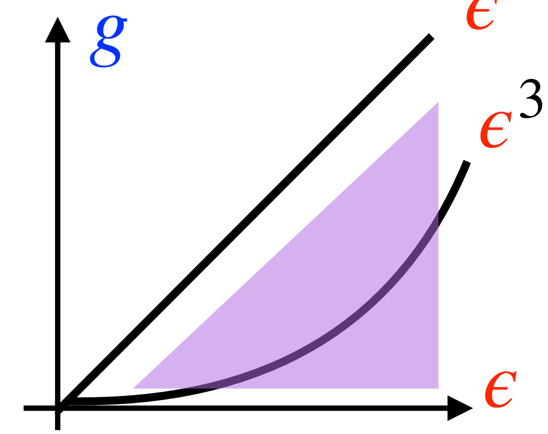

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Slow Drive Regime

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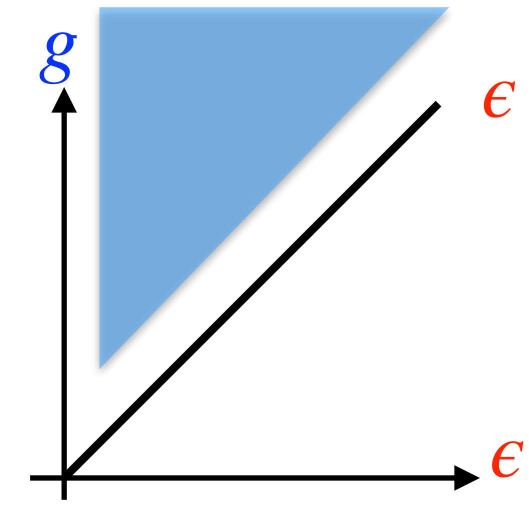

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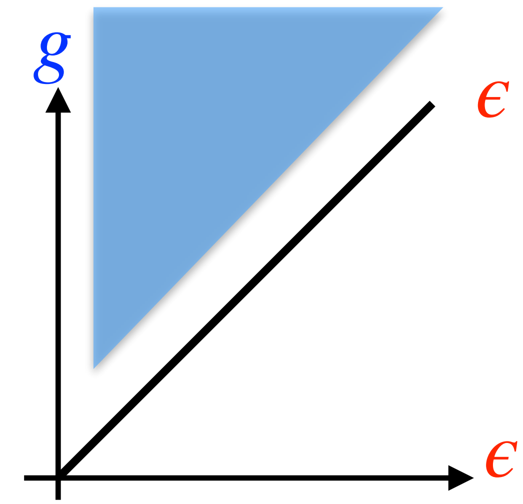

Hyp1:

$\sigma(H(t))$ simple, distinct Bohr freq. $\{e_j(t) - e_k(t)\}_{j \neq k}$

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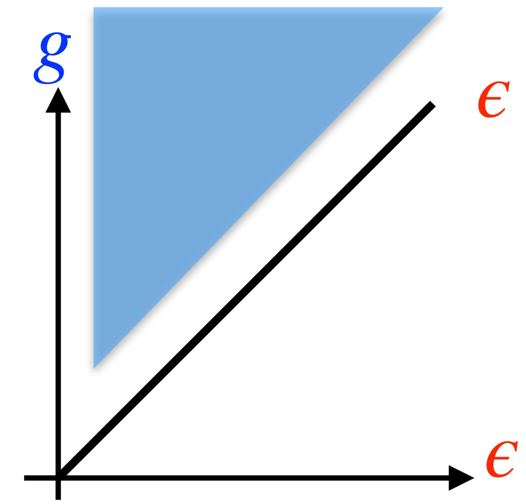

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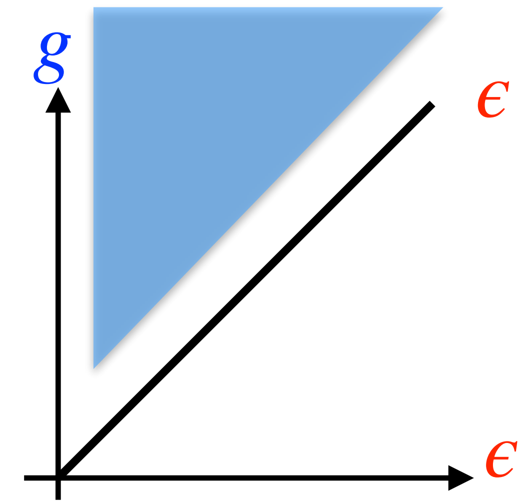

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$$\mathcal{P}_0(t)(A) = \sum_{1 \leq j \leq d} P_j(t) A P_j(t)$$

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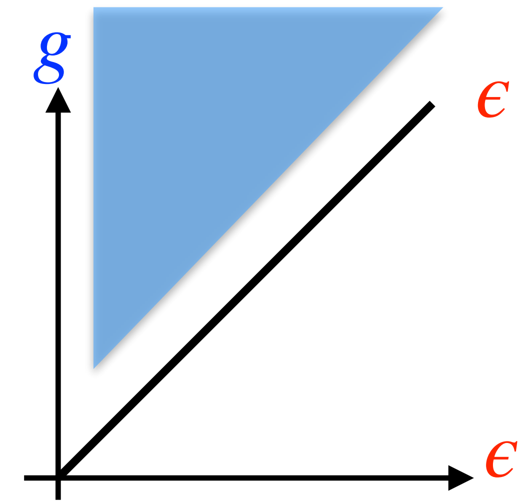

