Resonances in wave reflection from a disordered medium: nonlinear $\sigma-$model approach [1]

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[1] Based on YVF, M. Skvortsov & K. Tikhonov, arXiv:2211.03376

Quantum particle in a disordered media, models:

Consider a **classical** particle moving with speed v in a domain $\mathcal{D} \subset \mathbb{R}^d$ with randomly placed scattering centers separated by the mean-free path l, The motion is then characterized by a **diffusion constant** $D \sim vl \sim l^2/\tau$.

To describe a **quantum** analogue of such a system one may consider the Hamiltonian:

$$H = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}A(\mathbf{r})\right)^2 + \mathbf{V}(\mathbf{r}) \text{ + Dirichlet b.c. at } \partial\mathcal{D}$$

where $A(\mathbf{r})$ is a potential of a magnetic field $B = \nabla \times A$ and $\mathbf{V}(\mathbf{r})$, $\mathbf{r} \in \mathcal{D}$ is a (short-correlated) Gaussian random potential, in the simplest case:

$$\langle \mathbf{V}(\mathbf{r})\mathbf{V}(\mathbf{r}')\rangle = \frac{1}{2\pi\nu\tau}\delta(\mathbf{r} - \mathbf{r}')$$

where in the semiclassical "weak disorder" limit $\lambda \sim k^{-1} \ll l$ with energies $E \sim (\hbar k)^2/2m \gg \hbar\tau^{-1}$, the density of energy levels ν per unit volume is given by $\nu \sim mk^{d-2}$.

Alternatively, one can consider its lattice analogue, the **Anderson** model, defined for $\mathbf{x} \in \mathbf{\Lambda} \subset \mathbb{Z}^d$:

$$H = \sum_x \mathbf{V}_x \left|\mathbf{x}\right\rangle \left\langle\mathbf{x}\right| + \sum_{x\sim y}\left(t_{xy}e^{i\phi_{xy}} \left|\mathbf{x}\right\rangle \left\langle\mathbf{y}\right| + c.c.\right)$$

with phases $\phi_{xy} = -\frac{e}{\hbar}\int_{\mathbf{x}}^{\mathbf{y}} A(\mathbf{r}) \cdot d\mathbf{l}$ and $\langle \mathbf{V}(\mathbf{x})\mathbf{V}(\mathbf{x}')\rangle = W^2\delta_{\mathbf{x}\mathbf{x}'}$.

Physicists view, nonlinear $\sigma-$model description:

To get **quantitative** understanding of quantum particle motion in a random potential one may study e.g. the probability $P(\mathbf{r},\mathbf{r}',t)$ of transiting from $\mathbf{r}$ to $\mathbf{r}'$ in time t.
Its Fourier-transform can be written as

$$P(\mathbf{r},\mathbf{r}',\omega) = \frac{1}{2\pi\nu}\lim_{\eta\to 0}\left\langle G_{\mathbf{r}',\mathbf{r}}(E-\omega/2+i\eta)G_{\mathbf{r},\mathbf{r}'}(E+\omega/2-i\eta)\right\rangle$$

in terms of the resolvent

$$G_{\mathbf{r}',\mathbf{r}}(E+i\eta) = (E+i\eta-H)^{-1}_{\mathbf{r}',\mathbf{r}}\,, \quad \eta>0$$

The most efficient computational framework for investigating **universal** features of such objects is provided by a (super)matrix **nonlinear** $\sigma-$**model** introduced by **Wegner**'79 and fully developed by **Efetov**'82.

It maps computations of resolvent moments to studying a **non-random** model involving interacting **supermatrices** $Q_{\mathbf{x}}$, associated with every site on a lattice $\mathbf{x}\in\Lambda\subset\mathbb{Z}^d$ and satisfying constraints $Q_{\mathbf{x}}^2=1$ and $StrQ_{\mathbf{x}}=0$. Namely:

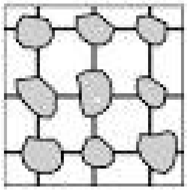
$$\left\langle f(G_{\mathbf{x}_i,\mathbf{x}_j}(E_1+i\eta),G_{\mathbf{x}_k,\mathbf{x}_l}(E_2-i\eta)\right\rangle_H \longrightarrow \int \mathcal{F}^{\mathbf{x}_k,\mathbf{x}_l}_{\mathbf{x}_i,\mathbf{x}_j}(Q)\,e^{-\mathcal{S}[Q]}\prod_{\mathbf{x}}\mathcal{D}\mu\left(Q_{\mathbf{x}}\right)$$

with the action

$$\mathcal{S}[Q] = \frac{\alpha}{2}\sum_{\mathbf{x}\sim\mathbf{y}} StrQ_{\mathbf{x}}Q_{\mathbf{y}} + (\eta-i\omega)\sum_x Str\left(\Lambda Q_{\mathbf{x}}\right), \quad \alpha>0,\, \omega=\frac{E_2-E_1}{2}.$$

Granular matter and banded matrices:

Physically, such nonlinear $\sigma-$model can be interpreted as describing a system of **metallic granules** on a lattice.



