Fishnets, Yangians and Calabi-Yaus

Florian Loebbert

Bethe Center For Theoretical Physics
University of Bonn

based on arXiv:2209.05291 + work in progress with
C. Duhr, A. Klemm, C. Nega, F. Porkert

Elliptics 2023
ETH Zürich

Zamolodchikovs ~~Fishnets~~, Yangians and Calabi-Yaus

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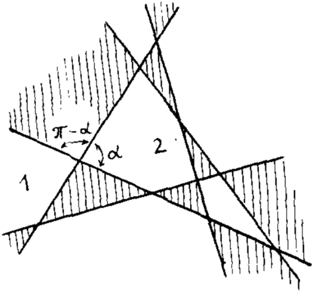
“FISHING-NET” DIAGRAMS AS A COMPLETELY INTEGRABLE SYSTEM

A.B. ZAMOLODCHIKOV
The Academy of Sciences of the USSR, L.D. Landau Institute for Theoretical Physics, Chernogolovka, USSR

Received 29 July 1980



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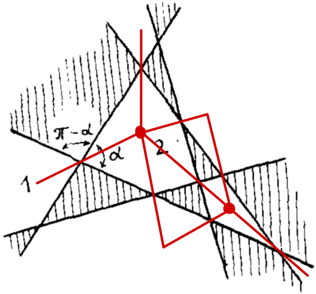
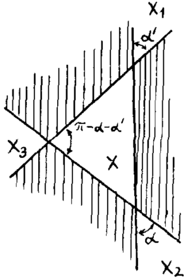


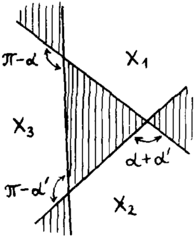
Vertex  : $\int \mathrm{d}^D x$

Propagator : $\frac{1}{x_{jk}^{2\alpha}} \equiv \frac{1}{(x_j - x_k)^{2\alpha}}$ $\qquad (x^2 = x^\mu x_\mu)$

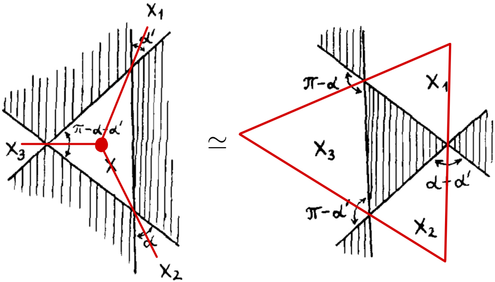
Integrability and Conformal Symmetry

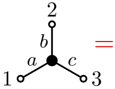


$\simeq$

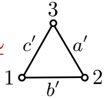


Integrability and Conformal Symmetry



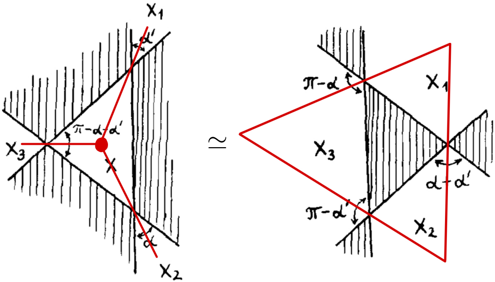


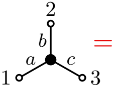
$$= \int \frac{\mathrm{d}^D x_0}{x_{10}^{2a} x_{20}^{2b} x_{30}^{2c}} \overset{a+b+c=D}{=} \frac{X_{abc}}{x_{12}^{2c'} x_{23}^{2a'} x_{31}^{2b'}} \simeq$$



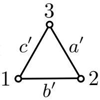
with $X_{abc} = \pi^{\frac{D}{2}} \frac{\Gamma_{a'}\Gamma_{b'}\Gamma_{c'}}{\Gamma_a \Gamma_b \Gamma_c}$ and $a' = \frac{D}{2} - a$