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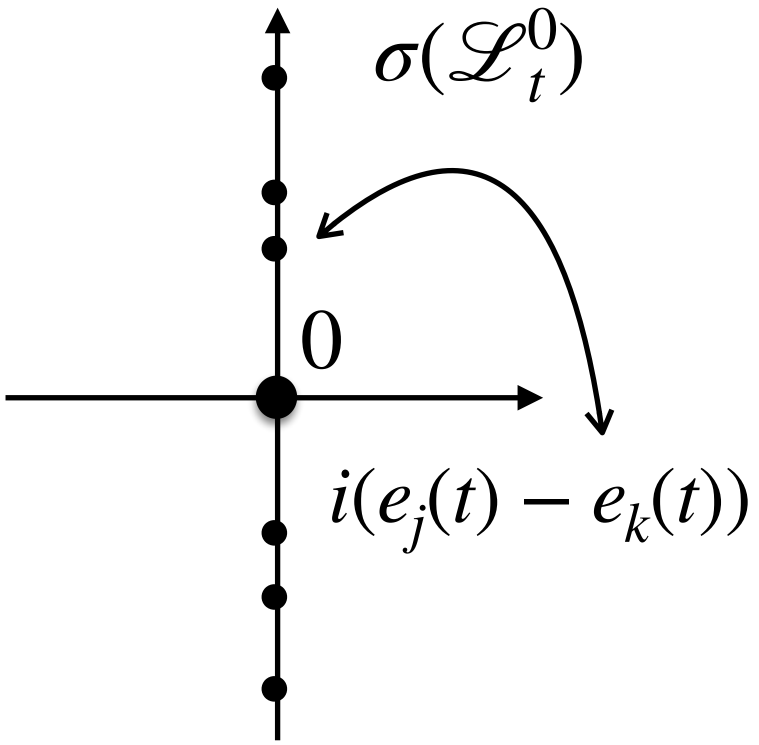
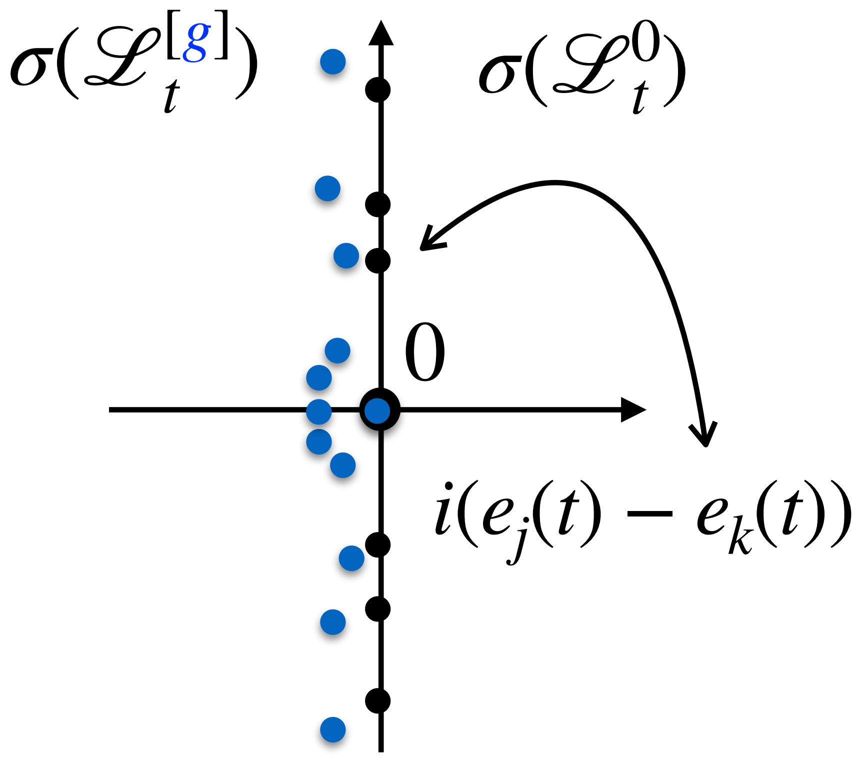
Slow Drive Regime



Slow Drive Regime

- Perturbation by $g\mathscr{L}_t^1$



Slow Drive Regime

- Perturbation of $0 \in \sigma(\mathscr{L}_t^0)$ by $g\mathscr{L}_t^1$

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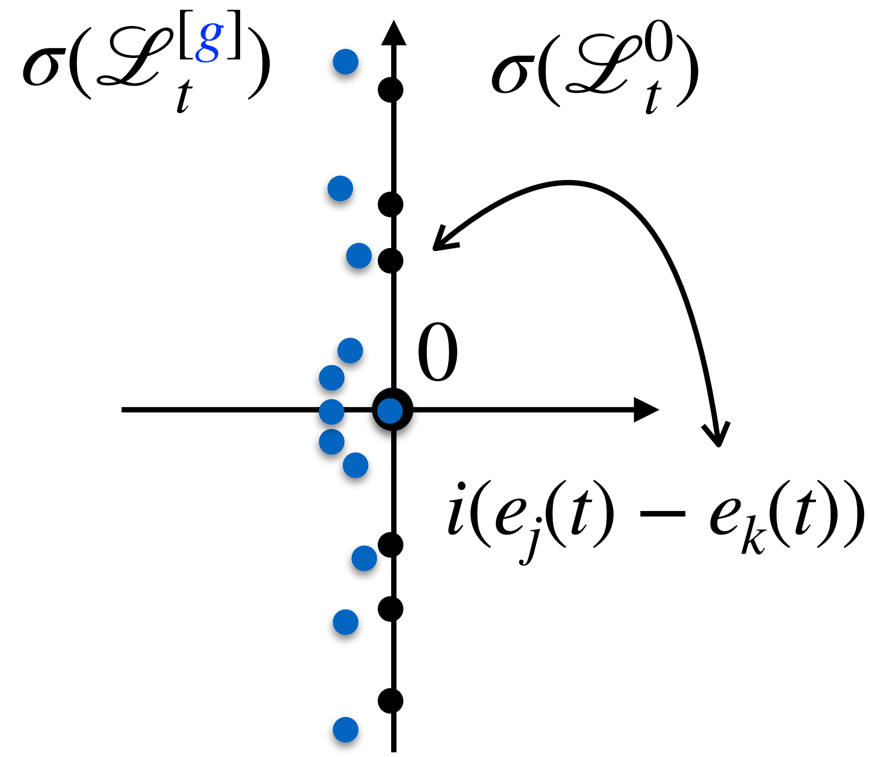