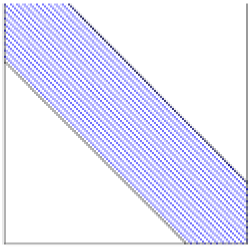
Each isolated granula represents a fully ergodic "**zero-dimensional**" system and is described by a single $Q-$matrix via the "action" $\mathcal{S}[Q] = (\eta - i\omega)\, Str\, (\Lambda Q)$. Allowing nearest neighbors to interact via tunneling creates the term $\frac{\alpha}{2}\sum_{\mathbf{x}\sim\mathbf{y}} StrQ_{\mathbf{x}}Q_{\mathbf{y}}$.

Remark. After appropriate scaling one can pass from the lattice to continuum. For example, setting for simplicity $\omega = 0$ and taking $d = 1$ one gets

$$\mathcal{S}[Q] = -Str \int_0^L \left[\frac{\pi\nu D}{4}\left(\frac{\partial Q}{\partial x}\right)^2 - \pi\nu\eta\,(\Lambda Q(x))\right]\, dx$$

where D is the familiar **diffusion constant**, and ν in the sample of length L defines the energy level spacing via $\Delta = (\nu L)^{-1}$.

Such form can be alternatively (and rigorously) derived from the model of **Banded Random Matrices** (**YF-Mirlin**'91-'94; **Shcherbina-Shcherbina** '18), where $L \sim N$ and $D \sim b^2 \gg 1$.



OPF in $\sigma-$model description:

In such a framework computation of all physical quantities characterizing statistics of eigenfunctions, energy levels, transport properties, etc. is reduced to studying expectations of the form $\int \mathcal{D}\mu(Q)(...) \exp -\mathcal{S}[Q]$.

In particular, one of the most useful objects in the theory is the "**order parameter function**" (OPF) introduced originally in **Zirnbauer**'86 via

$$Y_{\mathbf{x}}(Q;\eta) = \int_{Q_{\mathbf{x}}=Q} e^{-\mathcal{S}[Q_{\mathbf{y}}]} \prod_{\mathbf{y}\neq\mathbf{x}} \mathcal{D}\mu\left(Q_{\mathbf{y}}\right)$$

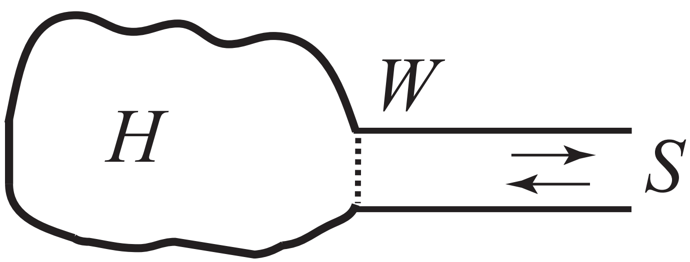
In the simplest example of systems with **broken time-reversal** symmetry the OPF actually depends only on two real *Cartan variables* parametrizing the $Q-$ matrices:

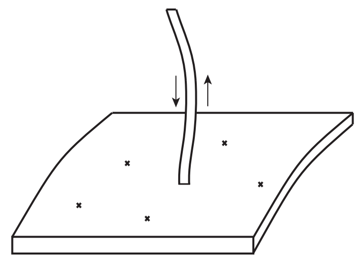
$$Y_{\mathbf{x}}(Q;\eta) := \mathcal{Y}(\lambda,\lambda_1;\eta) \text{ with } \lambda \in [-1,1] \text{ and } \lambda_1 \in [1,\infty].$$

Remark 1. This function is **conceptually important** as it describes the Anderson (de)localization transition as spontaneous symmetry breaking phenomenon: in the two phases $\mathcal{Y}(\lambda,\lambda_1;\eta)$ has very different dependence on the **non-compact** variable λ_1 as $L \to \infty$ accompanied with the limit $\eta \to 0$, hence the name.

