# Phase transition in the Integrated Density of States of the Anderson model arising from a supersymmetric sigma model.

joint work with V. Rapenne, C. Rojas-Molina and X. Zeng

Margherita DISERTORI

University of Bonn & Hausdorff Center for Mathematics

Mathematical Aspects of Condensed Matter Physics,
ETH Zürich 2023

- some facts about Anderson model and integrated density of states
- Anderson model arising from the $H^{2|2}$ supersymmetric sigma model/vertex reinforced jump process
- Integrated density of states in this special case

## Anderson model (discrete random Schrödinger)

electronic transport/wave propagation in disordered systems

**the model on $\mathbb{Z}^d$ : $H = -\Delta + \lambda V \in \mathbb{R}_{sym}^{\mathbb{Z}^d \times \mathbb{Z}^d}$ random op.**

- $-\Delta$ = lattice Laplacian $(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$, $-\Delta f(j) = \sum_{|k-j|=1} (f(j) - f(k))$
- $(Vf)(j) = V_j f(j)$, $\{V_j\}_{j \in \mathbb{Z}^d}$ random variables with prob. distr. $d\mathbb{P}(V)$
  $\to \mathrm{Var}(V_j) = 1$, $\mathbb{P}$ translation inv. and $V$ (almost) indep
- $\lambda > 0$ parameter

heuristics:

- $\lambda = 0$ $H = -\Delta$ a.c. spectrum, delocalized eigenvectors
- $\lambda \gg 1$ $H \simeq \lambda V$ p.p. spectrum, localized eigenvectors

*Question:* intermediate $\lambda$?

## Some results

- $d = 1$ localization $\forall\lambda > 0$
- $d \geq 2$ localization
  - $\forall\lambda > 0$ at the *edge* of the spectrum
  - for $\lambda \gg 1$ in the *bulk* of the spectrum
- *phase transition* on the *Bethe lattice:*
  - for $\lambda \gg 1$ localization
  - for $\lambda \ll 1$ delocalization in the bulk

## Major open conjecture: on $\mathbb{Z}^d$ $d \geq 3$

for $\lambda \ll 1$ : delocalization in the bulk of the spectrum

## Integrated Density of States: $N(E,H) := \lim_{L\to\infty} N(E,H_{\Lambda_L}) = \lim_{L\to\infty} \mathbb{E}[N(E,H_{\Lambda_L})]$

$\Lambda_L = [-L,L]^d \cap \mathbb{Z}^d, \quad H_{\Lambda_L} = H_{|\Lambda_L} \qquad N(E,H_{\Lambda_L}) = \frac{\#\text{eigenvalues}_{\leq E}}{|\Lambda_L|} = \frac{\mathrm{Tr}\,(\mathbf{1}_{(-\infty,E)}(H_{\Lambda_L}))}{|\Lambda_L|}$

- $V$ **i.i.d.** $d\mathbb{P}(V) = \prod_{j\in\mathbb{Z}^d} \mu(V_j)dV_j$
  - universality at the edge of the spectrum: *Lifschitz tails*

    if $\sigma(H) = [E_0,\infty)$ a.s. then for $E \simeq E_0$

    $N(E,H) = c_1 e^{-c_2(E-E_0)^{-\frac{d}{2}}} \ll N(E,-\Delta-E_0) \propto (E-E_0)^{\frac{d}{2}}$

    mechanism: many $V_j$ small $\to$ very unlikely
  - information on the spectral type?
    - Lifschitz tails $\Rightarrow$ localization near $E_0$, $\forall\lambda > 0$
    - $N(E,H)$ cannot see the phase transition a.c/p.p :

  $E \mapsto \mu(E)$ 'regular' $\Rightarrow E \mapsto N(E,H)$ 'regular' $\forall\lambda > 0$
- $V$ **not i.i.d.** Lifschitz tails may disappear

  $\to$our case: 2$-$ independent $V_j$

## Anderson model arising from $H^{2|2}$

**the model on** $\mathbb{Z}^d$ : $H_\beta = 2\beta - P^W \in \mathbb{R}^{\mathbb{Z}^d\times\mathbb{Z}^d}_{sym}$ **random op.**

- $(P^W)_{ij}=W_{ij}P_{ij}$, $P_{ij}=\mathbf{1}_{|i-j|=1}$ $W_{ij}=W_{ji}>0$ edge weight
- $(\beta f)(j)=\beta_j f(j)$, $\{\beta_j\}_{j\in\mathbb{Z}^d}$ random variables with prob. distr. $d\mathbb{P}(\beta)$