Integrability and Conformal Symmetry





$$= \int \frac{\mathrm{d}^D x_0}{x_{10}^{2a} x_{20}^{2b} x_{30}^{2c}} \overset{a+b+c=D}{=} \frac{X_{abc}}{x_{12}^{2c'} x_{23}^{2a'} x_{31}^{2b'}} \simeq$$



with $X_{abc} = \pi^{\frac{D}{2}} \frac{\Gamma_{a'}\Gamma_{b'}\Gamma_{c'}}{\Gamma_a \Gamma_b \Gamma_c}$ and $a' = \frac{D}{2} - a$

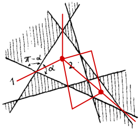
Do these graphs look like fishnets?

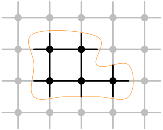
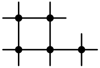
Fishnets in 4D

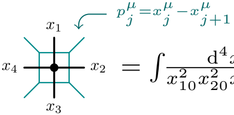
Euclidean integrals made from four-point vertices:

Propagator: j —— k : $\frac{1}{x_{jk}^{2a}} = \frac{1}{(x_j - x_k)^{2a}}$ with $a = 1$

Vertex:  : $\int \mathrm{d}^4 x$ with $\sum_{j=1}^{n} a_j = 4$



→ e.g.  → 

Simplest example: cross integral [Ussyukina '93, Davydychev]: 

$= \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$

Fishnets in 4D

Euclidean integrals made from four-point vertices:

Propagator: j —— k : $\frac{1}{x_{jk}^{2a}} = \frac{1}{(x_j - x_k)^{2a}}$ with $a = 1$

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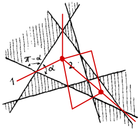
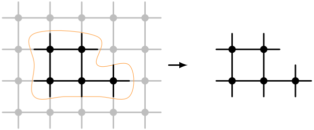
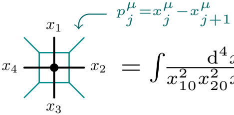


→ e.g.

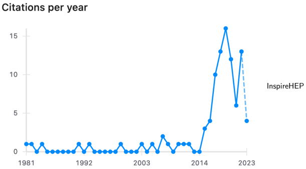


Simplest example: cross integral [Ussyukina '93, Davydychev]:



$$= \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$$

Fishnets and AdS/CFT integrability:



Fishnet Graphs as Correlation Functions

Double-scaling limit in AdS/CFT:

$\underline{\mathcal{N}=4\text{ SYM}}$		$\underline{\gamma\text{-Deformation}}$		<u>Fishnets</u>
$\mathcal{L}_{\mathcal{N}=4}$	$\xrightarrow{X\,Y\to e^{i\gamma_j(\dots)}XY}$	$\mathcal{L}^{\gamma}_{\mathcal{N}=4}$	$\xrightarrow[\xi=ge^{-i\gamma_3/2}\text{ fix}]{g\to 0,\ \gamma_3\to i\infty}$	$\mathcal{L}_{\mathsf{F}}$

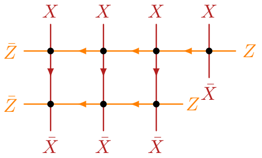
Resulting bi-scalar fishnet theory: [Gürdoğan Kazakov 2015]

$$\mathcal{L}_{\mathsf{F}} = N_{\mathsf{c}}\,\mathrm{tr}\big(-\partial_\mu \bar{X}\partial^\mu X - \partial_\mu \bar{Z}\partial^\mu Z + \xi^2 \bar{X}\bar{Z}XZ\big)$$

- Correlators given by single fishnet Feynman graphs.
- Fishnet integrals inherit conformal Yangian symmetry $Y[\mathfrak{so}(1,5)]$:

differential operator

$$\widehat{\mathrm{J}}^a$$



$$= 0.$$

[Chicherin, Kazakov, FL Müller, Zhong 2017]

Integrability and the Yangian

- The Yangian is an infinite dimensional extension of a Lie algebra $\mathfrak{g}$.
- It underlies rational quantum integrable models (rational S-matrix).