Remark 2. In what follows it turns out that the behaviour of the OPF $\mathcal{Y}(\lambda,\lambda_1;\eta)$ at finite η plays a central role. Note that η in the theory may be given an exact meaning: it represents a uniform **absorption rate** for particles in the medium.

"Heidelberg Approach" to wave scattering in chaotic/disordered systems:





Consider a model of **quantum particle/wave** reflection from a random medium via a single waveguide with M open channels characterized by the $M \times M$ energy-dependent scattering matrix of the form (**Weidenmueller et al** 85')

$$S(E) = \frac{\mathbf{1} - iK}{\mathbf{1} + iK} \qquad \text{where} \qquad K_{ab} = \sum_{\mathbf{x},\mathbf{y}} \overline{W}_{a\mathbf{x}} \left(E + i\eta - H\right)^{-1}_{\mathbf{x}\mathbf{y}} W_{\mathbf{y}b}$$

A self-adjoint H is to be chosen to describe **closed** system with a **random medium** inside, e.g.

$$H = -\Delta + \mathbf{V}(\mathbf{x}) \quad \text{or} \quad H = \sum_{\mathbf{x}} \mathbf{V}_{\mathbf{x}} \left|\mathbf{x}\right\rangle \left\langle \mathbf{x}\right| + \sum_{\mathbf{x}\sim\mathbf{y}} \left(t_{xy} \left|\mathbf{x}\right\rangle \left\langle \mathbf{y}\right| + c.c.\right)$$

with random potential $\mathbf{V}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$ or $\mathbf{x} \in \mathbb{Z}^d$, $|\Lambda| = N$.

The **coupling amplitudes** $\mathbf{w}_a = (W_{a1}, \ldots, W_{aN})$ of N inner states to M open channels are taken as *fixed orthogonal* energy-independent vectors $\mathbf{w}_a$

$$\mathbf{w}_a^{\dagger} \mathbf{w}_b = \gamma_a \delta_{ab}, \qquad \gamma_a > 0 \ \forall a = 1, \ldots, M.$$

Effective non-Hermitian Hamiltonian, resonances:

Equivalently, defining $z = E + i\eta$ entries of the scattering matrix can be rewritten as

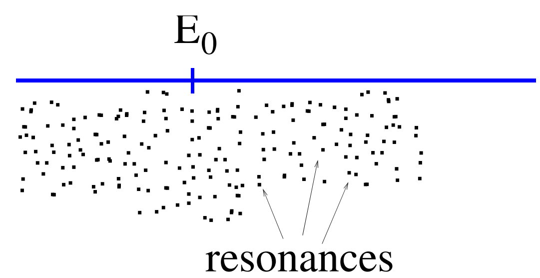
$$S_{ab}(z) = \delta_{ab} - 2i \sum_{\mathbf{x},\mathbf{y}} W^*_{a\mathbf{x}} \left[\frac{1}{z - \mathcal{H}_{eff}}\right]_{\mathbf{xy}} W_{\mathbf{y}b},$$

with an effective **non-Hermitian** Hamiltonian

$$\mathcal{H}_{eff} = H - i\Gamma, \quad \Gamma = \sum_{a=1}^{M} \mathbf{w}_a \otimes \mathbf{w}_a^\dagger \geq 0 - \text{ rank M}$$

whose N complex eigenvalues $z_n = E_n - i\Gamma_n$ provide **poles** of the scattering matrix in the lower half of $z-$plane, known as **RESONANCES**.

Main question:
Given the **mean level spacing** Δ for disordered medium in the **closed** scattering region, what can be said about the mean density of **S-matrix poles**



$$\rho_E(y) := \Delta \left\langle \sum_{n=1}^{N} \delta(E - E_n) \delta\left(y - 2\pi\Gamma_n/\Delta\right) \right\rangle$$

in the **open** medium, especially of the statistics of imaginary parts $y_n = 2\pi\Gamma_n/\Delta$?

Remark. The quantities Γ_n are traditionally called **resonance widths**. Their values are expected to reflect the **decay times** of quantum states from the medium to continuum via open channels, and hence are of special interest.