*finite volume marginal* of $d\mathbb{P}(\beta)$: $\Lambda \subset \mathbb{Z}^d$ finite [Sabot-Tarrès-Zeng 2015]

$$d\nu_{W,\Lambda}(\beta)=\left(\frac{2}{\pi}\right)^{|\Lambda|}\mathbf{1}_{H_{\beta,\Lambda}>0}\frac{1}{(\det H_{\beta,\Lambda})^{1/2}}e^{-\frac{1}{2}\left(\langle 1,H_{\beta,\Lambda}1\rangle+\langle\eta,H^{-1}_{\beta,\Lambda}\eta\rangle-2\langle\eta,1\rangle\right)}\prod_j d\beta_j$$

- $H_{\beta,\Lambda} = (H_\beta)_{|\Lambda}$, $\langle f,g\rangle=\sum_{j\in\Lambda} f_j g_j$
- $\eta_j=\sum_{k\notin\Lambda,|k-j|=1} W_{jk}$ "wired boundary conditions"

*finite volume Laplace transform:*

$$\mathbb{E}_\Lambda\left[e^{\langle f,\beta\rangle}\right]=\prod_{j\in\Lambda}\frac{e^{-\eta_j(\sqrt{1+f_j}-1)}}{\sqrt{1+f_j}}\prod_{|i-j|=1}e^{-W_{ij}(\sqrt{1+f_i}\sqrt{1+f_j}-1)},\quad f_j\geq -1$$

## some properties

- the r.v. $\beta$ are independent at distance 2
- $\exists!\mathbb{P}(\beta)$ on $\mathbb{R}^{\mathbb{Z}^d}$ with marginals $d\nu_{W,\Lambda}(\beta)$ (Kolmogorov ext. thm.)
- $H_{\beta,\Lambda}>0, \beta_i>0\ \forall i, H_{0,\Lambda}=-P^W$: hence $0<\beta_i\ll 1 \Rightarrow \beta_j \gg 1$ for some $j\neq i$

  $\Rightarrow$ main mechanism creating Lifschitz tails broken

## Constant weights $W_{ij}=W>0\ \forall |i-j|=1$.

- $H_\beta$ is ergodic and $\sigma(H_\beta)=[0,\infty)$ a.s. [Sabot,Zeng 2019][Rapenne 2023]
- $H_\beta=2\beta-WP=W(-\Delta+\frac{2\beta}{W}-2d)=W(-\Delta+\lambda V)\to$ we consider $\frac{H_\beta}{W}=\frac{2\beta}{W}-P$
- heuristics: here $W$ enters in $\mathbb{P}(\beta)$

$$\mathbb{E}\left[\frac{2\beta_j}{W}\right]=2d+\frac{1}{W}\simeq\begin{cases}2d & W\gg 1\\ \frac{1}{W}\gg 1 & W\ll 1\end{cases}\qquad \mathrm{Var}\left[\frac{2\beta_j}{W}\right]=\frac{2d}{W}+\frac{2}{W^2}\begin{cases}\ll 1 & W\gg 1\\ \gg 1 & W\ll 1\end{cases}$$

strong disorder $W\ll 1$: $\frac{H_\beta}{W}\simeq\frac{2\beta}{W}$ weak disorder $W\gg 1$: $\frac{H_\beta}{W}\simeq 2d-P=-\Delta$

$$\to\frac{H_\beta}{W}\equiv-\Delta+\lambda V,\ \lambda=\frac{1}{W}$$

## Origin: supersymmetric nonlinear sigma model $H^{2|2}$ [Zirnbauer 1996]

originally introduced as a toymodel for quantum diffusion

### Model in finite volume $\Lambda\subset\mathbb{Z}^d$

- spin $S$ is a (super)vector $S=(x,y,z,\xi,\eta)$,

  $x,y,z$ even, $\xi,\eta$ odd elements in a real Grassmann algebra
- $\langle S,S'\rangle=xx'+yy'-zz'+\xi\eta'-\eta\xi'$
- nonlinear constraint $\langle S,S\rangle=-1\Rightarrow z=\sqrt{1+x^2+y^2+2\xi\eta}$
- $\eta_j\geq 0$ magnetic field in direction $e=(0,0,1,0,0)$
- "Gibbs" measure:

  $d\mu_\Lambda(S)=\prod_{|i-j|=1} e^{W_{ij}(\langle S_i,S_j\rangle+1)}\prod_j e^{\eta_j(\langle e,S_j\rangle+1)}\delta(\langle S_j,S_j\rangle+1)\ dS_\Lambda$