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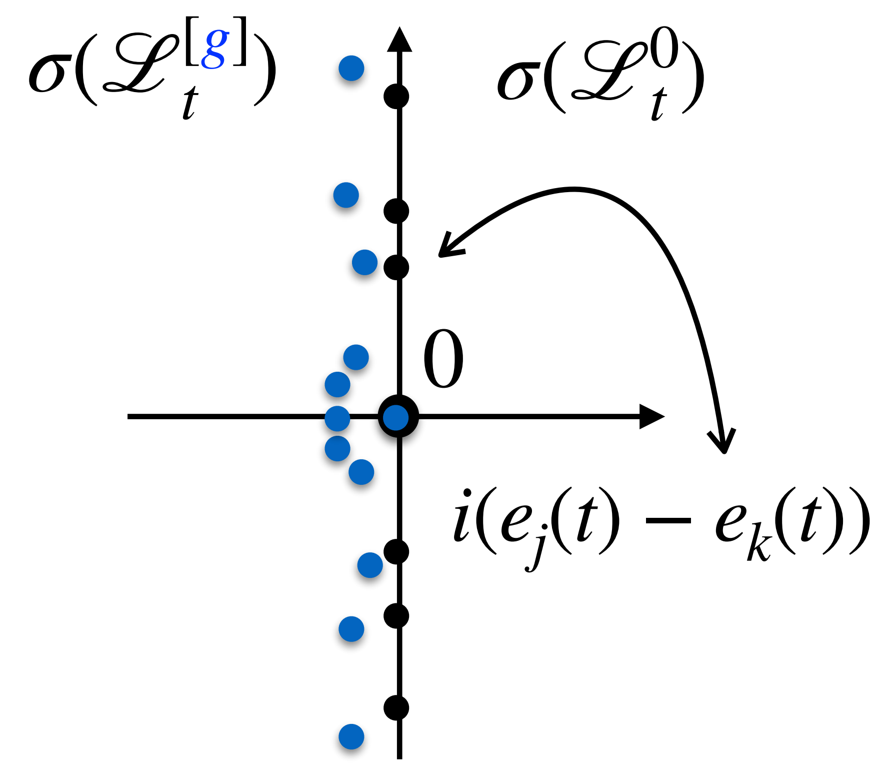
  Hyp2: maximal splitting

  $\sigma(\widetilde{\mathscr{L}}_t^1|_{\mathrm{Ker}\mathscr{L}_t^0})$ is simple $\forall t \in [0,1]$



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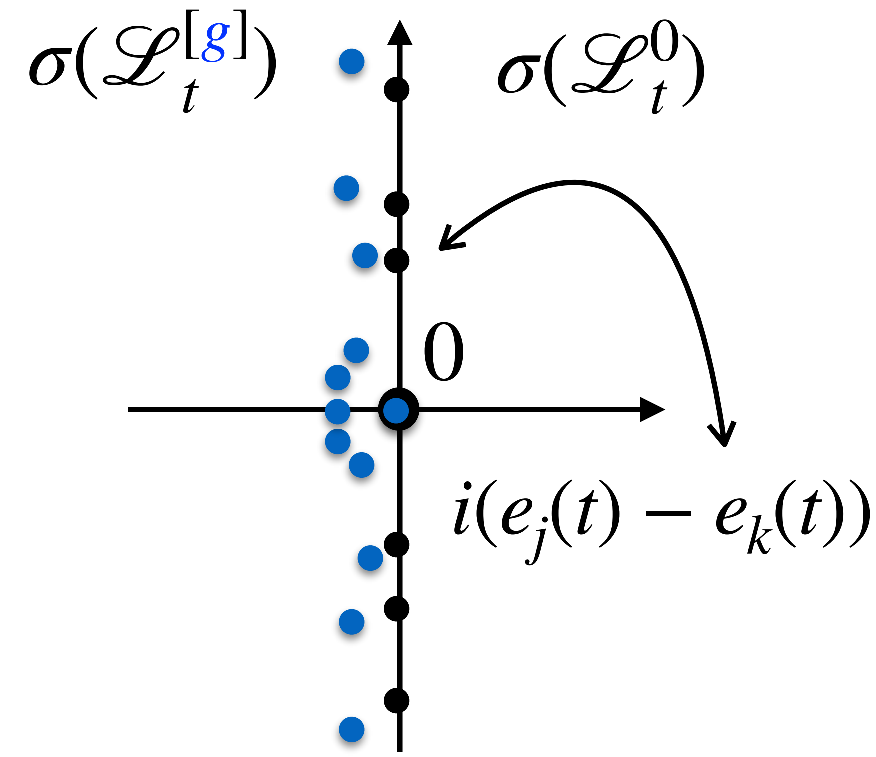

$\exists!$ state $\tilde{\nu}_0(t) = \mathscr{P}_0(t)\tilde{\nu}_0(t) \in \mathscr{B}(\mathscr{H})$ s.t. $\widetilde{\mathscr{L}}_t^1(\tilde{\nu}_0(t)) = 0$

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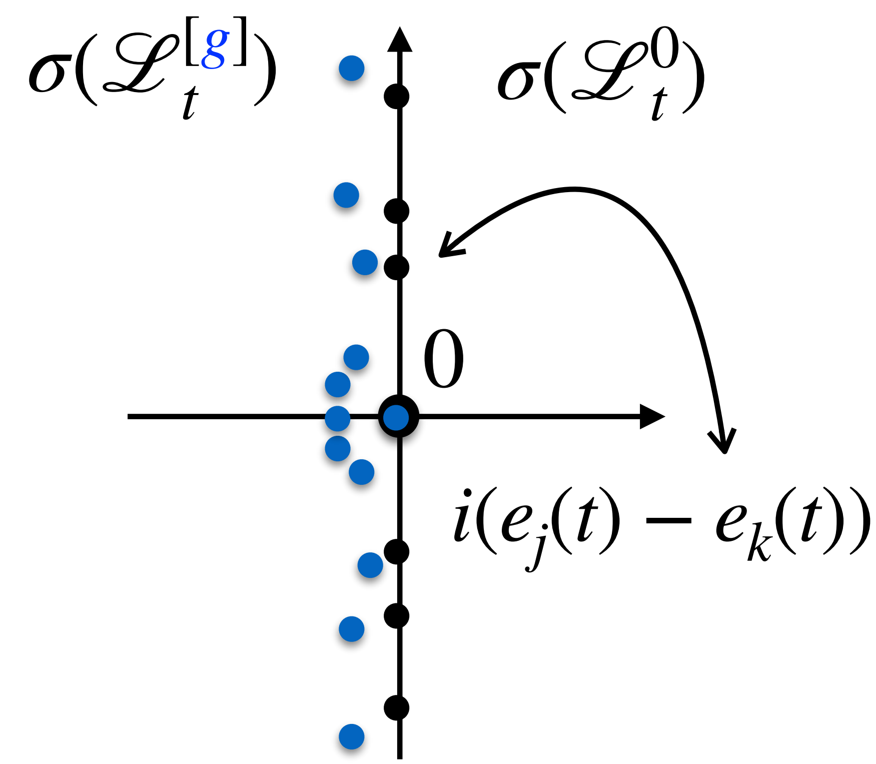