Resonances via OPF in $\sigma-$model description:

It turns out the **mean resonance density** for $M-$channel single-lead reflection from a **disordered medium** can be explicitly related to the associated **Order Parameter Function** $\mathcal{Y}(\lambda,\lambda_1;\eta)$ of the associated $\sigma-$ model with **uniform absorption** $\eta>0$:

$$\rho_E(y)=\frac{(-1)^{M-1}}{2(M-1)!}\frac{\partial^2}{\partial y^2}\int_{-1}^{1}(g-\lambda)^M\,\frac{\partial^{M-1}}{\partial g^{M-1}}\left[\frac{\mathcal{Y}(\lambda,g;\eta)}{(g-\lambda)^2}\right]_{\eta=\Delta y/2\pi},\quad g=\frac{1}{2\pi\nu}(\gamma+\gamma^{-1})$$

Remark: derivation proceeds through finding the density of **complex** eigenvalues of an $M\times M$ **non-Hermitian resolvent** matrix $(E+i\eta-H)^{-1}_{\mathbf{r}_1,\mathbf{r}_2},\quad \eta>0$, with M distinct but close points $\mathbf{r}_i\;\; i=1,\ldots,M$.

The simplest case is that of the **zero-dimensional** σ-model in the ergodic regime describing a **fully chaotic system**, with Hamiltonian essentially equivalent to a single random **GUE matrix**. For such a case it is well-known that $\mathcal{Y}(\lambda,\lambda_1;\eta)=e^{-\eta(\lambda_1-\lambda)}$ implying the resonance density (**YF - Sommers** '96; **YF - Khoruzhenko** '99, see also **Shcherbina-Shcherbina** '21.

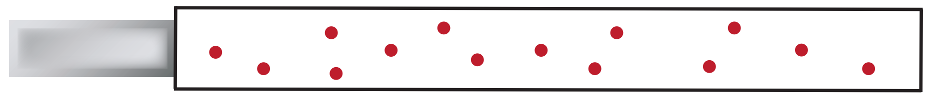
$$\rho_E(y)=\frac{(-1)^M y^{M-1}}{(M-1)!}\frac{\partial^M}{\partial y^M}\left(e^{-yg\frac{\sinh y}{y}}\right).$$

In particular, for **perfect coupling** $g=1$ the tail is powerlaw: $\rho(y\gg 1)\sim M/y^2$.
Confirmed in experiment: **L Chen, S. M. Anlage $\&$ YF** *Phys Rev Lett* **127** *, 204101(2021)*

Resonances in reflection from quasi-1D disordered media:

The **one-dimensional** σ-model with the action

$$\mathcal{S}[Q] = -Str \int_0^L dx \left[\frac{\pi\nu D}{4} \left(\frac{\partial Q}{\partial x}\right)^2 - \pi\nu\eta\left(\Lambda Q(x)\right)\right]$$

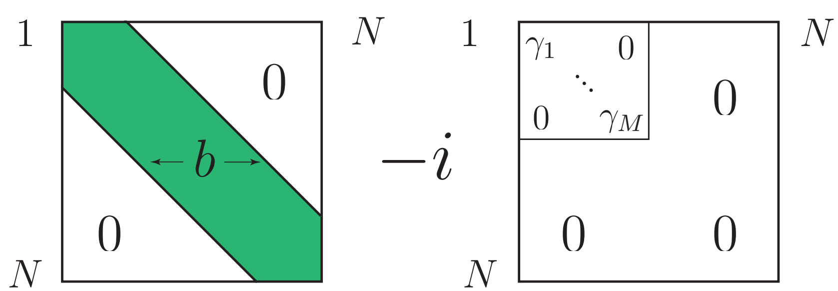


describes a **disordered wire** (**Efetov-Larkin**'83) of length L and the localization length $\xi = 2\pi\nu D$. Attaching an $M-$channel waveguide to one end provides the first truly nonperturbative test of our theory.

The OPF $\mathcal{Y}(\lambda, \lambda_2; \eta)$ as a function of length L satisfies an evolution equation

$$\frac{\partial}{\partial L}\mathcal{Y} = -\mathcal{H}\mathcal{Y} \text{ with the initial condition } \mathcal{Y}(\lambda, \lambda_1; \eta)|_{L=0} = 1$$

and $\mathcal{H}$ being a certain second-order differential operator with respect to λ, λ_1. The associated **scattering** problem can be modelled using the effective **banded** non-Hermitian Hamiltonian:



Resonances in reflection from 1D disordered media:

Define Δ_ξ to be the level spacing for a medium with $L = \xi$. For $L >> \xi$ it is natural to measure resonance widths Γ_n in units of Δ_ξ, defining $y_n = 2\pi\Gamma_n/\Delta_\xi$.

The associated OPF in the limit $L/\xi \to \infty$ has been found explicitly:

$$\mathcal{Y}(\lambda,\lambda_1;y) = K_0(p)qI_1(q) + I_0(q)pK_1(p), \quad \textbf{Skvortsov-Ostrovsky}\ '07$$

where $q = \kappa\sqrt{(\lambda+1)/2},\ p = \kappa\sqrt{(\lambda_1+1)/2}$, with $\kappa = \sqrt{\frac{8y}{\pi}}$ and $K_n(\kappa),\ I_n(\kappa)$ being modified Bessel functions.

