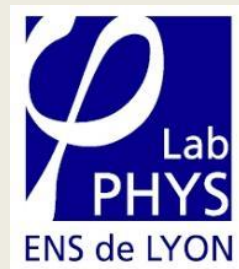


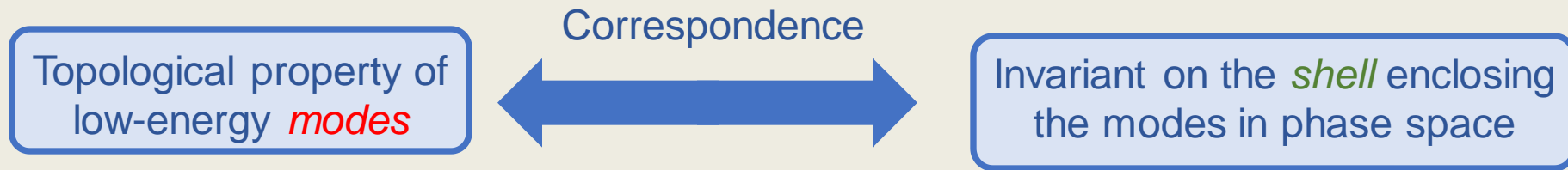
MODE-SHELL CORRESPONDENCE: A UNIFYING THEORY IN TOPOLOGY

Lucien Jezequel

PhD supervisor: Pierre Delplace



Mode-shell correspondance in short

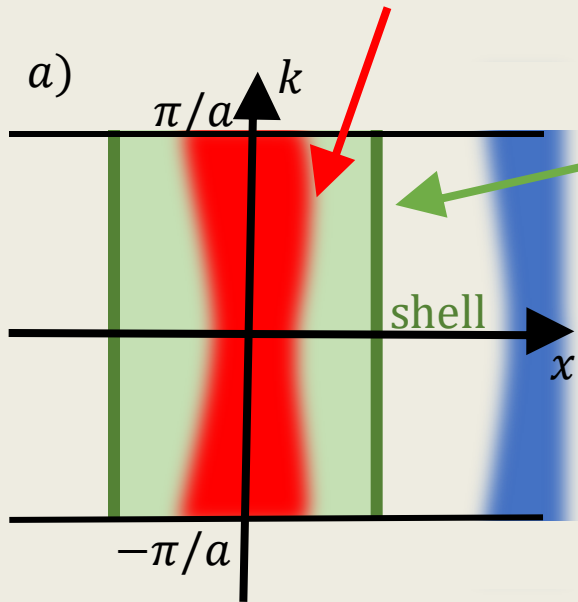
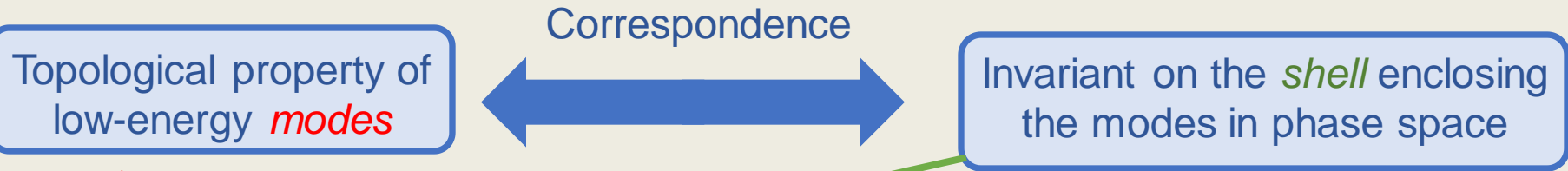


Phase space (x,k)

position + wavenumber → Wigner-Weyl transform

$$H(x, k) = W(\hat{H}) = \int dx' \left\langle x + \frac{x'}{2} \left| \hat{H} \right| x - \frac{x'}{2} \right\rangle e^{-ikx'}$$

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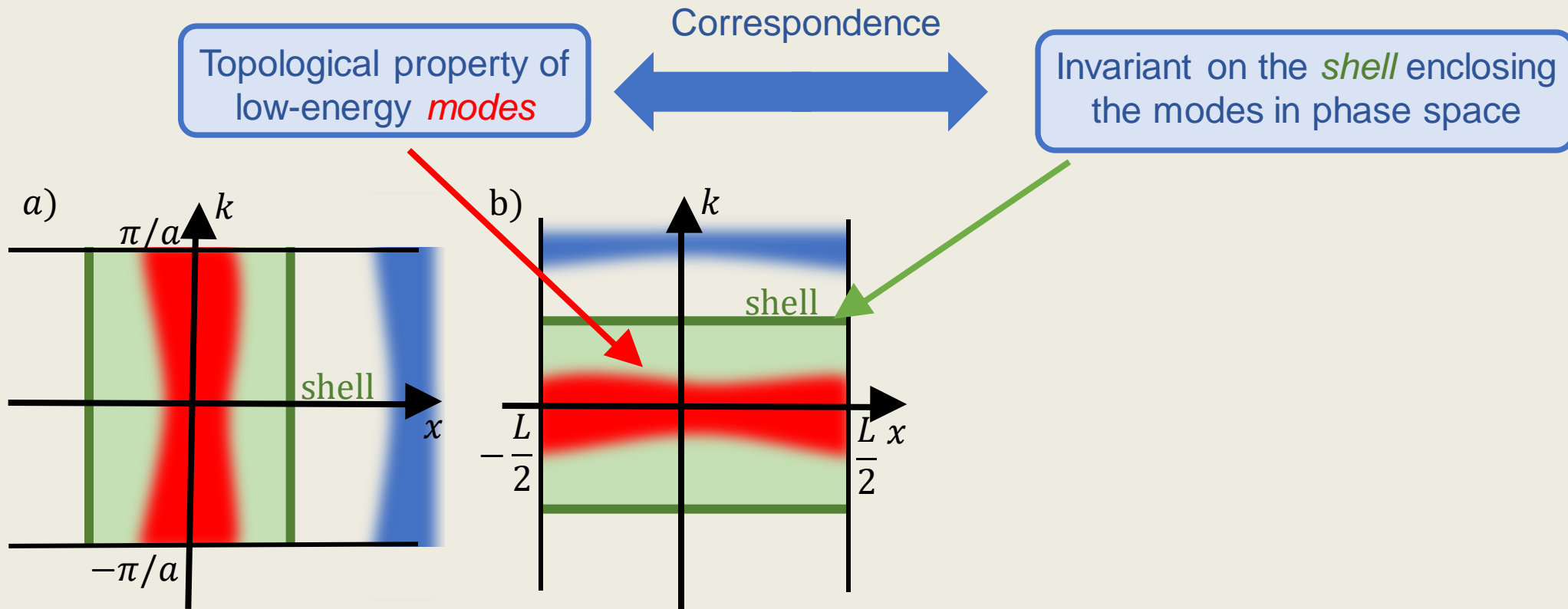
- a) bulk-edge correspondance in position
- b) Low-high wavenumber correspondance
- c) Mixed correspondance in position/wavenumber
- d) Higher-order correspondance

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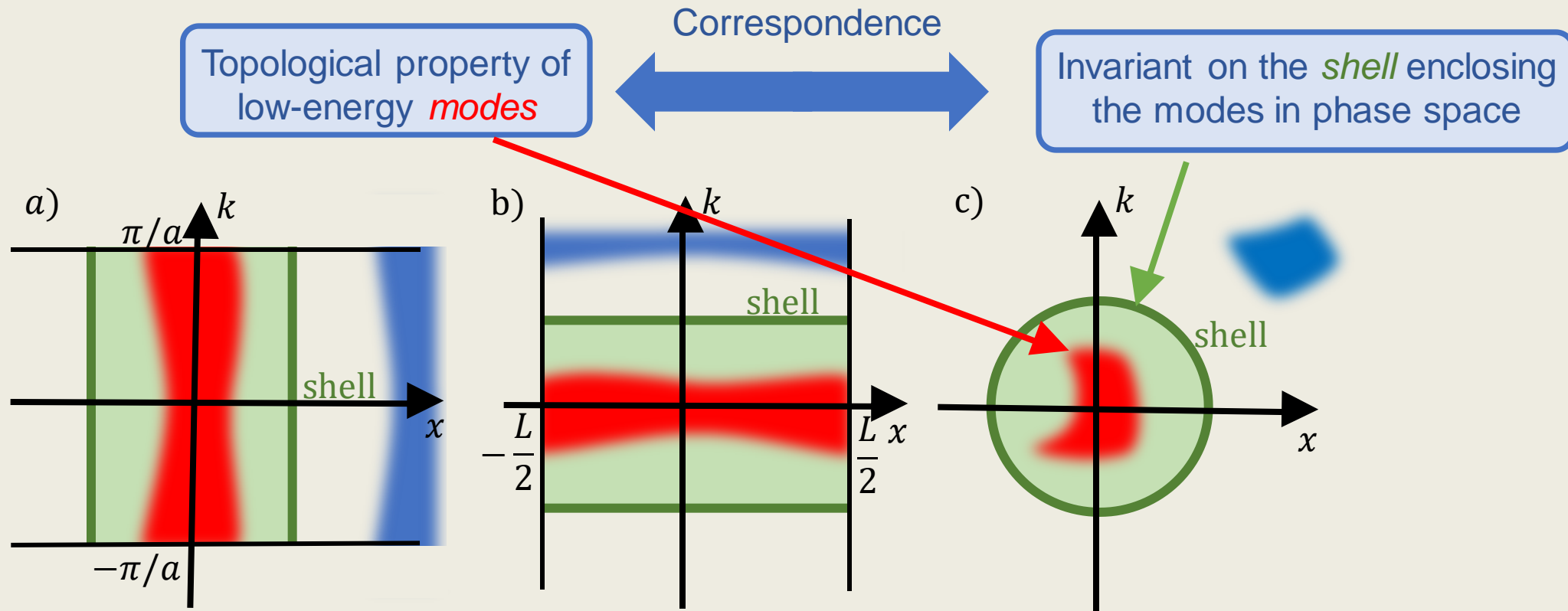
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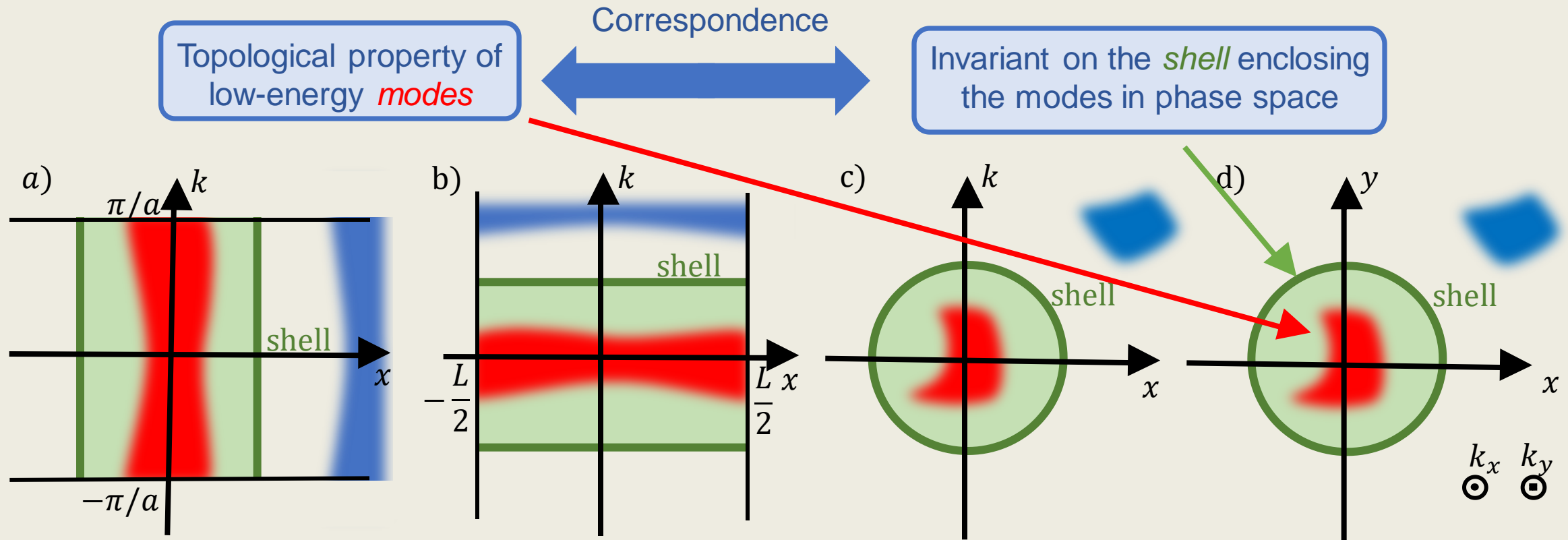
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The chiral number of zero modes