## From $H^{2|2}$ to $H_\beta$

- *horospherical coordinates:* $(x,y,\xi,\eta)\to(u,s,\bar\psi,\psi)$

$d\mu_\Lambda(S)\to d\mu_\Lambda(u,s,\bar\psi,\psi)=$

$$e^{-\sum_{|i-j|=1} W_{ij}(\cosh(u_i-u_j)-1)-\sum_j \eta_j(\cosh u_j-1)]}e^{-\frac{1}{2}(s,Ms)-(\bar\psi,M\psi)}\;(du e^{-u} ds d\bar\psi d\psi)^\Lambda$$

$$(s,Ms)=\sum_{|i-j|=1} W_{ij}e^{u_i+u_j}(s_i-s_j)^2+\sum_{j\in\Lambda}\eta_j e^{u_j}s_j^2$$

- *u−marginal:*

$$d\rho_\Lambda(u)=e^{-\sum_{|i-j|=1} W_{ij}(\cosh(u_i-u_j)-1)-\sum_j \eta_j(\cosh u_j-1)]}\sqrt{\det H_{\beta(u)}}du_\Lambda$$

$$2\beta(u)_j=\sum_{|k-j|=1} W_{kj}e^{u_k-u_j}+\eta_j e^{-u_j}$$

- $H^{2|2}\leftrightarrow H_\beta$: $\mathbb{E}^u_\Lambda[f(\beta(u))]=\mathbb{E}^\beta_\Lambda[f(\beta)]$

## From $H^{2|2}$ to vertex reinforced jump process (VRJP)

*VRJP (Werner 2000):* continuous time process $(X_t)_{t\geq 0}$ on $\mathbb{Z}^d$.

$$\mathbb{P}\big(X_{t+dt}=j|\ X_t=i,(X_s)_{s\leq t}\big)=dt\ W_{ij}(1+L_j(t))+o(dt)$$

$L_j(t)=\int_0^t \mathbf{1}_{X_s=j}ds=$ total time spent at $j$ up to time $t$

- $H^{2|2} \leftrightarrow$ *VRJP:*

VRJP in finite volume $\Lambda$=mixture of Markov jump processes

$$\mathbb{P}_\Lambda^{VRJP}[\cdot]'='\int \mathbb{P}_\Lambda^{W(u)}[\cdot]d\rho_\Lambda(u), \qquad W(u)_{ij}=W_{ij}e^{u_i+u_j}$$

$\to u-$marginal=mixing measure for VRJP [Sabot-Tarrès-Zeng]

## three related models: $H^{2|2}$, VRJP, $H_\beta$

### Phase transitions in $d \geq 3$

- $H^{2|2}$ : disordered for $W \ll 1$ 'ordered' for $W \gg 1$

[D.-Spencer, D.-Spencer-Zirnbauer 2010]

- VRJP: recurrent for $W \ll 1 \leftrightarrow$ transient for $W \gg 1$

[Sabot-Tarrès 2013]

### Conjecture: phase transition for $H_\beta$ in $d \geq 3$

- localization $W \ll 1 \leftrightarrow$ extended for $W \gg 1$

only localization proved [Collevecchio-Zeng 2021]

# IDS for $\frac{H_\beta}{W}$

**Theorem** [D.-Rapenne-Rojas Molina-Zeng 2023]

- phase transition in $d \geq 3$ : $N(E) \simeq \sqrt{E}$ for $W \ll 1$ $\leftrightarrow$ $N(E) \leq E$ for $W \gg 1$
- no Lifschitz tails in

  $d=1$ (any disorder $W > 0$)

  $d \geq 2$ (strong disorder $W \ll 1$)

Precisely $\forall d \geq 1$ we have:

$$N(E) \leq 2\sqrt{\frac{W}{\pi}}\sqrt{E} \qquad \forall W>0, E>0$$

$$N(E) \geq c_W (|\log E|)^{-d}\sqrt{E} \qquad \forall 0<W<W_c(d),\ 0<E<E_0(W,d)$$

Note: $W_c(1)=\infty,\ W_c(d)=\frac{C}{d}, d \geq 2,\quad C=\frac{\sqrt{\pi}}{\Gamma(1/4)2^{3/4}}$

On the contrary $\forall d \geq 3$ we have:

$$N(E,H_\beta) \leq C'E \qquad \forall E>0, W>W_0(d)$$

## lower bound: strategy of the proof

- $N(E,H_\beta/W)=N(EW,H_\beta)\geq\frac{1}{|\Lambda_L|}\mathbb{P}\Big((H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq\frac{1}{WE}\Big)\quad\forall L>0$