This allows us to find the probability density of the scaled resonance widths for M **perfectly coupled** channels attached to the 'edge" of the wire as

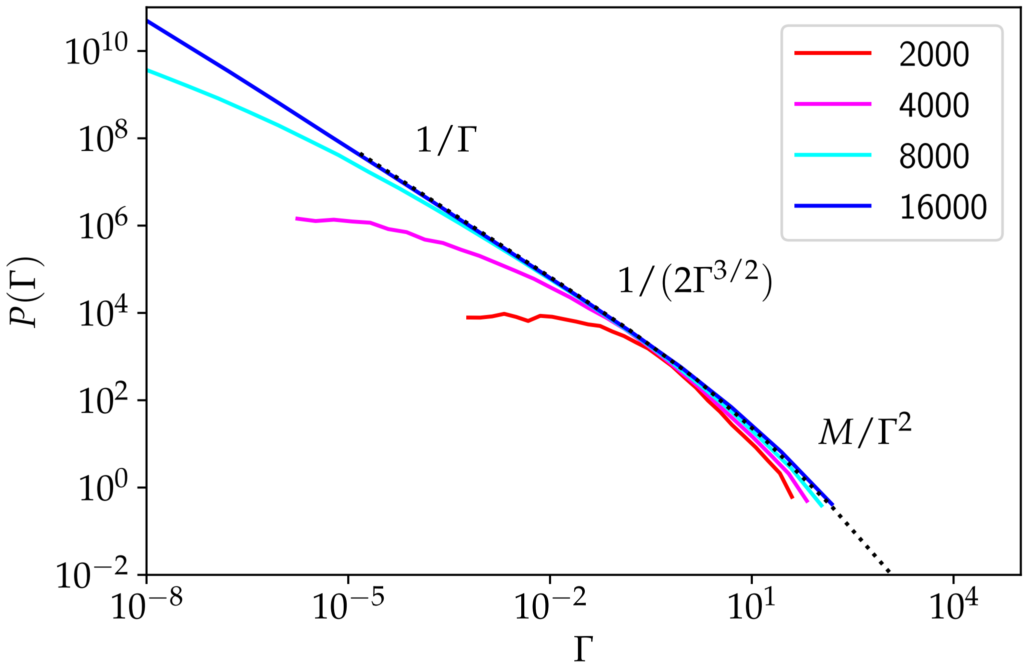
$$\rho(y) = -\frac{4}{\pi^2\kappa}\frac{\partial}{\partial\kappa}\left[\frac{1}{\kappa}\sum_{n=0}^{M-1} K_n(\kappa)I_{n+1}(\kappa)\right], \text{ where } \kappa = \sqrt{\frac{8y}{\pi}},$$

Analysis at $M \gg 1$ shows that

$$\rho(\Gamma) \sim \begin{cases} \Delta_\xi/\Gamma & \text{for} \quad \Gamma \ll \Delta_\xi & \textbf{localization} \\ (\Delta_\xi/\Gamma)^{3/2} & \text{for} \quad \Delta_\xi \ll \Gamma \ll M^2\Delta_\xi & \textbf{diffusion} \\ M\,(\Delta_\xi/\Gamma)^2 & \text{for} \quad \Gamma \gg M^2\Delta_\xi & \textbf{ergodic decay} \end{cases}$$

The behaviour $\rho(\Gamma) \sim \Gamma^{-3/2}$ was first reported numerically for **quantum kicked rotator** in **Borgonovi, Guarneri, and Shepelyansky** '91, see also **Kottos** '05, **Skipetrov - van Tiggelen** '06

The analytic results agree well with numerical simulations for **banded matrices**:



The resonance density $\rho(\Gamma)$ for banded random matrices with $b = 30$ of different sizes (from $N = 2000$ to $N = 16000$) and $M = 10$ open channels. In the limit $N \to \infty$ the numerical curves approach the dashed black line, which is computed using the analytic formula.

Future aims: to provide analysis of $\rho(\Gamma)$ for $d > 1$ in various regimes, including the vicinity of **Anderson localization** transition; to extend from $\sigma-$**model** to **random Schrödinger**, e.g. in pure 1D: $-\frac{d^2}{dx^2} + V(x)$. See **Kunz-Shapiro**'08 $\&$ **Feinberg**'09 in Physics and **Klopp**'16 in Math context.