We restrict to the case of hamiltonian with chiral symmetry

$$\hat{H} = \begin{pmatrix} 0 & \hat{h}^\dagger \\ \hat{h} & 0 \end{pmatrix} \quad \hat{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \hat{C}\hat{H} + \hat{H}\hat{C} = 0$$

Zero modes has positive/negative chirality

A topological invariant:
the chiral number of zero modes

$$I_{modes} = \dim \ker(\hat{h}) - \dim \ker(\hat{h}^\dagger) = \text{Ind}(\hat{h})$$

Zero modes of positive chirality



Zero modes of negative chirality



The chiral number of zero modes

Generalisation of the index:

- More robust in finite system
- Coincide for infinite system
- Selection in phase space

$$I_{modes} = Tr(\hat{C}(1 - \hat{H}_F^2)\hat{\theta}_\Gamma)$$

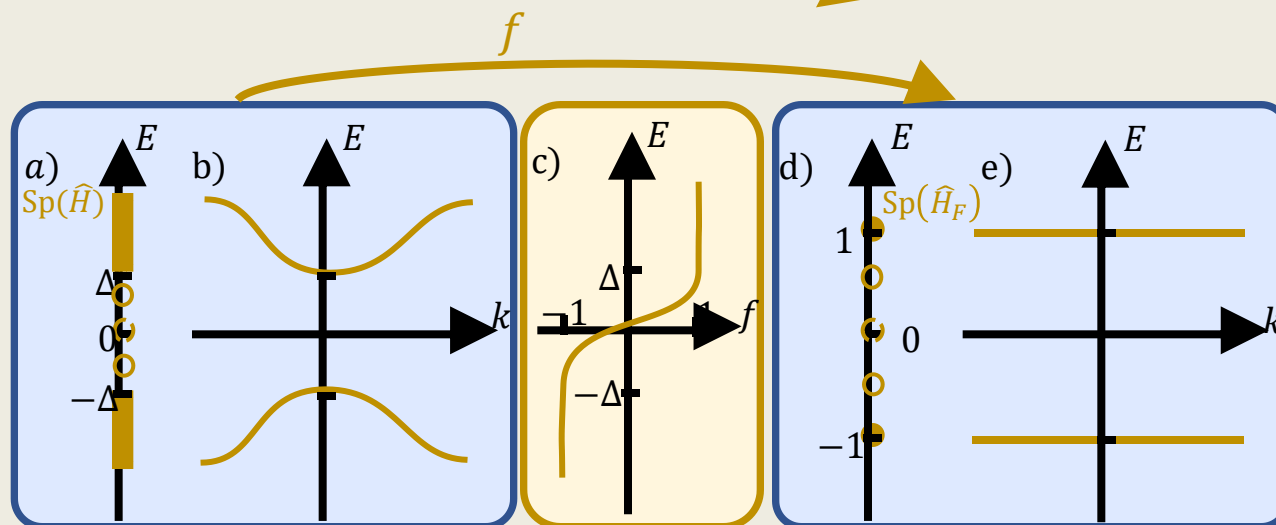
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Energy filter by smooth flattening of the gapped bands



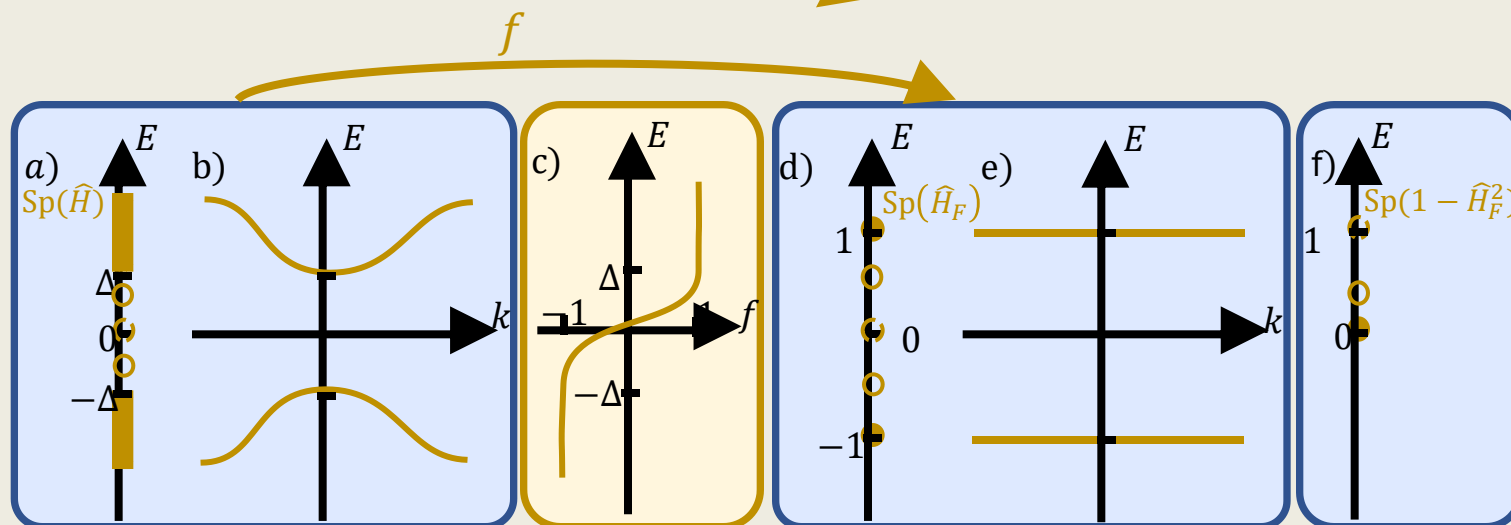
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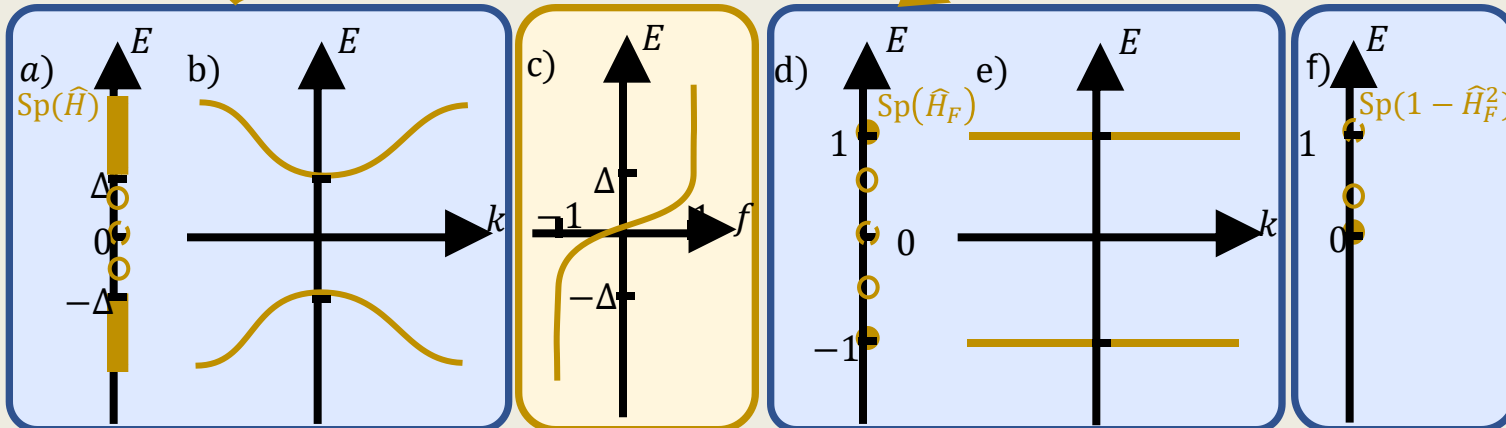
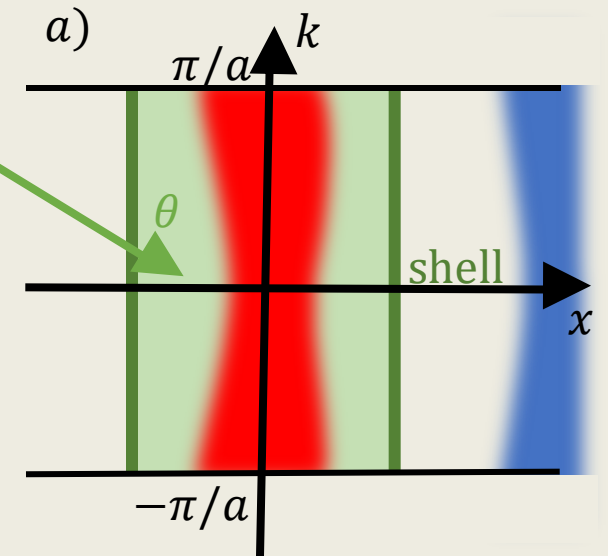
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$$I_{modes} = \text{Tr}(\hat{C}(1 - \hat{H}_F^2)\hat{\theta}_\Gamma)$$

Energy filter by smooth flattening of the gapped bands

Selection in phase space by a cut-off

$$\hat{\theta}_\Gamma = e^{-\frac{x^2}{\Gamma^2}} \quad \Gamma \rightarrow \infty$$

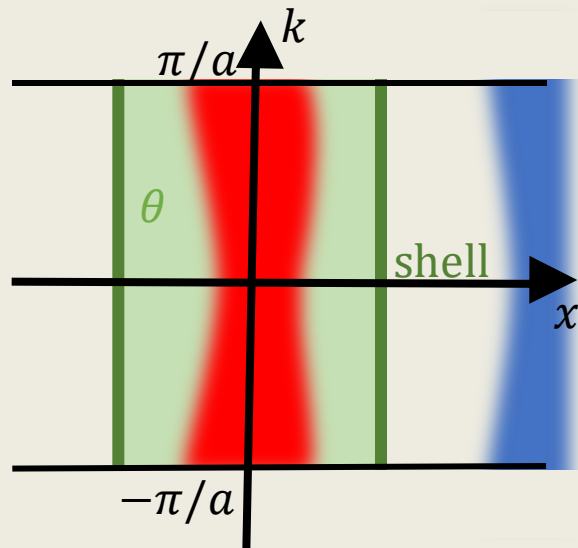


The cut-off can be anything

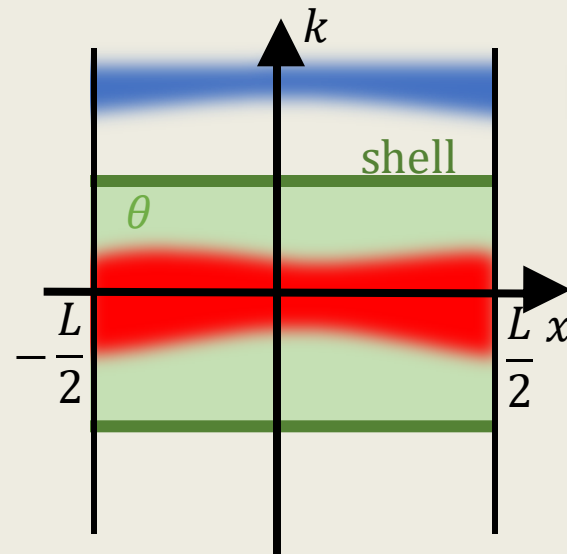
$$I_{modes} = Tr(\hat{C}(1 - \hat{H}_F^2)\hat{\theta}_\Gamma)$$

Shell: Transition region of the cut-off.