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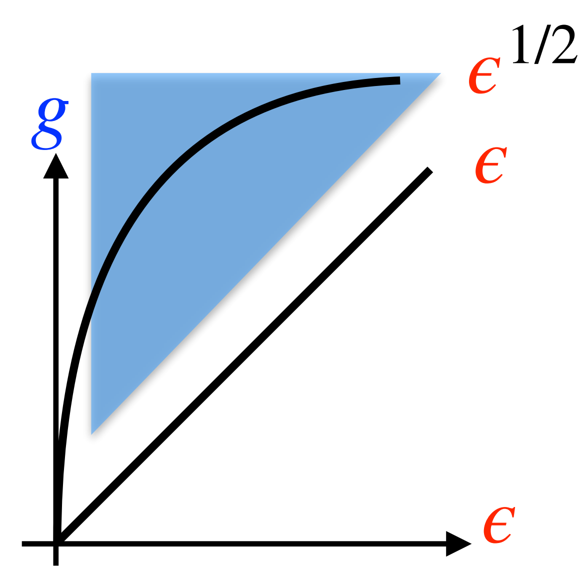
Thm:

if $\epsilon \ll g \ll \epsilon^{1/2}$ and $\rho_j = P_j(0)$ then

$$\mathrm{tr}\,(P_k(t)\mathscr{U}(t,0)(P_j(0))) = \mathrm{tr}\,(P_k(t)\tilde{\nu}_0(t)) + O(g^2/\epsilon + \epsilon/g)$$



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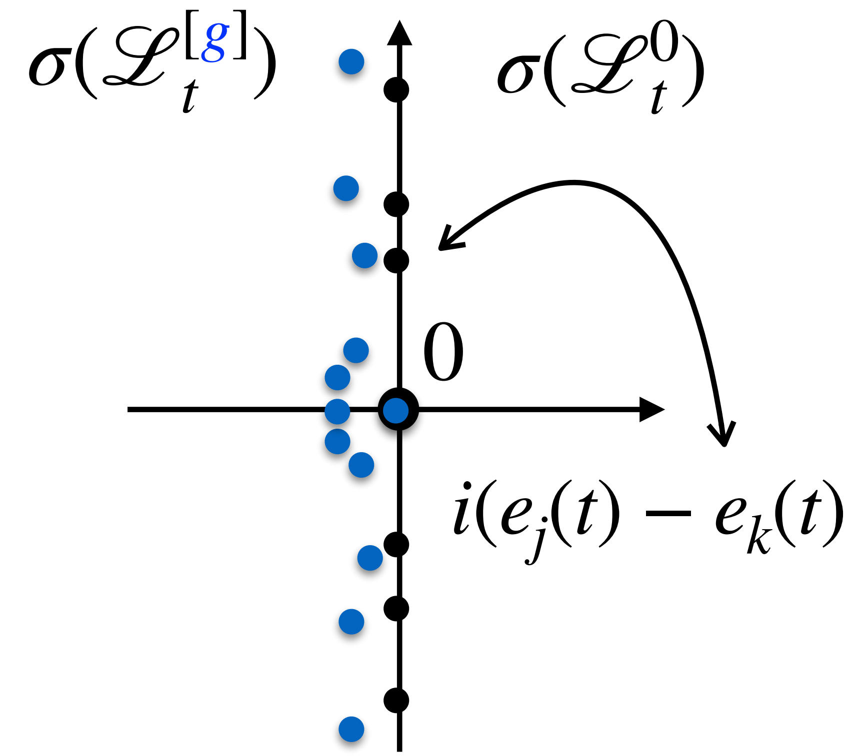
- Perturbation of $0 \in \sigma(\mathscr{L}_t^0)$ by $g\mathscr{L}_t^1$

  governed by $\widetilde{\mathscr{L}}_t^1 := \mathscr{P}_0(t)\mathscr{L}_t^1\mathscr{P}_0(t)$

  Hyp2: maximal splitting

  $\sigma(\widetilde{\mathscr{L}}_t^1|_{\mathrm{Ker}\mathscr{L}_t^0})$ is simple $\forall t \in [0,1]$

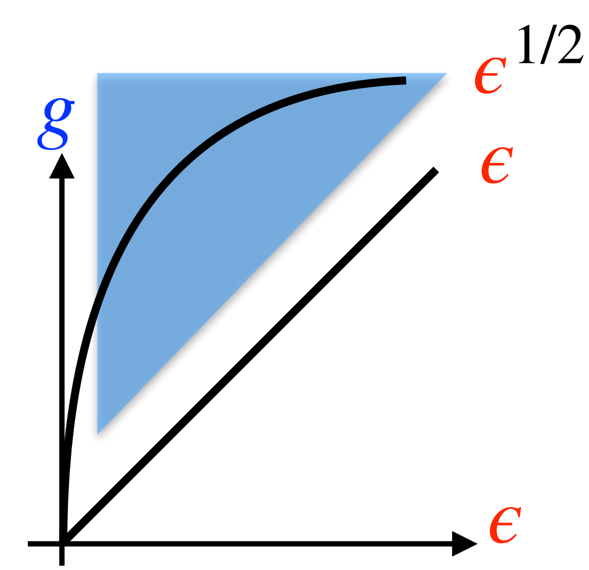


$\exists!$ state $\tilde{\nu}_0(t) = \mathscr{P}_0(t)\tilde{\nu}_0(t) \in \mathscr{B}(\mathscr{H})$ s.t. $\widetilde{\mathscr{L}}_t^1(\tilde{\nu}_0(t)) = 0$

Thm:

if $\epsilon \ll g \ll \epsilon^{1/2}$ and $\rho_j = P_j(0)$ then

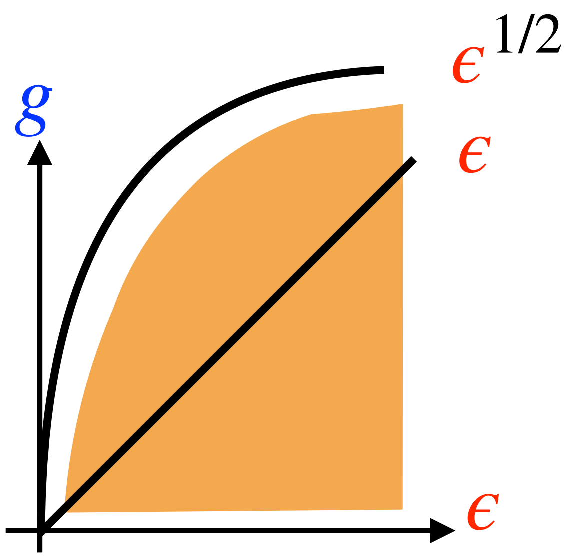
$$\mathrm{tr}\,(P_k(t)\mathscr{U}(t,0)(P_j(0))) = \mathrm{tr}\,(P_k(t)\tilde{\nu}_0(t)) + O(g^2/\epsilon + \epsilon/g)$$