Yangian algebra $Y[\mathfrak{g}]$ (first realization): [Drinfeld 1985]

Level 0 : $$\mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}_k^a$$

Level 1 : $$\widehat{\mathrm{J}}^a = f^a{}_{bc} \sum_{j<k=1}^{n} \mathrm{J}_j^c \, \mathrm{J}_k^b$$

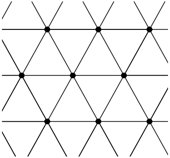
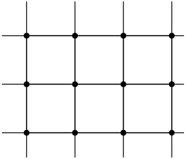
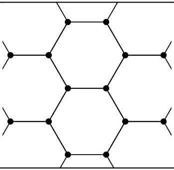
Serre relations: $$[\widehat{\mathrm{J}}_a, [\widehat{\mathrm{J}}_b, \mathrm{J}_c]] - [\mathrm{J}_a, [\widehat{\mathrm{J}}_b, \widehat{\mathrm{J}}_c]] = \mathcal{O}(\mathrm{J}^3).$$

Examples:

- AdS/CFT: $\mathfrak{g} = \mathfrak{psu}(2,2|4)$
- fishnet integrals: $\mathfrak{g} = \mathfrak{so}(1, D+1)$

Regular Tilings of the Plane

Similar Yangian symmetry for other graph structures [Chicherin, Kazakov, FL Müller, Zhong 2017]

Dimension	$D = 3$	$D = 4$	$D = 6$
Propagator	$\lvert x_{ij} \rvert^{-1}$	$\lvert x_{ij} \rvert^{-2}$	$\lvert x_{ij} \rvert^{-4}$
Scalar Fishnet			

- Works also for parametric propagator powers a_k with conformal condition [Chicherin, Kazakov, FL Müller, Zhong 2017][FL, Müller Münkler 2019]

$$\sum_{k \in \text{vertex}} a_k = D.$$

- Recently generalized to Zamolodchikov's Baxter lattices ("looms") [Kazakov Olivucci 2022] [Kazakov, Levkovich-Maslyuk Mishnyakov 2023]

Yangian PDEs for Feynman Integrals

Level 0:

[FL, Müller Münkler 2019]

$$\mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}_k^a \qquad \text{with} \qquad \mathrm{J}^a \in \begin{cases} \mathrm{D} = -ix_\mu \partial^\mu - i\Delta, \\ \mathrm{L}_{\mu\nu} = ix_\mu \partial_\nu - ix_\nu \partial_\mu, \\ \mathrm{P}_\mu = -i\partial_\mu, \\ \mathrm{K}_\mu = ix^2 \partial_\mu - 2ix_\mu x^\nu \partial_\nu - 2i\Delta x_\mu. \end{cases}$$

$\Rightarrow I_n = V_n \, \phi$ with $\phi(z_1, z_2, \dots)$ function of conformal cross ratios.

Level 1: additional non-local generators $\widehat{\mathrm{J}}^a = f^a{}_{bc} \sum_{j<k} \mathrm{J}_j^c \, \mathrm{J}_k^b$ e.g.

$$\widehat{\mathrm{P}}^\mu = \sum_{j<k=1}^{n} \left[\left(\mathrm{L}_j^{\mu\nu} + \eta^{\mu\nu} \mathrm{D}_j \right) \mathrm{P}_{k,\nu} - (j \leftrightarrow k) \right] + \sum_{k=1}^{n} s_k \mathrm{P}_k$$

Yangian invariance: $$0 = \widehat{\mathrm{P}}^\mu I_n = V_n \sum_{j<k=1}^{n} \frac{x_{jk}^\mu}{x_{jk}^2} \, \mathrm{PDE}_{jk} \phi$$