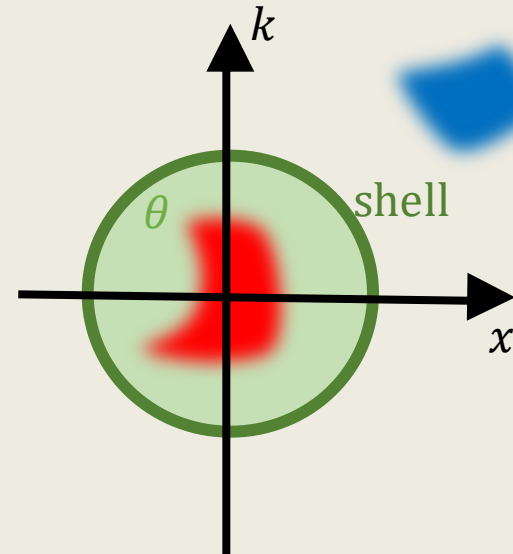
$$\hat{\theta}_\Gamma = e^{-\frac{x^2}{\Gamma^2}}$$



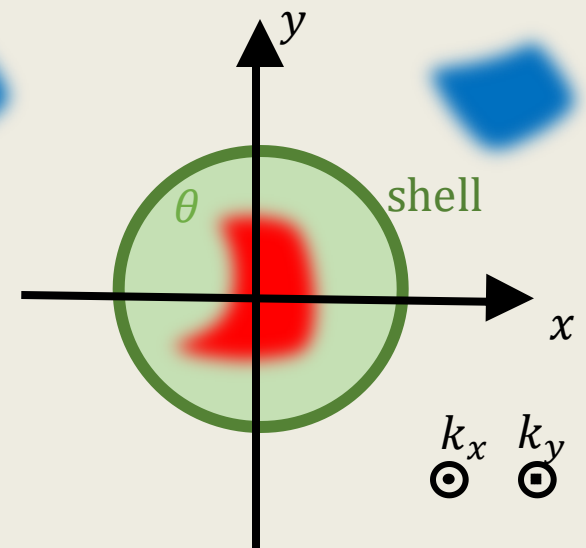
$$\hat{\theta}_\Gamma = e^{\frac{\partial_x^2}{\Gamma^2}} \approx e^{-\frac{k^2}{\Gamma^2}}$$



$$\hat{\theta}_\Gamma = e^{\frac{-x^2 + \partial_x^2}{\Gamma^2}} \approx e^{-\frac{x^2 + k^2}{\Gamma^2}}$$



$$\hat{\theta}_\Gamma = e^{-\frac{x^2 + y^2}{\Gamma^2}}$$



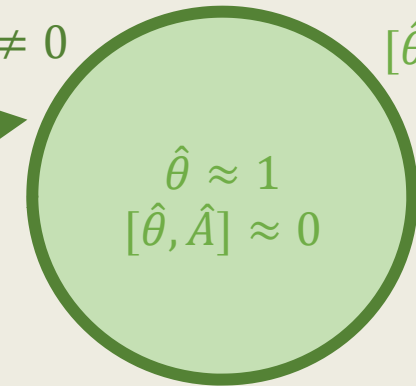
The mode-shell correspondance

Up to a rearrangement of the terms, the index can be re-expressed as a quantity on the shell

$$\begin{aligned}
 I_{modes} &= \text{Tr}(\hat{C}(1 - \hat{H}_F^2)\hat{\theta}_\Gamma) \\
 &= \text{Tr}(\hat{C}\hat{\theta}_\Gamma) + \frac{1}{2}\text{Tr}(\hat{C}\hat{H}_F[\hat{\theta}_\Gamma, \hat{H}_F]) \stackrel{\text{def}}{=} I_{shell}
 \end{aligned}$$

Supported on the shell

$$[\hat{\theta}, \hat{A}] \neq 0$$



$$\begin{aligned}
 \hat{\theta} &\approx 0 \\
 [\hat{\theta}, \hat{A}] &\approx 0
 \end{aligned}$$

summed chirality of site/d.o.f weighted by the cut-off. Often zero

$$\sum_{\lambda \in d.o.f} c_\lambda \langle \lambda | \hat{\theta}_\Gamma | \lambda \rangle$$

Semi-classical approximation

Semi-classical approximation:

$$W(\hat{A}\hat{B})(x, k) \approx A(x, k)B(x, k)$$

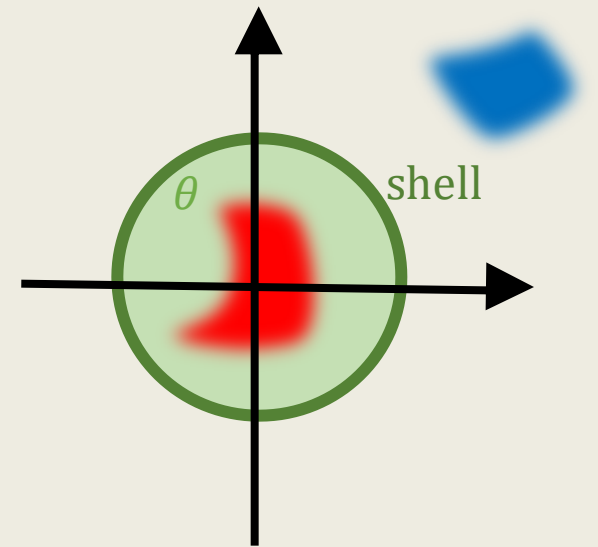
$$W([\hat{A}, \hat{B}])(x, k) \approx \{A(x, k), B(x, k)\}$$

When $H(x, k)$ varies slowly in x or in k in the shell

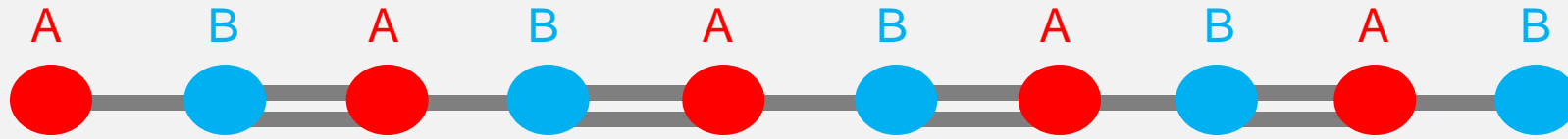
→ shell invariant reduced to a (higher) winding number

$$I_{shell} \xrightarrow{sc \text{ lim}} \frac{2(D)!}{(2D)! (2i\pi)^D} \int_{shell} Tr^{int} ((U^\dagger dU)^{2D-1}) = w_{2D-1}$$

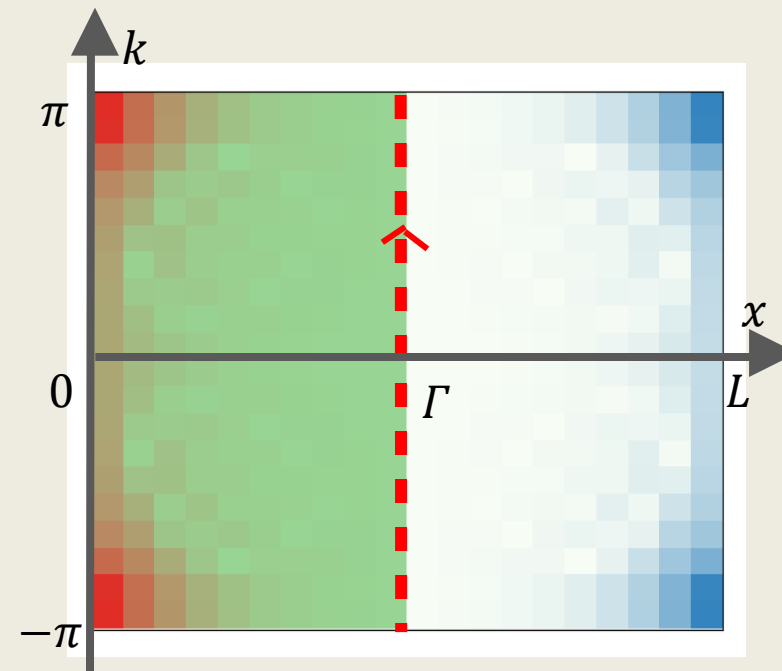
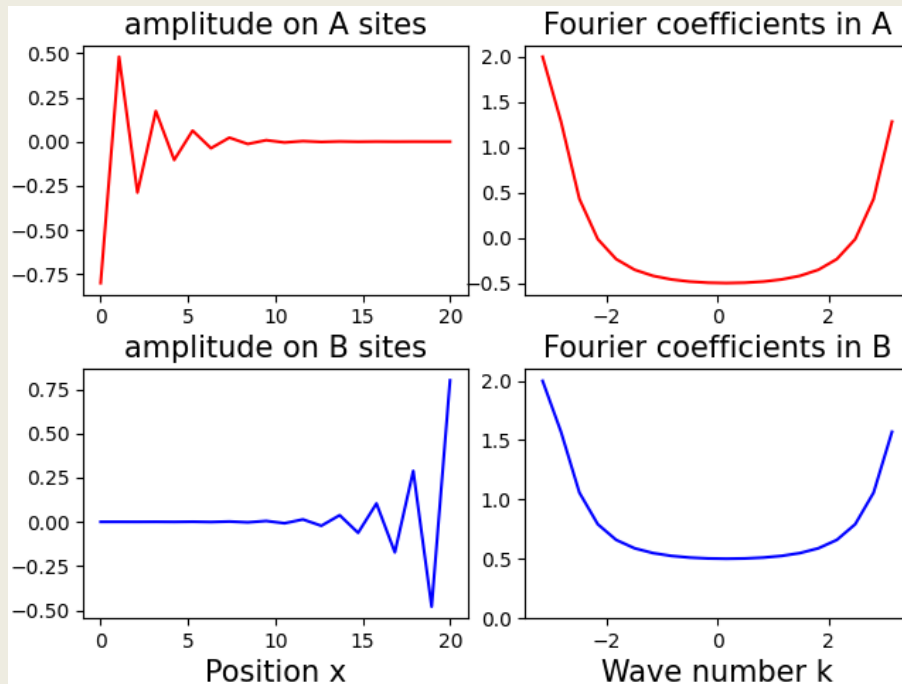
Where U is defined as $H_F(x, k) = \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix} (x, k)$