- Actually: $\mathscr{U}(t,0)(P_j(0)) = \tilde{\nu}_0(t) + O(g^2/\epsilon + \epsilon/g)$

Transition Regime

- $g \ll \epsilon^{1/2} \ll 1$

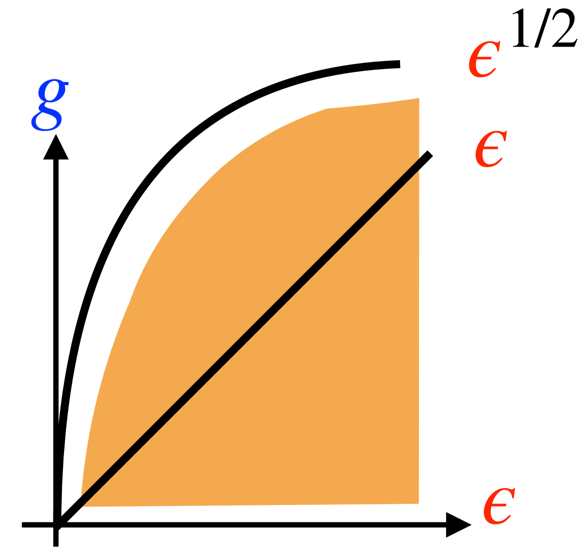


Transition Regime

- $g \ll \epsilon^{1/2} \ll 1$

Recall $\widetilde{\mathscr{L}}_t^1 := \mathscr{P}_0(t)\mathscr{L}_t^1\mathscr{P}_0(t)$

Define the Reduced Dynamics: $\Psi_\delta(t,s) \in \mathscr{B}(\mathscr{B}(\mathscr{H}))$



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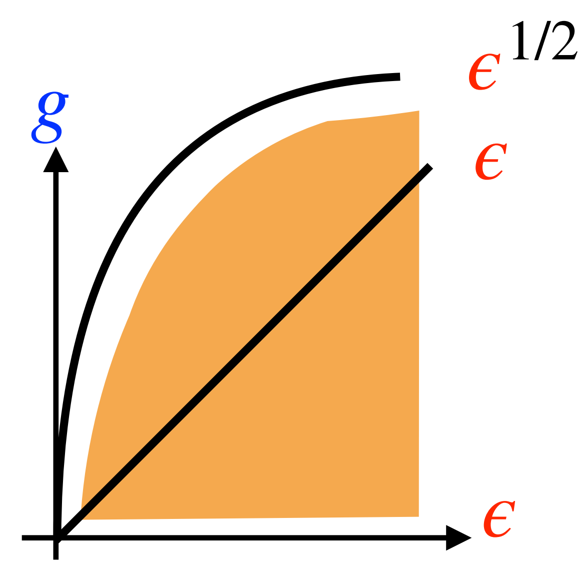
$$\begin{cases} \delta \partial_t \Psi_\delta(t,s) = \mathscr{W}_0(0,t)\widetilde{\mathscr{L}}_t^1 \mathscr{W}_0(t,0)\Psi_\delta(t,s), \\ \Psi_\delta(s,s) = \mathbb{I},\ 0 \le s \le t \le 1 \end{cases}$$

where $\delta > 0$



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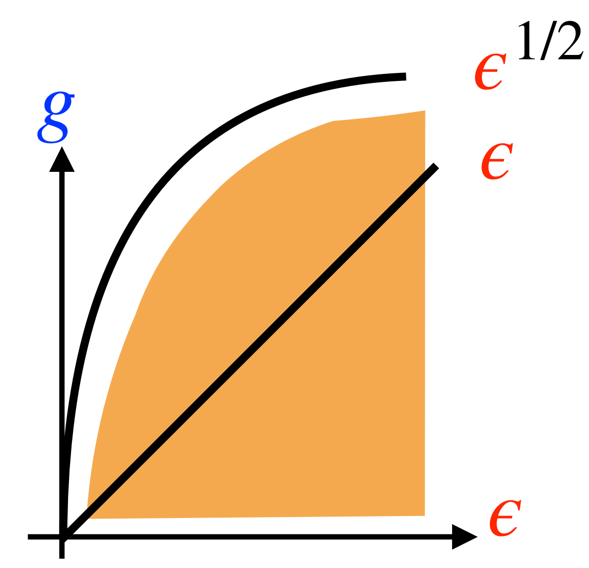
Prop: $[\Psi_\delta(t,s), \mathscr{P}_0(0)] \equiv 0$,



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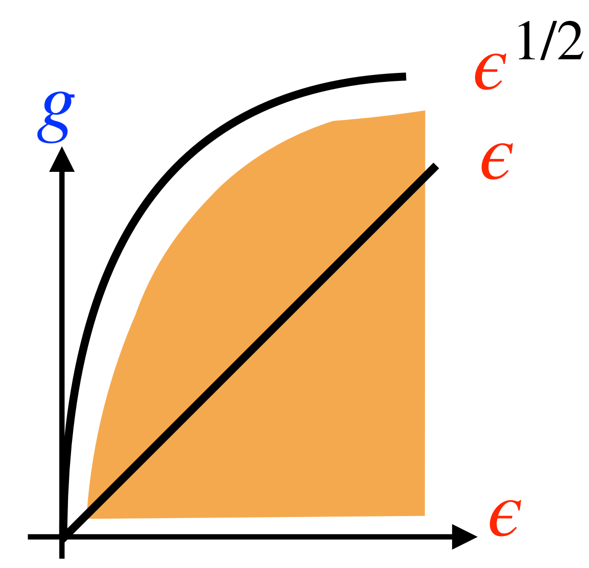

Transition Regime

- $g \ll \epsilon^{1/2} \ll 1$

Recall $\widetilde{\mathscr{L}}^1_t := \mathscr{P}_0(t)\mathscr{L}^1_t\mathscr{P}_0(t)$