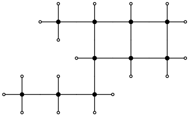
Leads to **system of Yangian PDEs** in the cross ratios:

$$\mathrm{PDE}_{jk} \, \phi = 0, \qquad 1 \le j < k \le n.$$

Fishnets in Lower Dimensions

Conformal Fishnets in 1D and 2D

Fishnet integrals in lower dimensions with conformal choice of powers in propagators $|x|^{-2a_j}$: [Duhr, Klemm, FL; Nega, Porkert, in progress]



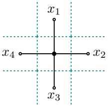
$$1\text{D} : a_j = \tfrac{1}{4}, \qquad 2\text{D} : a_j = \tfrac{1}{2}, \qquad \sum_{j=1}^{4} a_j = D.$$

Integrals are correlators in D-dimensional fishnet theory: [Kazakov, Olivucci 2018]

$$\mathcal{L} = N_c \, \mathrm{tr}\Big[X(-\partial_\mu \partial^\mu)^{\frac{D}{4}} \bar{X} + Z(-\partial_\mu \partial^\mu)^{\frac{D}{4}} \bar{Z} + \xi^2 X Z \bar{X} \bar{Z}\Big]$$

Simplest example: Cross integral in 1D (here $x_1 < x_2 < x_3 < x_4$):

$$I_4^{1\text{D}} =$$



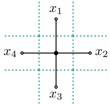
$$= \frac{4}{\sqrt{x_{13}x_{24}}}\big[K(z) + K(1-z)\big], \qquad z = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

with elliptic K integral: $K(z) = \int_0^{\pi/2} \mathrm{d}\theta \frac{1}{\sqrt{1 - z\sin^2\theta}}$.

Geometry of 1D Cross

Simplest example: Cross integral in 1D

$$I_4^{\mathsf{1D}} =$$



$$= \int \frac{\mathrm{d}x_0}{x_{10}^{2\frac{1}{4}}\, x_{20}^{2\frac{1}{4}}\, x_{30}^{2\frac{1}{4}}\, x_{40}^{2\frac{1}{4}}} \xrightarrow{\text{conf. transf.}} \int \frac{\mathrm{d}x}{\sqrt{x(x-1)(x-z)}}$$

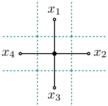
Natural geometry given by Legendre family of elliptic curves

$$y^2 = x(x-1)(x-z) = P_{\mathsf{cross}}(x,z)$$

1D Box from Yangian Bootstrap

Consider Yangian over 1D conformal algebra $Y[\mathfrak{sl}(2,\mathbb{R})]$ on one-loop box:

$$I_4 = $$



$$= \frac{1}{\sqrt{x_{13}x_{24}}}\phi(z)$$

Yangian differential equation (= Legendre equation):

$$0 = \phi(z) + 4(2z-1)\phi'(z) + 4(z-1)z\phi''(z), \qquad z = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

Two solutions:

1 power series	1 single-log solution
$K(z) = \sum_j c_j z^j$	$K(1-z) = \log(z)\sum_j c_j z^j + \ldots$

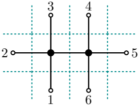
For $\vec{\Pi} = (K(z), K(1-z))$ integral must be given by

$$\phi(z) = \vec{v}\cdot\vec{\Pi}.$$

Fix linear combination using e.g. numerics to find $\vec{v} = (4,4)$.

1D Double Box from Yangian Bootstrap

Two loops:

$I_6 =$ 

$= \dfrac{1}{\sqrt{x_{14}x_{26}x_{35}}}\phi(z_1, z_2, z_3)$

Set of homogeneous second-order PDEs generated by Yangian symmetry $\widehat{\mathrm{P}} I_6 = 0$, e.g.