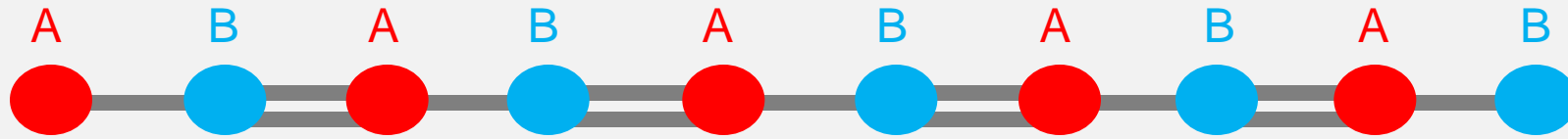
Bulk-edge correspondence



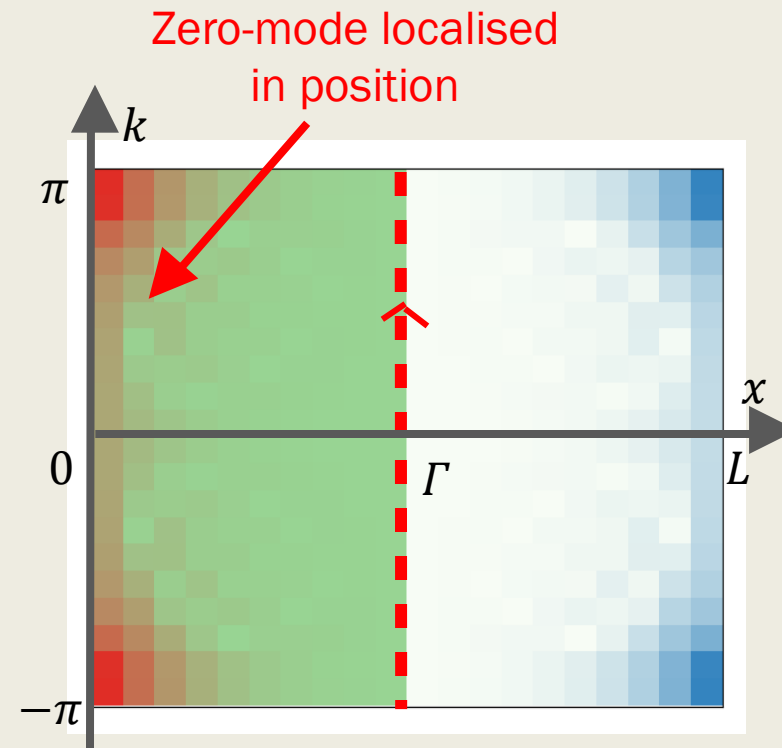
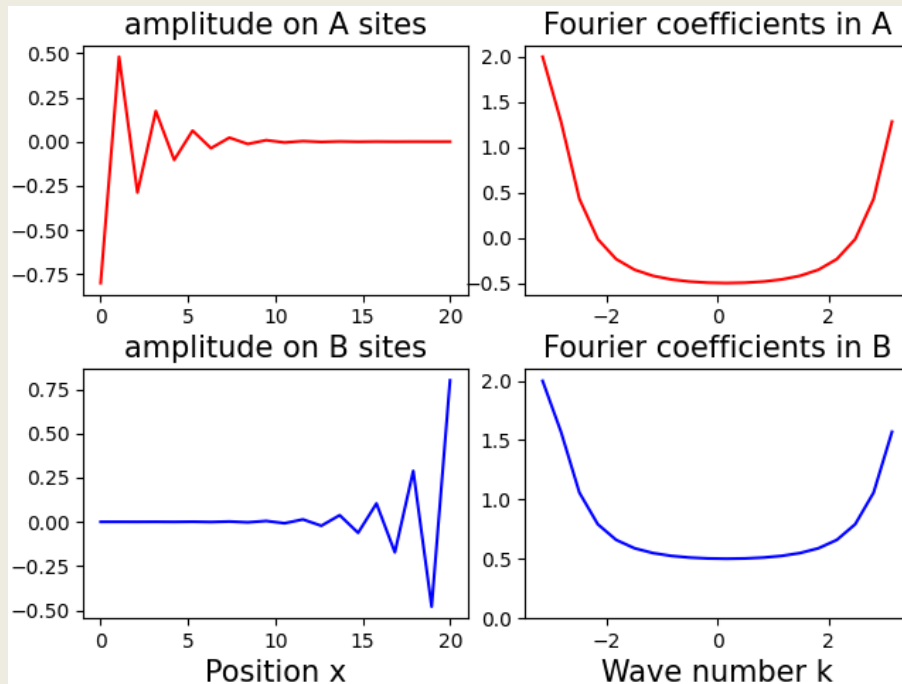
Dimer (SSH) model