Define the Reduced Dynamics: $\Psi_\delta(t, s) \in \mathscr{B}(\mathscr{B}(\mathscr{H}))$

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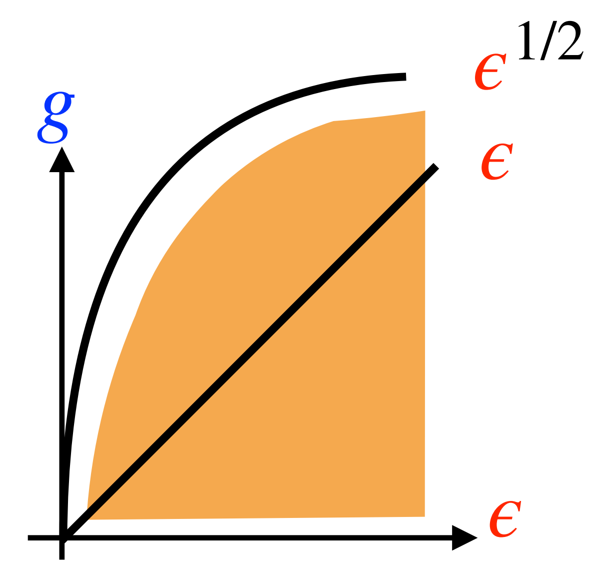
Thm: if $g \ll \epsilon^{1/2} \ll 1$

$$\mathscr{U}(t,0)\mathscr{P}_0(0) = \mathscr{W}_0(t,0)\Psi_{\epsilon/g}(t,0)\mathscr{P}_0(0) + O(\epsilon + g + g^2/\epsilon)$$



Rem: $\dim P_j(t)$ arbitrary

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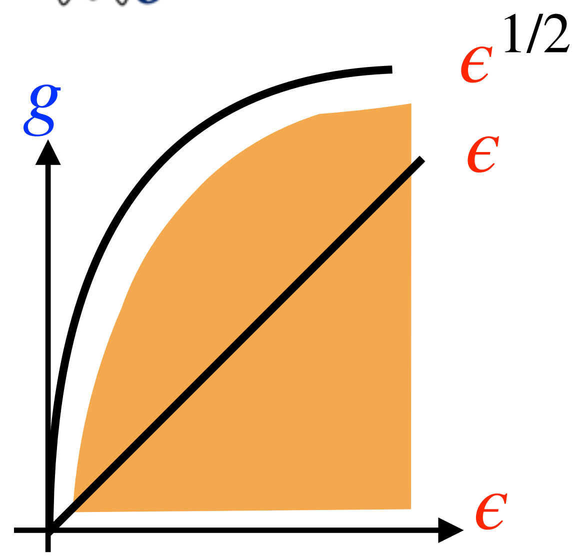
Transition Regime

Thm: if $g \ll \epsilon^{1/2} \ll 1$ for $\rho_j = P_j(0)\rho_j P_j(0)$ a state

$$\mathrm{tr}\,(P_k(t)\mathcal{U}(t,0)(\rho_j)) = \mathrm{tr}\,(P_k(0)\Psi_{\epsilon/g}(t,0)(\rho_j)) + O(\epsilon + g + g^2/\epsilon)$$



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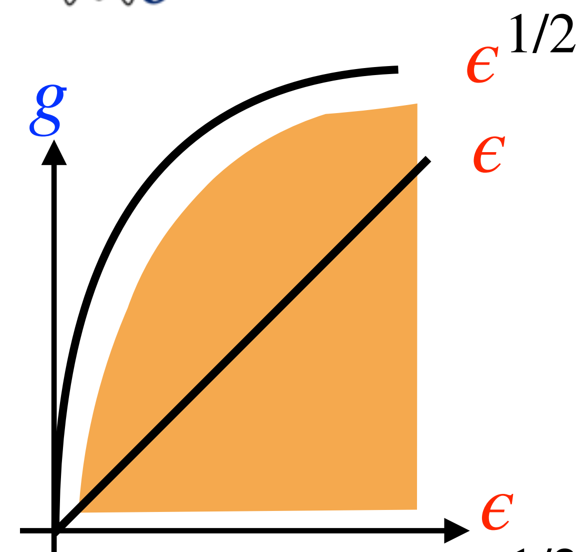
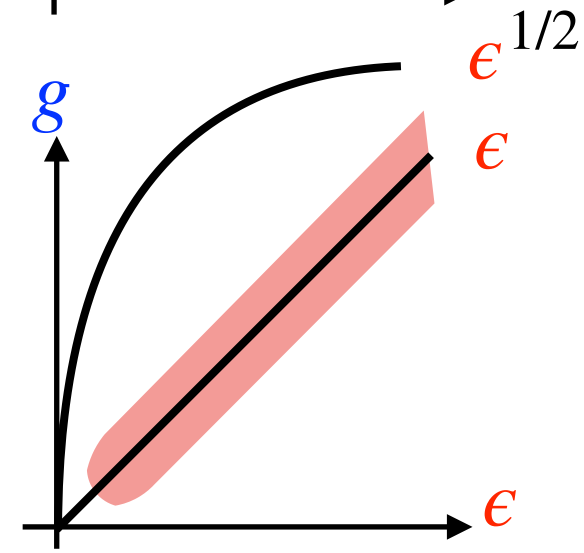


Remarks:

- $g = \epsilon$: $\Psi_{\epsilon/g}(t,0) = \Psi_1(t,0)$ s.t.

$$\mathrm{tr}\,(P_k(t)\mathcal{U}(t,0)(\rho_j)) = \mathrm{tr}\,(P_k(0)\Psi_1(t,0)(\rho_j)) + O(\epsilon)$$



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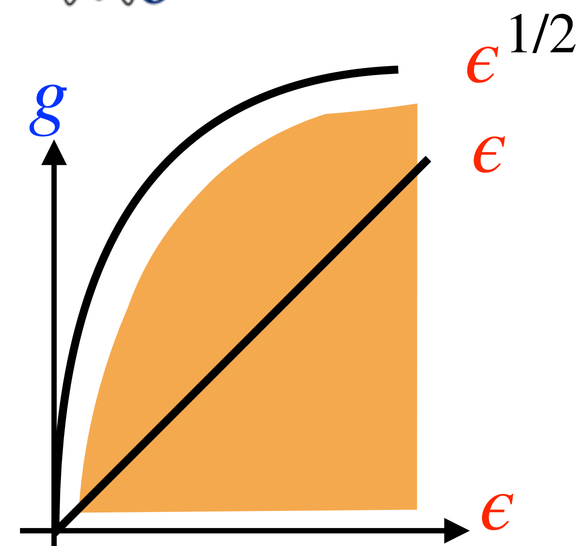