  $H^D_{\beta,\Lambda_L}=H_{\beta,\Lambda_L}+M_{2d-n},\qquad M_{2d-n}f(j)=(2d-n_j)f(j),\ n_i=\sum_{|k-i|=1,k\in\Lambda}1$

*no direct info* on $(H^D_{\beta,\Lambda_L})^{-1}(0,0)\rightarrow$ use info on $H^{-1}_{\beta,\Lambda_L}(0,0)$

- $0<(H^D_{\beta,\Lambda_L})^{-1}(0,j)\leq H^{-1}_{\beta,\Lambda_L}(0,j)\forall j$ (rand. walk repr.) hence

$$(H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq H^{-1}_{\beta,\Lambda_L}(0,0)-(2d-1)W\sum_{j\in\partial\Lambda_L}H^{-1}_{\beta,\Lambda_L}(0,j)^2$$

- $W\ll 1\Rightarrow\mathbb{P}(\Omega_{loc}):=\mathbb{P}\Big(H^{-1}_{\beta,\Lambda_L}(0,j)\leq e^{-c|j|}\forall j\in\partial\Lambda_L\Big)\simeq 1$

- on $\Omega_{loc},H^{-1}_{\beta,\Lambda_L}(0,0)\geq\frac{1}{E}\Rightarrow(H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq\frac{1}{E}-L^{d-1}e^{-c2L}\geq\frac{1}{2E}$

$$\Rightarrow\mathbb{P}\Big((H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq\frac{1}{WE}\Big)\geq\mathbb{P}\Big(H^{-1}_{\beta,\Lambda_L}(0,0)\geq\frac{2}{WE}\cap\Omega_{loc}\Big)$$

## lower bound: strategy of the proof

- conditioned on $\beta_{0^c}$ $(H_{\beta,\Lambda_L})^{-1}(0,0)=\frac{1}{y}$ $d\rho_a(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{(a-y)^2}{2y}}\frac{1}{\sqrt{y}}dy$

$$a=a(\beta_{0^c})=\sum_{j\in\partial\Lambda_L}\frac{H^{-1}_{\beta,\Lambda_L}(0,j)}{H^{-1}_{\beta,\Lambda_L}(0,0)}\eta_j$$

$$\Rightarrow\mathbb{P}\Big((H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq\frac{1}{WE}\Big)\geq\mathbb{E}\Big[\mathbf{1}_{\Omega_{loc}}\int_0^{WE}d\rho_a(y)\Big]$$

add to $\Omega_{loc}$ : $(H_{\beta,\Lambda_L})^{-1}(0,0)\leq e^{cL/2}$

$$\Rightarrow 0<a\leq Ke^{-cL/2},\quad \mathbb{P}\Big((H^D_{\beta,\Lambda_L})^{-1}(0,0)\geq\frac{1}{WE}\Big)\geq\mathbb{E}\Big[\mathbf{1}_{\Omega_{loc}}\int_{We^{-cL}}^{WE}d\rho_a(y)\Big]$$

using the bound on $a$

$$\int_{We^{-cL}}^{WE}d\rho_a(y)=\frac{1}{\sqrt{2\pi}}\int_{We^{-cL}}^{WE}e^{-\frac{(a-y)^2}{2y}}\frac{1}{\sqrt{y}}dy\simeq\int_{We^{-cL}}^{WE}\frac{1}{\sqrt{y}}dy\simeq\sqrt{E}$$

## upper bound: strategy of the proof

- in finite volume $N(EW,H_{\beta,\Lambda_L}) \leq \frac{1}{|\Lambda_L|} \sum_{j\in\Lambda_L} \mathbb{E}_{W,\Lambda_L}\left[\mathcal{L}_{\rho_{a_j}}(2WE)\right]$

$\mathcal{L}_{\rho_{a_j}} =$ Lévy concentration of the conditional measure $\rho_{a_j(\beta_{j^c})}$

$$\mathcal{L}_\mu(\varepsilon) := \sup_x \mu([x, x+\varepsilon])$$

- $d \geq 1$ : use $\mathcal{L}_{\rho_{a_j}}(\varepsilon) \leq c\sqrt{\varepsilon}$
- $d \geq 3$ : use $\rho_{a_j}(y) \leq \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a} + \frac{1}{\sqrt{a}}\right)$ together with (use relation with $H^{2|2}$ and $u-$marginal)

$$\mathbb{E}[1/a] = \mathbb{E}_u[e^{-u_0} H^{-1}_{\beta(u),\Lambda_L}(0,0)] \leq C/W$$

## Some open problems

- $d \geq 3$ : complete the phase diagram for IDS
- spectral phase transition for $H_\beta$ from the phase transition in the IDS?

THANK-YOU!