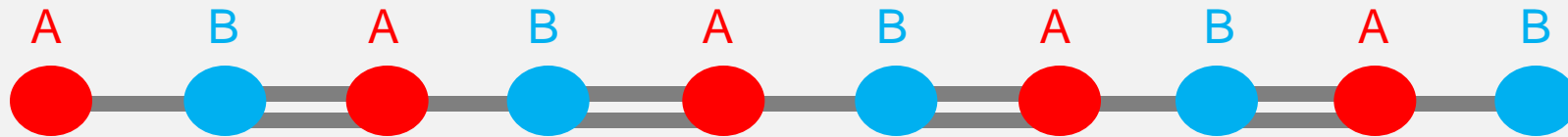
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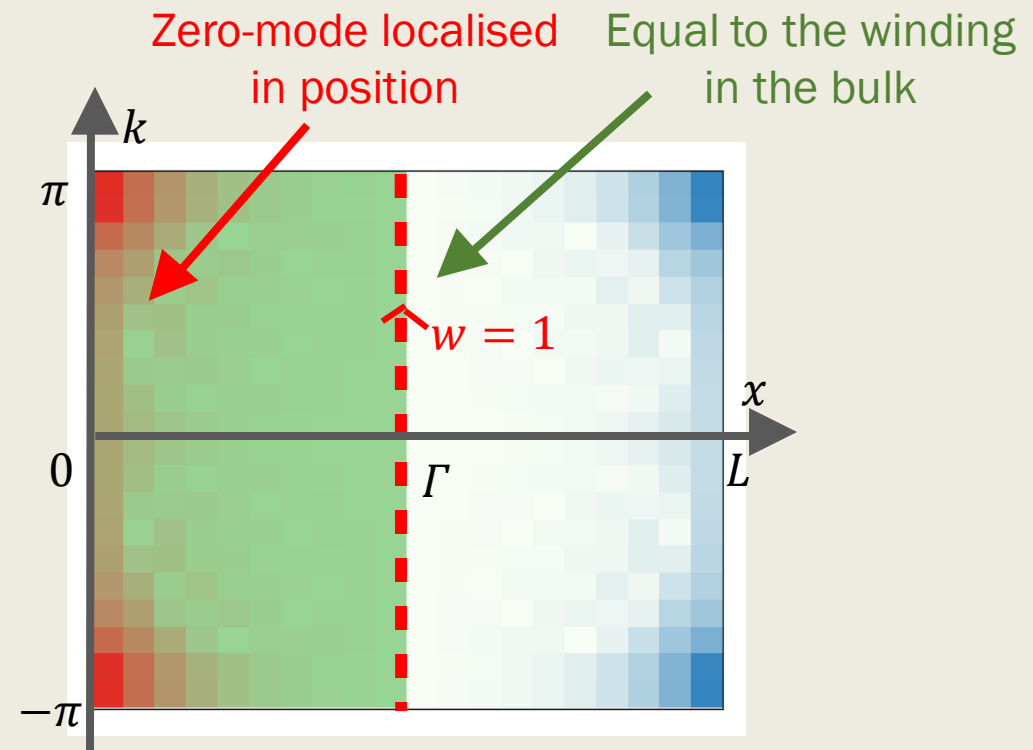
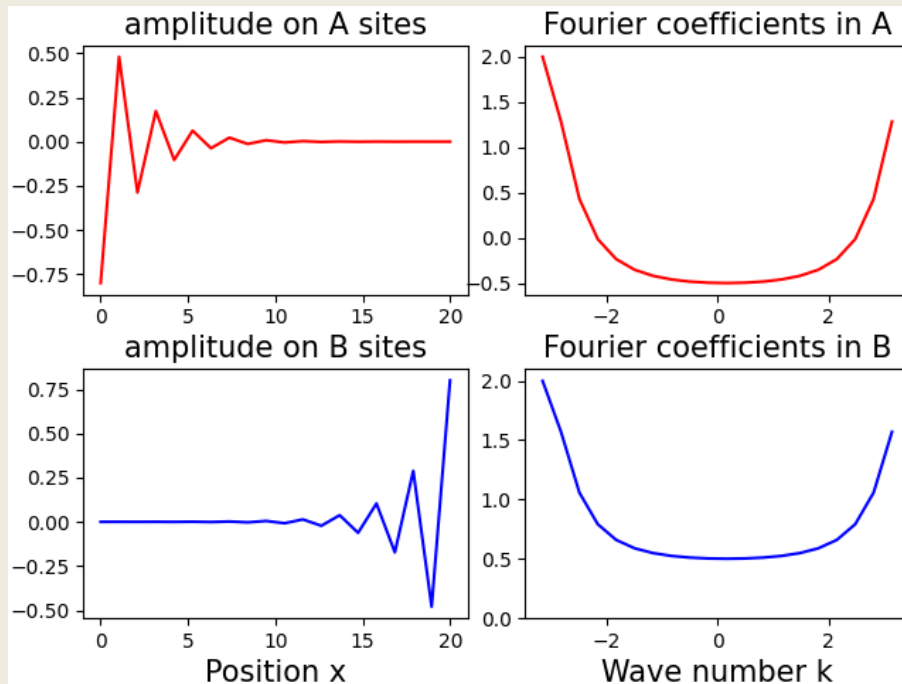
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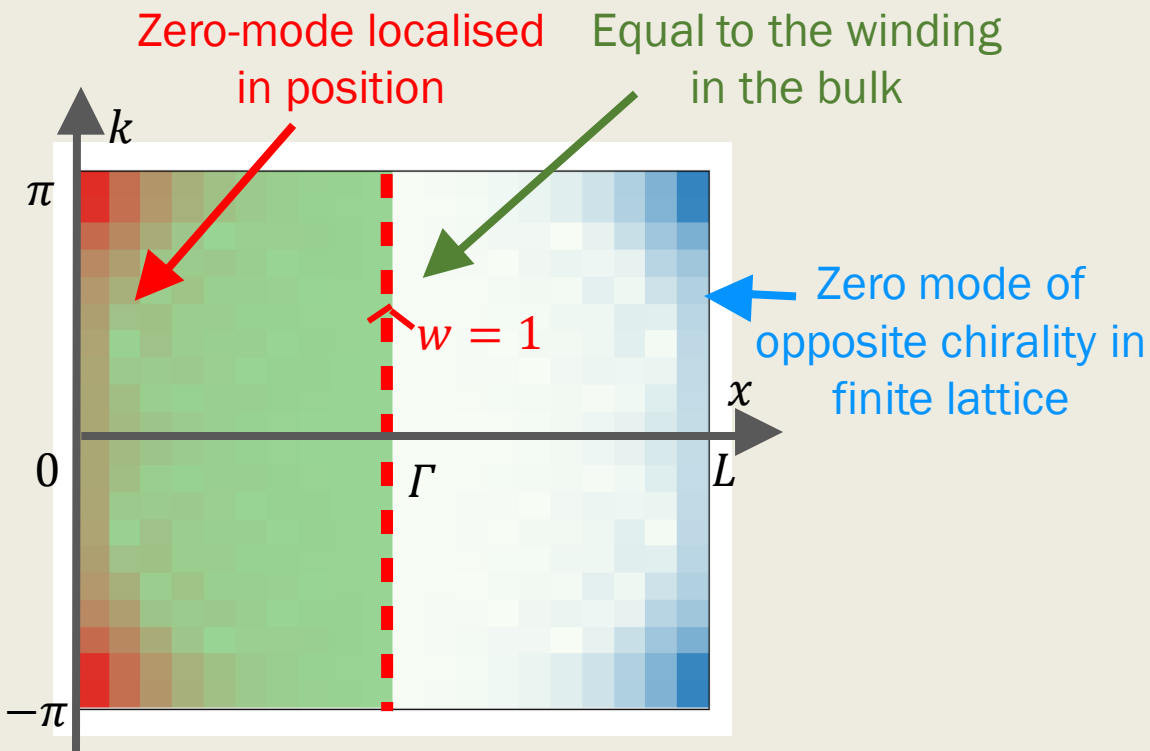
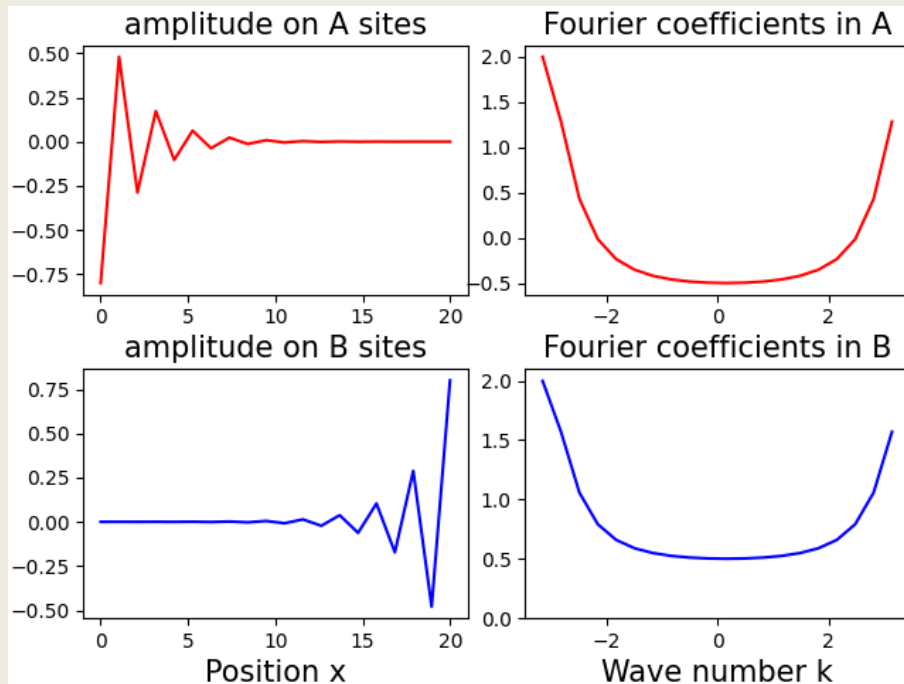
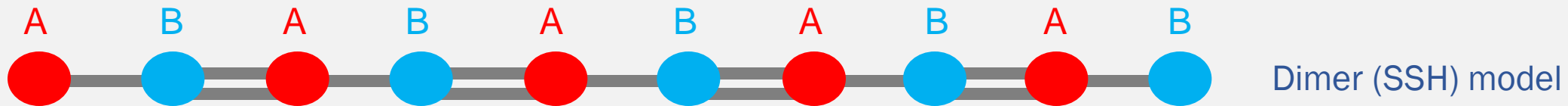
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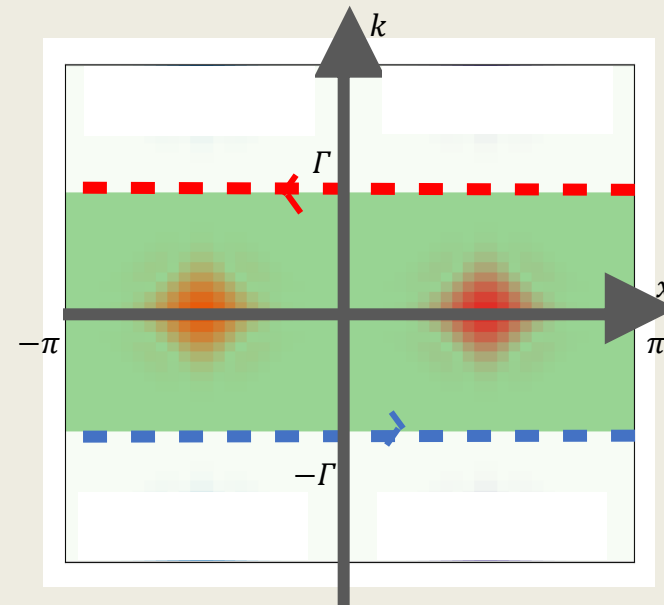
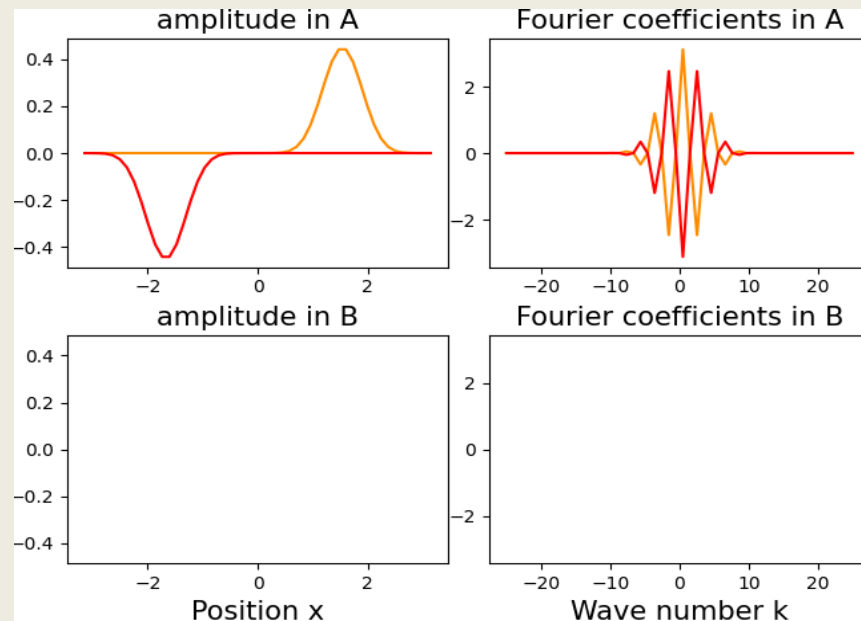
Bulk-edge correspondence



Low-high wavenumber correspondance

$$\hat{H} = \begin{pmatrix} 0 & V(x) + \varepsilon c(x)\partial_x \\ V(x) - \varepsilon\partial_x c(x) & 0 \end{pmatrix}$$

Wave system with a periodic local wave velocity $c(x) = \sin x$ and potential $V(x) = \cos x$

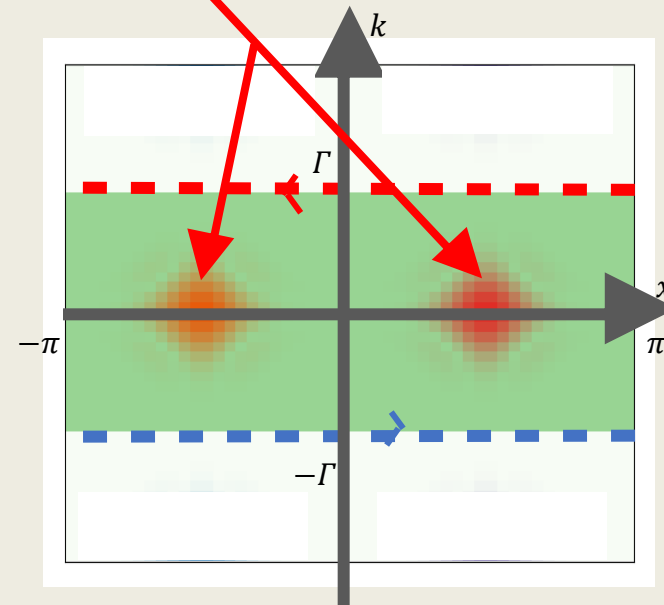
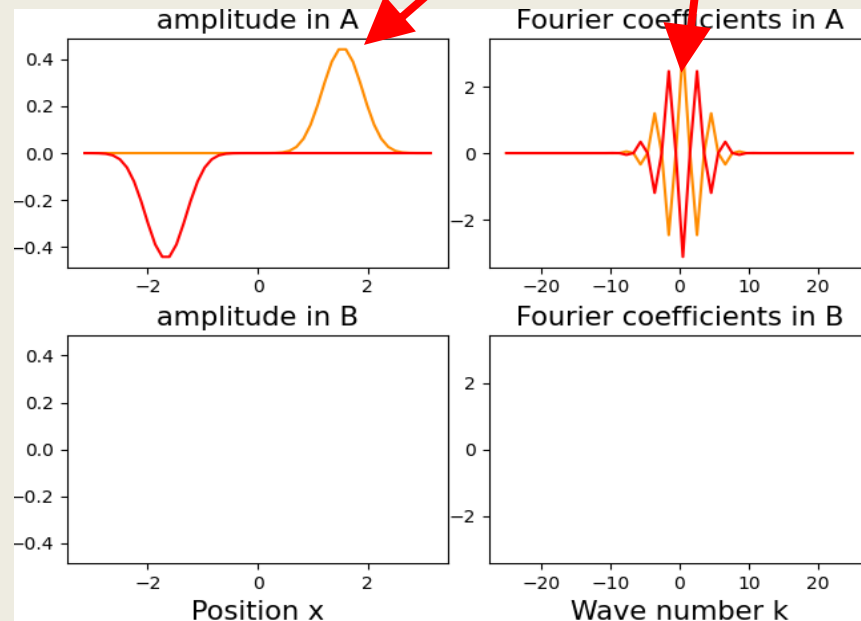


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Zero modes of positive
chirality at low-wave number