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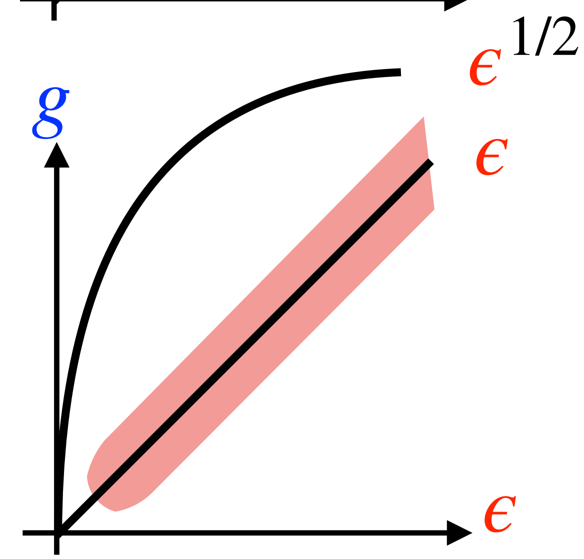

- $\Psi_{\epsilon/g}(t,0)$ "recovers" $\left\{ \begin{array}{ll} \text{perturbative} & g \ll \epsilon \ll 1 \\ \text{slow drive} & \epsilon \ll g \ll \epsilon^{1/2} \end{array} \right\}$ regimes

Transition Regime

Thm: if $g \ll \epsilon^{1/2} \ll 1$ for $\rho_j = P_j(0)\rho_j P_j(0)$ a state

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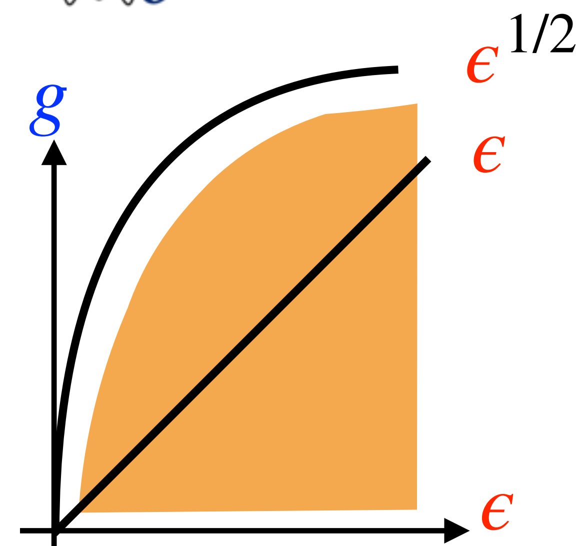

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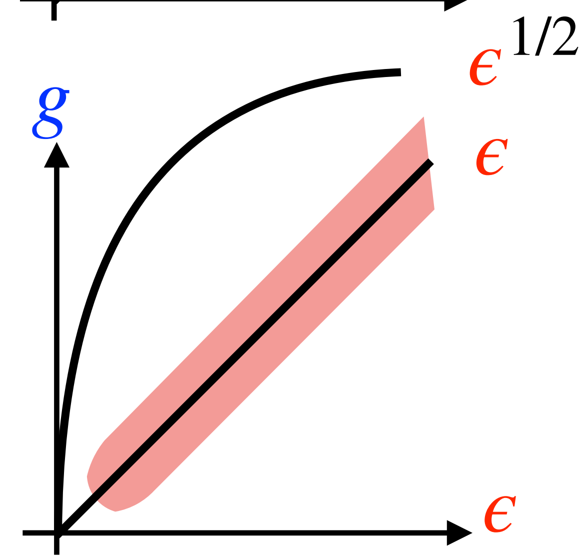

- $\Psi_{\epsilon/g}(t,0)$ "recovers" { perturbative $g \ll \epsilon \ll 1$; slow drive $\epsilon \ll g \ll \epsilon^{1/2}$ } regimes

- Natural Markov Process: if $\mathrm{Dim}\ P_j(t) \equiv 1$

$$\mathbb{P}(X_t = k \,|\, X_0 = j) = \mathrm{tr}\big(P_k(0)\Psi_{\epsilon/g}(t,0)(P_j(0))\big)$$

Methods of proofs

- Approximation of the evolution op. $\mathcal{U}(t,s)$

$$\begin{cases} \epsilon \partial_t \mathcal{U}(t,s) = (\mathcal{L}_t^0 + g\mathcal{L}_t^1)(\mathcal{U}(t,s)), \\ \mathcal{U}(s,s) = \mathbb{I}, \ 0 \le s \le t \le 1 \end{cases}$$

$$(\epsilon, g) \to (0,0)$$

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- Approximation of the evolution op. $\mathscr{U}(t,s)$

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- Dyson series in the perturbative regime $g \ll \epsilon \ll 1$

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- Dyson series in the perturbative regime $g \ll \epsilon \ll 1$

- Integration by parts in the slow drive regime $\epsilon \ll g \ll 1$

- Perturbation theory in the transition regime $g \ll \epsilon^{1/2}$

Concluding remarks

- Literature:

Adiabatics for open quantum systems

Davies-Spohn '78, Abou Salem-Fröhlich '05, J. '07, Teufel-Wachsmuth '12, Benoist-Fraas-Jaksic-Pillet '17, J.-Merkli-Spehner '20, Jaksic-Pillet-Tauber `22, J.-Merkli '23,...

Adiabatics for dephasing Lindbladians

Avron, Fraas, Graf, Grech '11, '12, Fraas, Hänggli '17,...

Perturbative results for Lindbladian dynamics

Ballestros, Crowford, Fraas, Fröhlich, Schubnel '21, Haack-J. '21, Benoist, Bernardin, Chétrite, Chhaibi, Najnudel, Pellegrini '21,...

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Thank you!