$$
\begin{aligned}
0 =& z_2\phi + 2\left(z_2-1\right)\left(z_1 z_2\left(z_3-1\right)+1\right) z_2^2 \phi^{(0,2,0)} \\
&- \left(z_3-1\right)\left(5z_3 + z_1 z_2\left(3z_3^2 - 5z_3 + 2\right) - 2\right) z_2 \phi^{(0,0,1)} \\
&- 2\left(z_1 z_2\left(z_3-1\right)+1\right)\left(z_3-1\right)^2 z_3 z_2 \phi^{(0,0,2)} \\
&+ \left(3z_1\left(z_3-1\right) z_2^2 + \left(5 - z_1\left(z_3-1\right)\right) z_2 - 3\right) z_2 \phi^{(0,1,0)} \\
&+ \left(z_2\left(z_3-1\right) z_1^2 + 3z_1 - 2\right) \phi^{(1,0,0)} \\
&+ 2\left(z_1-1\right) z_1\left(z_1 z_2\left(z_3-1\right)+1\right) \phi^{(2,0,0)}
\end{aligned}
$$

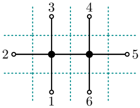
Yangian and Permutations

Note: In 1D, $\widehat{\mathrm{P}}$ yields less PDEs than in higher dimensions, cf. [FL, Müller Münkler 2019]

$$0 = \widehat{\mathrm{P}}^{\mu} I_n = V_n \sum_{j<k=1}^{n} \frac{x_{jk}^{\mu}}{x_{jk}^{2}} \, \mathrm{PDE}_{jk} \phi$$

Use Permutations: Yangian level-one generator not invariant under permutation symmetries of graph G, e.g. for the double box:

$$\sigma \in \{1 \leftrightarrow 2,\ 1 \leftrightarrow 3,\ 4 \leftrightarrow 5, \ldots$$

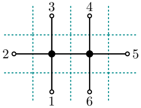


Permutations $\sigma \in \mathcal{S}_G$ generate further differential operators:

$$\sigma \circ \widehat{\mathrm{J}}^{a} = f^{a}{}_{bc} \sum_{j<k=1}^{n} \mathrm{J}^{c}_{\sigma(j)} \, \mathrm{J}^{b}_{\sigma(k)}$$

1D Double Box from Yangian Bootstrap cntd.

Two loops:

$$I_6 =$$  $$= \frac{1}{\sqrt{x_{14}x_{26}x_{35}}}\phi(z_1, z_2, z_3)$$

Full set of PDEs from Yangian and permutations $\sigma \in \mathcal{S}_G$:

$$\widehat{\mathrm{P}} I_6 = 0, \qquad (\sigma \circ \widehat{\mathrm{P}}) I_6 = 0$$

Frobenius Method: Ansatz yields 5-dimensional solution vector $\vec{\Pi}$

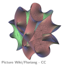
1 power series	3 single-log	1 double-log solution
$\sum_{jkl} c_{jkl} z_1^j z_2^k z_3^l$	$\log(z_a) \sum_{jkl} c_{jkl} z_1^j z_2^k z_3^l$	$\log(z_a)\log(z_b) \sum_{jkl} c_{jkl} z_1^j z_2^k z_3^l$

Fix linear combination e.g. by using numerics:

$$\phi(z_1, z_2, z_3) = \vec{v} \cdot \vec{\Pi}$$

$\ell = 1$: elliptic curve, $\ell = 2$ geometry? $\rightarrow$ need Calabi-Yaus

Mini Calabi-Yau Overview



A Calabi–Yau ℓ-fold is an ℓ-dimensional complex Kähler manifold with vanishing first Chern class.

Uniquely defined by triplet: $\left[\begin{array}{l} M \\ \Omega \\ \omega \end{array}\right.$ $\begin{array}{l} \leftarrow \text{complex, } \ell\text{-dimensional manifold} \\ \leftarrow (\ell, 0) \text{ form} \\ \leftarrow \text{Kähler } (1,1) \text{ form} \end{array}$

Integrating Ω over the cycles Γ_j of the CY yields a vector $\vec{\Pi}$ of associated periods $\Pi_j(z) = \int_{\Gamma_j} \Omega$

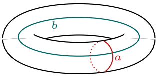
For every family of CYs there is a set of differential operators, the Picard–Fuchs Ideal (PFI), whose solutions are exactly the periods.