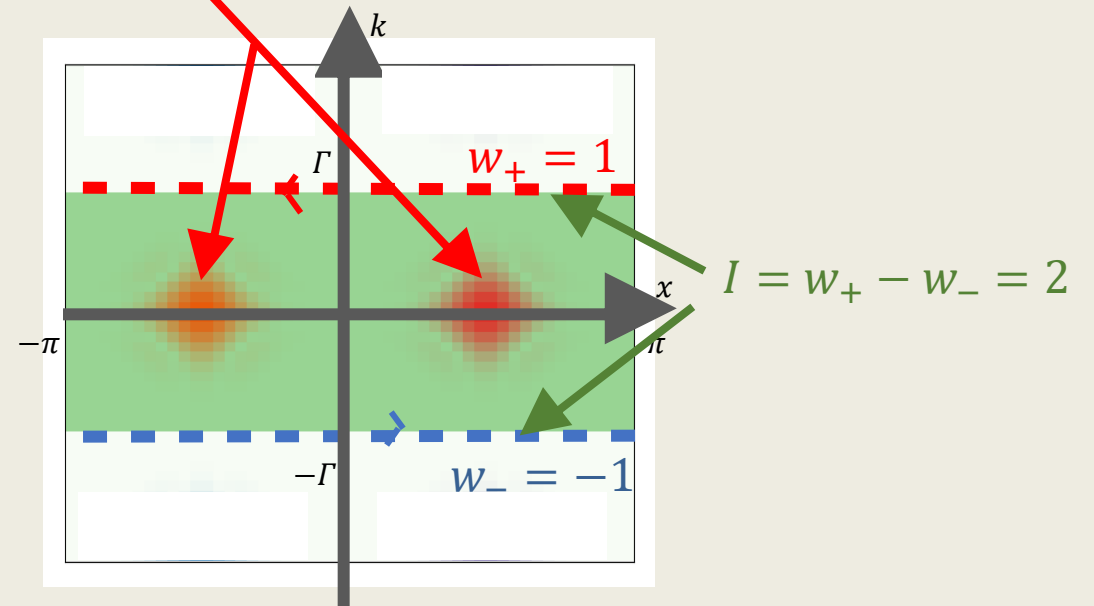
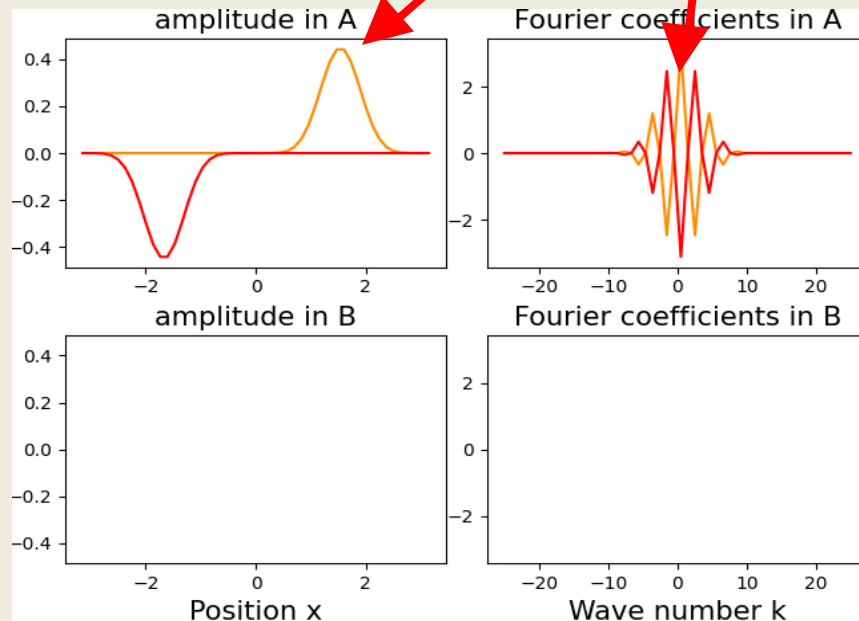


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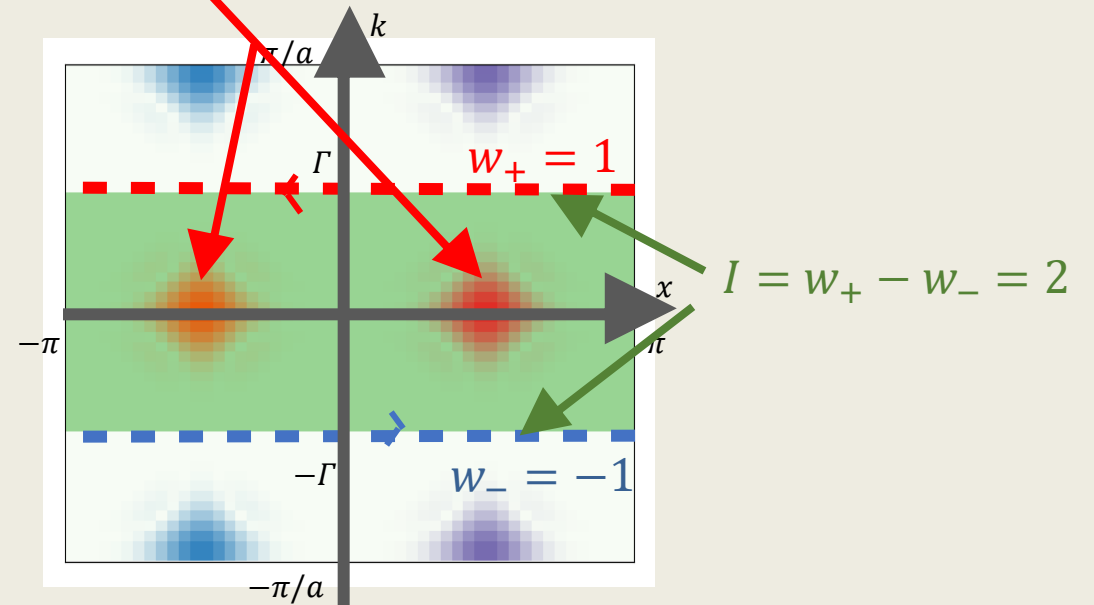
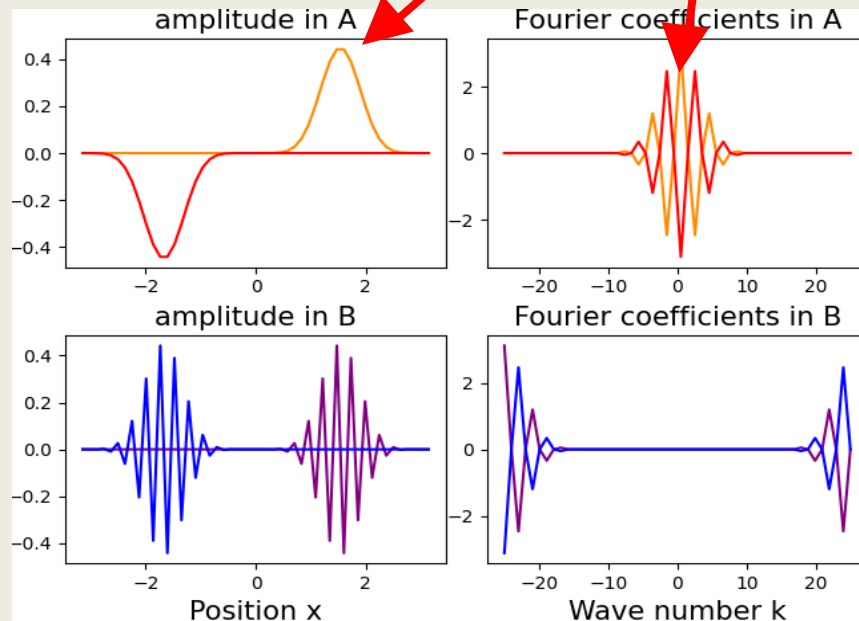
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Wave system with a periodic local wave velocity $c(x) = \sin x$ and potential $V(x) = \cos x$

Zero modes of opposite chirality

Protection from separation in wavenumber

Zero modes of positive chirality at low-wave number

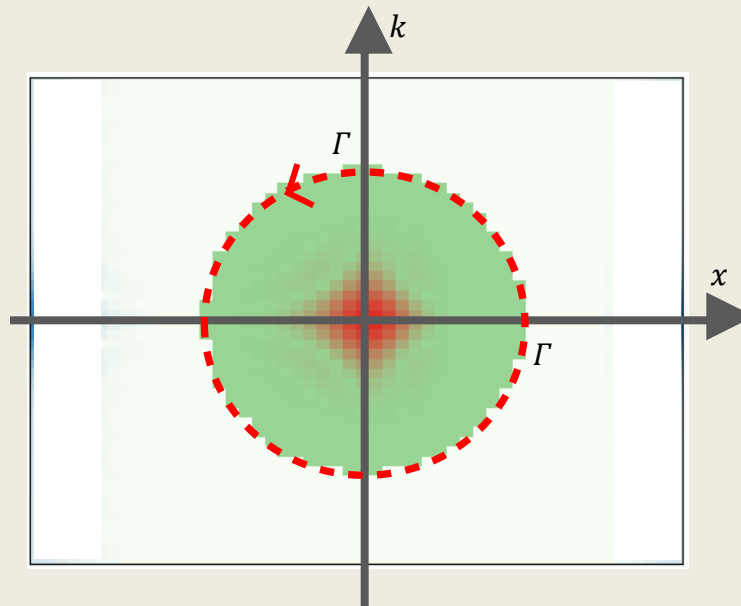


Mixed correspondance in position/wavenumber

$$\hat{H} = \begin{pmatrix} 0 & x + \partial_x \\ x - \partial_x & 0 \end{pmatrix}$$

Jackiw-Rebbi model

→ Continuous system with constant local wave velocity but unbounded in position



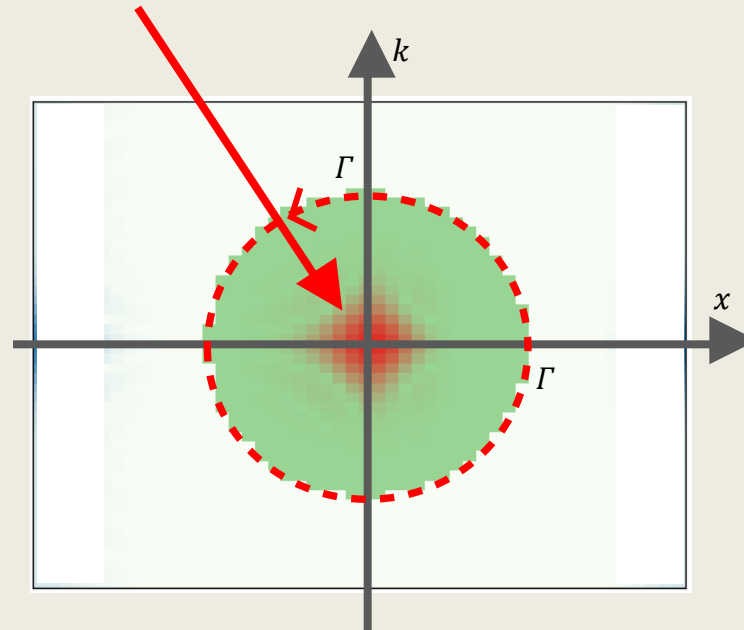
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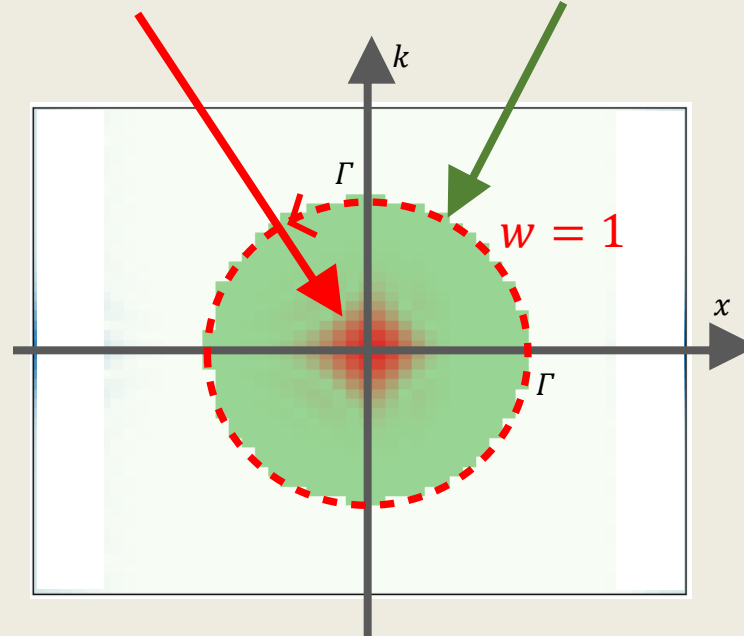
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Equal to the winding on the phase space circle