More on Reduced Dynamics

Recall: $\Psi_\delta(t,s) \in \mathscr{B}(\mathscr{B}(\mathscr{H})), \ \delta > 0$

$$\begin{cases} \delta \partial_t \Psi_\delta(t,s) = \mathscr{W}_0(0,t)\widetilde{\mathscr{L}}_t^1 \mathscr{W}_0(t,0)\Psi_\delta(t,s), \\ \Psi_\delta(s,s) = \mathbb{I},\ 0 \le s \le t \le 1 \end{cases}$$

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where $\widetilde{\mathscr{L}}_t^1 := \mathscr{P}_0(t)\mathscr{L}_t^1\mathscr{P}_0(t)$, $\mathscr{P}_0(0)$ proj. onto $\ker \mathscr{L}_0^0$

$$\mathscr{L}_t^{[g]}(\,\cdot\,) = -i[H(t),\cdot\,] + g\sum_j \Gamma_j(t)\cdot\Gamma_j^*(t) - \frac{1}{2}\{\Gamma_j^*(t)\Gamma_j(t),\cdot\,\}$$
$$\equiv \mathscr{L}_t^0(\,\cdot\,) + g\mathscr{L}_t^1(\,\cdot\,)$$

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$$\equiv \mathscr{L}_t^0(\cdot) + g\mathscr{L}_t^1(\cdot)$$

- Generator: $\mathscr{G}_t := \mathscr{W}_0(0,t)\mathscr{P}_0(t)\mathscr{L}_t^1\mathscr{P}_0(t)\mathscr{W}_0(t,0)$

$\{P_1(0), P_2(0), \dots, P_d(0)\}$ basis of $\mathscr{P}_0(0)\mathscr{B}(\mathscr{H})$

where $P_j(t) = |\varphi_j(t)\rangle\langle\varphi_j(t)|$ s.t. $H(t)\varphi_j(t) = e_j(t)\varphi_j(t)$

Classical Markov Process

Lemma: in this basis $\mathscr{G}_t$ is (the transp. of) a **Q-matrix**

$$\sum_l \begin{pmatrix} |\langle\varphi_1|\Gamma_l\varphi_1\rangle|^2 - \|\Gamma_l\varphi_1\|^2 & |\langle\varphi_1|\Gamma_l\varphi_2\rangle|^2 & & |\langle\varphi_1|\Gamma_l\varphi_d\rangle|^2 \\ |\langle\varphi_2|\Gamma_l\varphi_1\rangle|^2 & |\langle\varphi_2|\Gamma_l\varphi_2\rangle|^2 - \|\Gamma_l\varphi_2\|^2 & & |\langle\varphi_2|\Gamma_l\varphi_d\rangle|^2 \\ & & \ddots & \\ |\langle\varphi_d|\Gamma_l\varphi_1\rangle|^2 & |\langle\varphi_d|\Gamma_l\varphi_2\rangle|^2 & & |\langle\varphi_d|\Gamma_l\varphi_d\rangle|^2 - \|\Gamma_l\varphi_d\|^2 \end{pmatrix}$$

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Corollary:

$\Psi_\delta(t,0)|_{\mathrm{Span}\{P_1(0),\dots,P_d(0)\}}$ transp. of a **stochastic** matrix

Classical Markov Process

Lemma: in this basis $\mathscr{G}_t$ is (the transp. of) a Q-matrix

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Corollary:

$\Psi_\delta(t,0)|_{\mathrm{Span}\{P_1(0),\ldots,P_d(0)\}}$ transp. of a stochastic matrix

- Cont. Markov Process:

$(X_t)_{t\geq 0}$ on classical state space $\{P_1(0),\cdots,P_d(0)\} \equiv \{1,2,\ldots,d\}$

s.t. $\mathbb{P}(X_t = k \,|\, X_0 = j) = \mathrm{tr}\big(P_k(0)\Psi_\delta(t,0)(P_j(0))\big)$

Example for $d = 2$

Assume $\sum_{l} |\langle \varphi_1(t) | \Gamma_l(t) \varphi_2(t) \rangle|^2 = \sum_{l} |\langle \varphi_2(t) | \Gamma_l(t) \varphi_1(t) \rangle|^2$

Example for $d=2$

Assume $\sum_l |\langle \varphi_1(t) | \Gamma_l(t)\varphi_2(t)\rangle|^2 = \sum_l |\langle \varphi_2(t) | \Gamma_l(t)\varphi_1(t)\rangle|^2 := \gamma(t)$

$$\rightsquigarrow \quad \delta \partial_t \Psi_\delta(t,s) = \gamma(t)\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Psi_\delta(t,s)$$

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$$\Psi_\delta(t,0)|_{\mathrm{Span}\{P_1(0),P_2(0)\}} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{e^{-\frac{2}{\delta}\int_0^t \gamma(s)ds}}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

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For $\rho_0 = P_1(0)$

$$\mathscr{U}(t,0)(P_1(0)) = r_1(t)P_1(t) + r_2(t)P_2(t) + O(\epsilon + g + g^2/\epsilon) \quad \text{if} \quad g \ll \epsilon^{1/2} \ll 1$$

where $r_1(t) = (1 + e^{-\frac{2g}{\epsilon}\int_0^t \gamma(s)ds})/2, \quad r_2(t) = (1 - e^{-\frac{2g}{\epsilon}\int_0^t \gamma(s)ds})/2$

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Assume $\sum_l |\langle \varphi_1(t) | \Gamma_l(t) \varphi_2(t) \rangle|^2 = \sum_l |\langle \varphi_2(t) | \Gamma_l(t) \varphi_1(t) \rangle|^2 := \gamma(t)$

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For $\rho_0 = P_1(0)$

$$\mathcal{U}(t,0)(P_1(0)) = r_1(t)P_1(t) + r_2(t)P_2(t) + O(\epsilon + g + g^2/\epsilon) \quad \text{if} \quad g \ll \epsilon^{1/2} \ll 1$$

where $r_1(t) = (1 + e^{-\frac{2g}{\epsilon}\int_0^t \gamma(s)ds})/2, \quad r_2(t) = (1 - e^{-\frac{2g}{\epsilon}\int_0^t \gamma(s)ds})/2$

$\Rightarrow$

$$\mathcal{U}(t,0)(P_1(0)) = \begin{cases} P_1(t) - g/\epsilon \int_0^t \gamma(s)ds((P_1(t) - P_2(t)) + O(\epsilon + g^2/\epsilon^2), & g \ll \epsilon \\ \frac{1}{2}(P_1(t) + P_2(t)) + O(g^2/\epsilon^2 + (\epsilon/g)^\infty), & \epsilon \ll g \ll \epsilon^{1/2} \end{cases}$$