Example: 1D Box Integral (CY 1-fold, Elliptic Curve)

Triplet $(\mathcal{E}, \mathrm{d}a = \frac{\mathrm{d}x}{y}, A\, \mathrm{d}a \wedge \mathrm{d}\bar{a})$

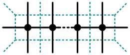
Periods $\vec{\Pi} = (K(z), K(1-z))$

PFI $= \{1 + 4(2z-1)\partial_z + 4z(z-1)\partial_z^2\}$



General Fishnets in 1D

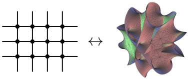
Traintracks are Calabi–Yau ℓ-folds. [Duhr, Klemm, FL, Nega, Porkert]



Loops	1	2	3	...
Geometry	Elliptic Curve	K3 surface	CY 3-fold	...
Periods	1|1	1|3|1	1|5|5|1	...

Note: In 4D it's $\ell - 1$-folds! [Bourjaily, He, Mcleod, Hippel, Wilhelm 2018]

Generic fishnets have CY structure!



Picture Wiki/Floriang - CC

$$y^2 = P_G(x)$$

P_G of degree 4 in integration variables

holomorphic $(\ell, 0)$ form: $\Omega_G = \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_\ell}{\sqrt{P_G(z)}}$

Conjecture

Yangian with permutations generates Picard–Fuchs ideal of differential operators with Calabi–Yau periods as solutions!

From 1 to 2 Dimensions

Double Copy in 2D

Split 2D Yangian into holomorphic and anti-holomorphic part:

$$Y[\mathfrak{sl}(2,\mathbb{R})] \oplus \overline{Y[\mathfrak{sl}(2,\mathbb{R})]}$$

Double Copy Structure: Same Yangian invariants $\vec{\Pi}$ as in 1D:

$$\phi(z) = \vec{\Pi}^\dagger \cdot \Sigma \cdot \vec{\Pi}$$

Indeed: Box integral in 2D given by linear combination of two factorized Yangian invariants [Derkachov, Kazakov Olivucci 2018] [Corcoran, FL Miczajka 2021]:

$$\begin{aligned}\phi(z,\bar{z}) &= 4\big[K(z)K(1-\bar{z}) + K(1-z)K(\bar{z})\big] \\ &= 4i \begin{pmatrix} K(z) & iK(1-z) \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} K(\bar{z}) \\ -iK(1-\bar{z}) \end{pmatrix}\end{aligned}$$

What is the role of the matrix Σ?

Intersection Matrix and Kähler Potential

- The intersection matrix Σ of the Calabi–Yau defines a natural bilinear pairing of the periods. We observe [Duhr, Klemm, FL Nega, Porkert 2022]

$$\phi(z) = \vec{\Pi}^\dagger \cdot \Sigma \cdot \vec{\Pi} = e^{-V}$$

Here V denotes the Kähler potential.

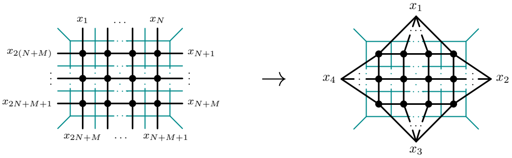
- Intersection matrix Σ can be computed explicitly (using the Griffiths transversality).

This structure persists for higher loop integrals!

Calabi-Yaus and Basso-Dixon Formula

Four-Point Limits of Fishnet Integrals

In 4D: Basso–Dixon (BD) found determinant representation for fishnet integrals in four-point coincidence limit of $M \times N$ fishnet [Basso Dixon 2017]



In 2D: generalization of [Derkachov, Kazakov Olivucci 2018] agrees with above structure

$$\phi_{MN} \simeq \det_{1\leq j,k\leq M}\left[(z\partial_z)^{j-1}(\bar{z}\partial_{\bar{z}})^{k-1}\partial_\varepsilon^{M+N-1}\Big|_{M+N+1}F_{M+N}(\varepsilon,z)\Big|^2\right]_{\varepsilon=0}$$