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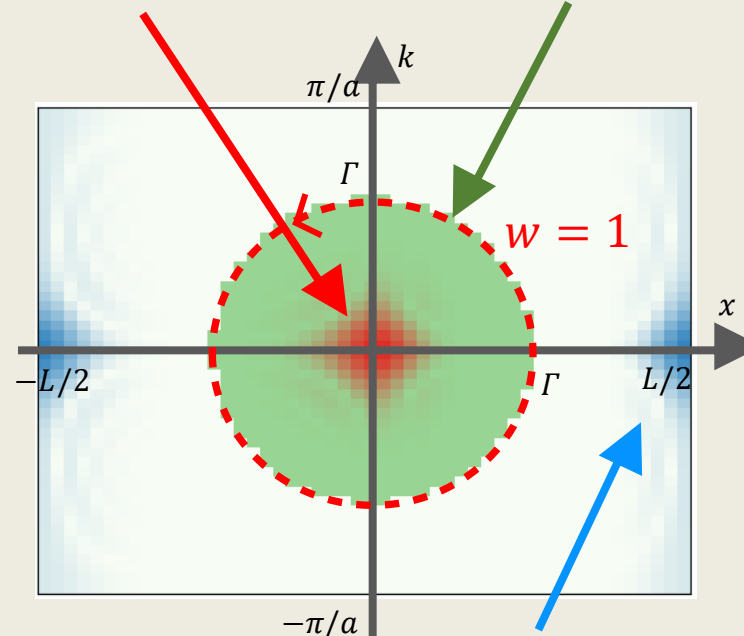
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In finite approximation of the system, zero modes appear elsewhere in phase space

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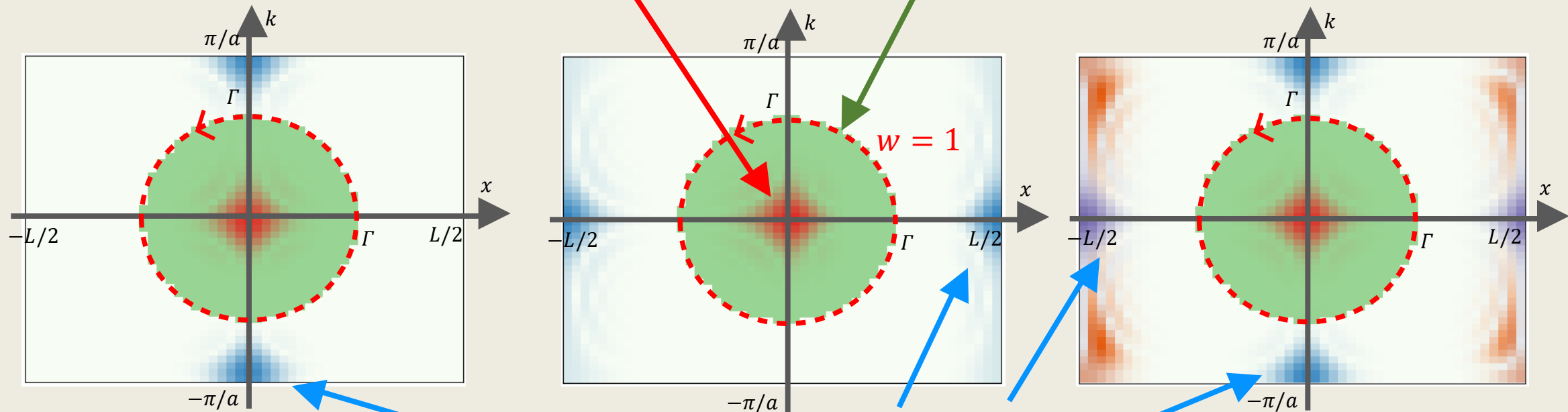
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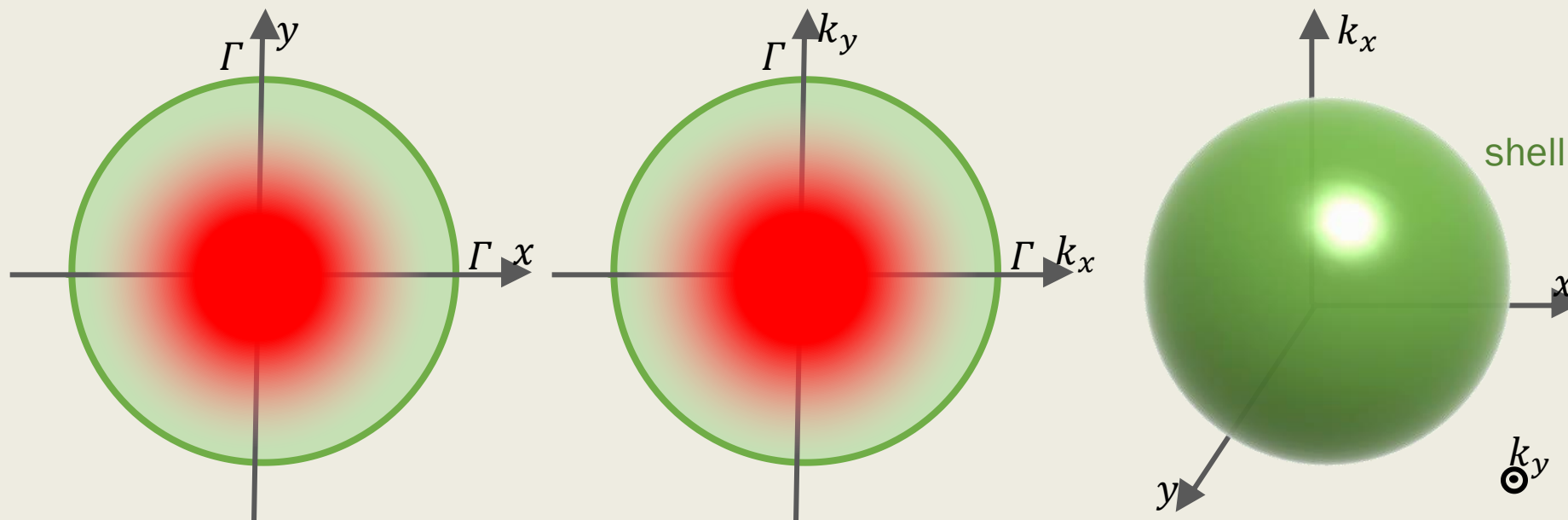


In finite approximation of the system, zero modes appear elsewhere in phase space

A higher-order correspondance

$$\hat{H} = \begin{pmatrix} 0 & x - \partial_x & y - \partial_y \\ x + \partial_x & -(y - \partial_y) & \\ y + \partial_y & x - \partial_x & 0 \end{pmatrix}$$

2D model: zero-mode localised in position
and wavenumber in 4D phase space



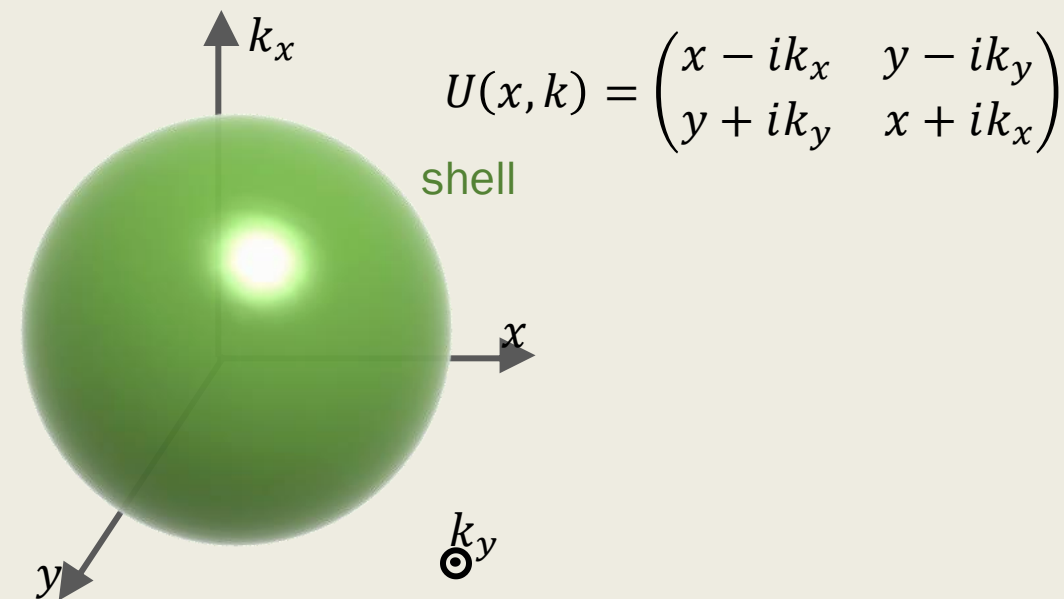
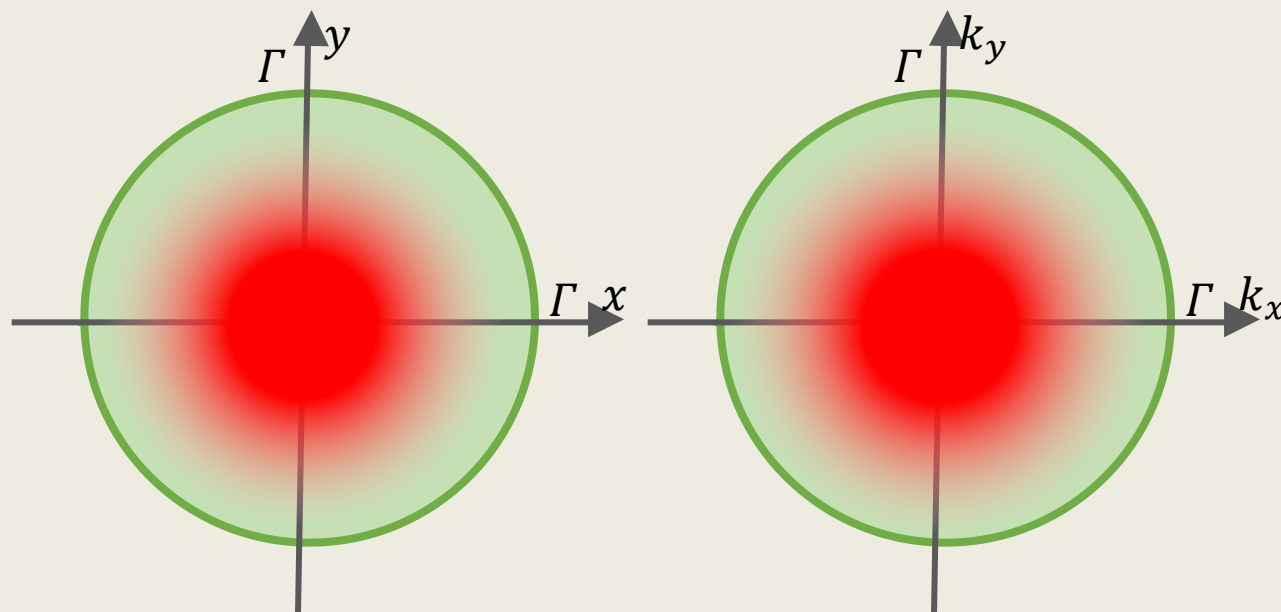
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2D model: zero-mode localised in position and wavenumber in 4D phase space

→ higher winding number on the 3D sphere in phase space

$$W_{2D-1} = \frac{-1}{24\pi^2} \int_{S^3} Tr^{int}((U^\dagger dU)^3)$$



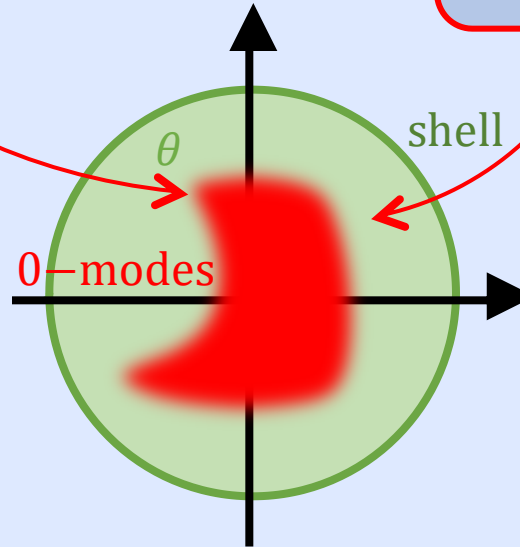
$$U(x, k) = \begin{pmatrix} x - ik_x & y - ik_y \\ y + ik_y & x + ik_x \end{pmatrix}$$

Conclusion

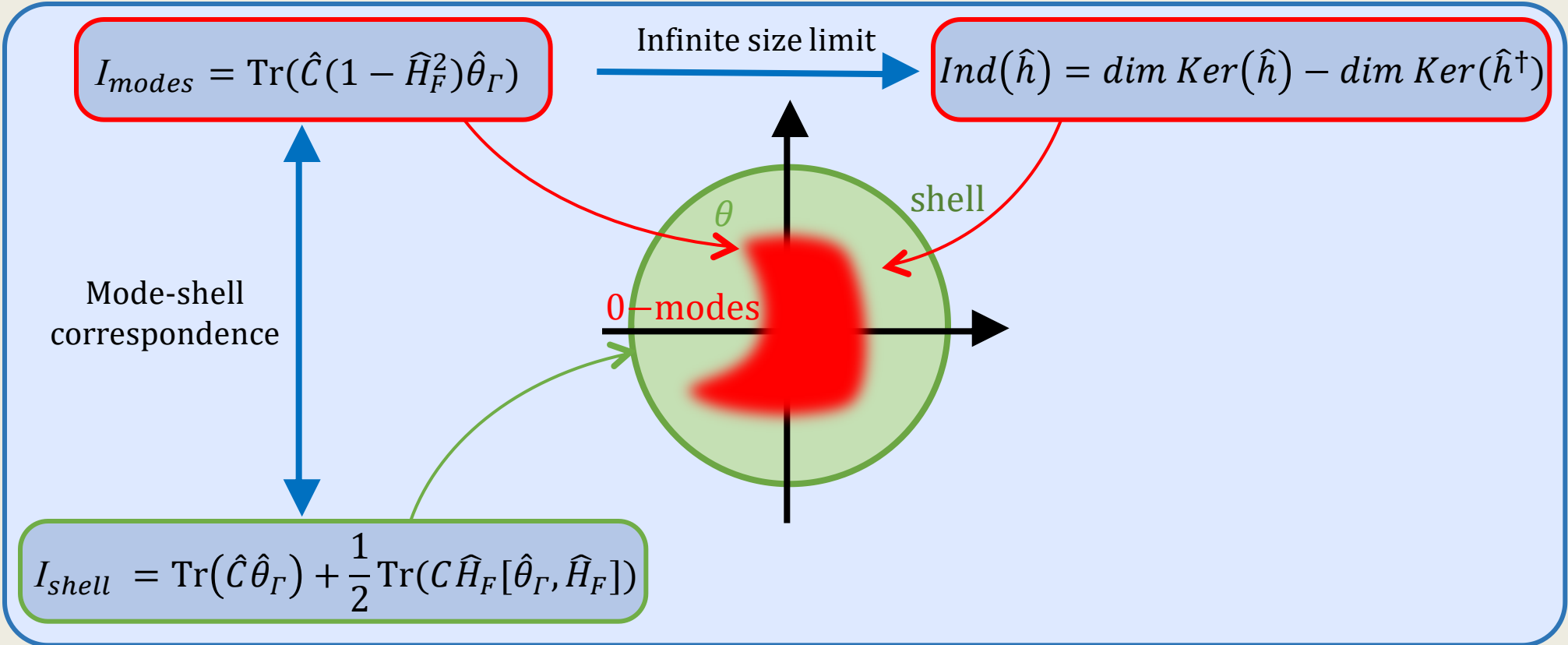
$$I_{modes} = \text{Tr}(\hat{C}(1 - \hat{H}_F^2)\hat{\theta}_\Gamma)$$

Infinite size limit

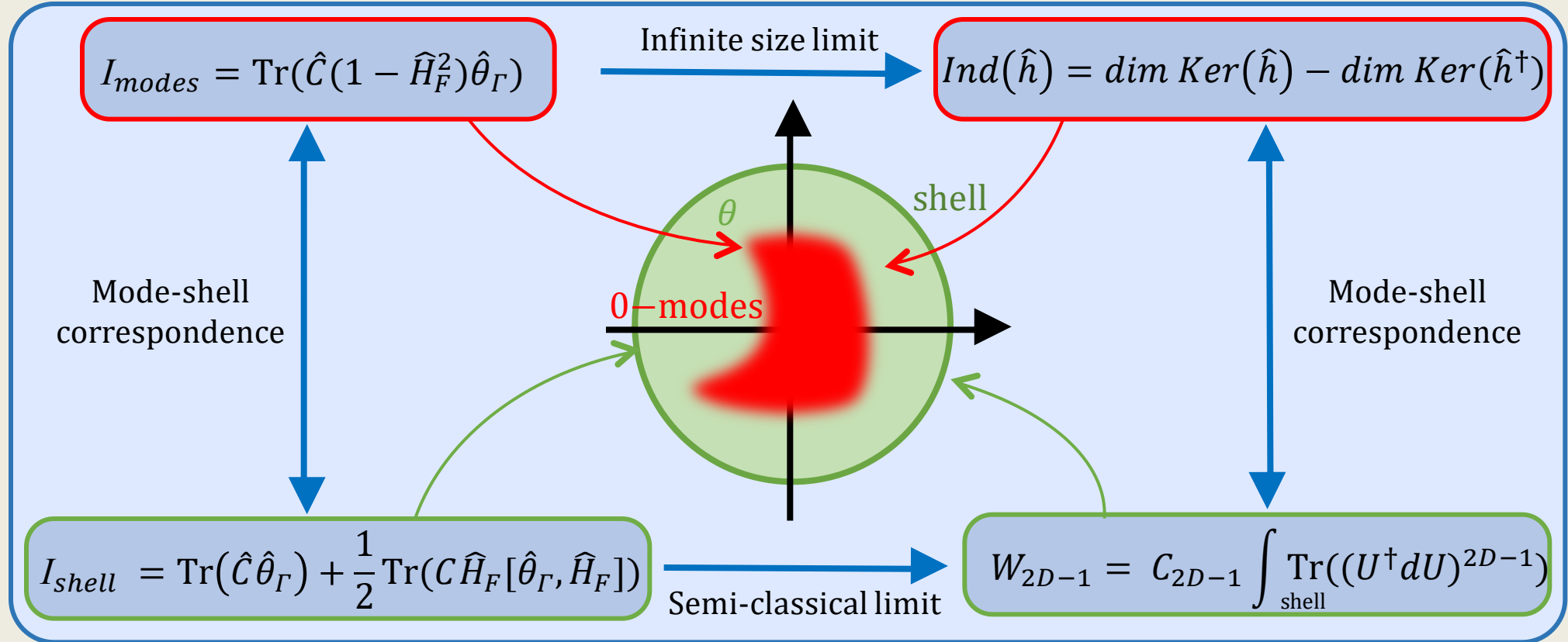
$$\text{Ind}(\hat{h}) = \dim \text{Ker}(\hat{h}) - \dim \text{Ker}(\hat{h}^\dagger)$$



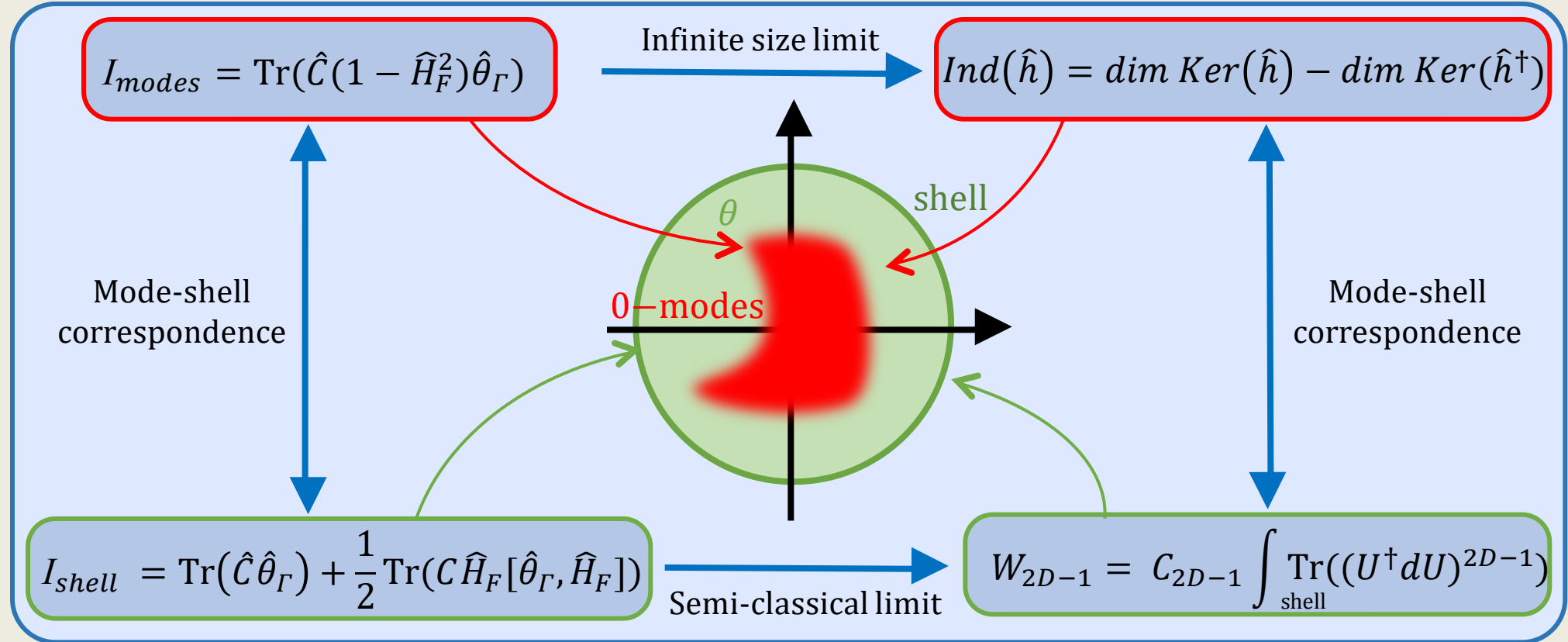
Conclusion



Conclusion



Conclusion



Other mode-shell correspondance:

- spectral flow (1D) \rightarrow Pierre Delplace's talk (Wed)
- number of Dirac-Weyl point (2D & 3D)