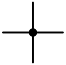
Yangian induces dimension-shift relations on BD integrals [Corcoran, FL Miczajka 2021]

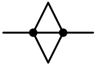
$$D_{uv}\,\phi_{MN}^{D} \simeq \phi_{MN}^{D+2}(a_j \to a_j+1).$$

$\Rightarrow$ get Picard–Fuchs ideal in coincidence limit by other means

Four-Point Graphs in 2D

Compact 1-/2-loop results: [Derkachov, Kazakov Olivucci 2018] [Corcoran, FL Miczajka 2021] [Duhr, Klemm, FL Nega, Porkert 2022]

Elliptic:  $= \frac{4}{\pi}(K(z)K(1-\bar{z}) + K(1-z)K(\bar{z})) = \vec{\Pi}^\dagger \cdot \Sigma \cdot \vec{\Pi}$

K3:  $= \frac{8}{\pi^2}\left(K_+\overline{K}_- + K_-\overline{K}_+\right)^2 = \vec{\Pi}^\dagger \cdot \Sigma \cdot \vec{\Pi},$

with $K_\pm = K\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1-z}\right)$.

Note:

- 2D $M \times N$ Basso–Dixon integrals depend on single variable $z \in \mathbb{C}$
- associated Picard–Fuchs ideal is generated by single differential operator $L_{M,N}$ that annihilates CY periods
- obtain $L_{M,N}$ from holomorphic series from contour integration

Picard–Fuchs Operators in 1 Variable

Notions for differential operator L: [Duhr, Klemm, FL / Nega, Porkert, to appear]

- p^{th} symmetric power:

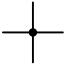
$$\mathrm{Sym}^p L := L_{\mathrm{Sym}^p(\mathrm{Sol}(L))}$$

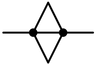
with $\mathrm{Sol}(L) =$ invariants of L with Frobenius basis $\{y_i\}$

- p^{th} exterior power: operator of minimal degree that annihilates determinants of the form

$$\begin{vmatrix} y_{i_1} & \cdots & y_{i_p} \\ \theta_z y_{i_1} & & \theta_z y_{i_p} \\ \vdots & \ddots & \vdots \\ \theta_z^{p-1} y_{i_1} & \cdots & \theta_z^{p-1} y_{i_p} \end{vmatrix}, \qquad \text{with } \theta_z = z\partial_z.$$

- differential operators obey relations, e.g. $L_2 = \mathrm{Sym}^2(L_1)$ [Doran 2000][M.Bogner 2013], which are inherited by their invariants, e.g.

$\phi_1 =$  $= \frac{4}{\pi}(K(z)K(1-\bar{z}) + K(1-z)K(\bar{z}))$

$\phi_2 =$  $= \frac{8}{\pi^2}\left(K_+\overline{K}_- + K_-\overline{K}_+\right)^2$

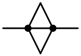
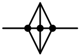
Ladders from Hadamar Product

Two holomorphic functions $f(z) = \sum_{i=0}^{\infty} f_i z^i \quad \text{and} \quad g(z) = \sum_{i=0}^{\infty} g_i z^i$ have Hadamard product

$$(f * g)(z) = \sum_{i=0}^{\infty} f_i g_i z^i \, .$$

Similarly for differential operators:

$$L_f * L_g := L_{f*g} \, .$$

Ladder Integrals:  ,  ,  , . . . [Duhr, Klemm, FL Nega, Porkert, tbp]

Picard-Fuchs operator: $L_N = L_0^{*(N+1)}, \qquad L_N = \theta_z^{N+1} - z\left(\theta_z + \frac{1}{2}\right)^{N+1}$

Holomorphic period: $\Pi_N = \Pi_0^{*(N+1)} = \sum_{i=0}^{\infty} \binom{-\frac{1}{2}}{i}^{N+1} z^i$

Basso–Dixon from Calabi–Yau

$W = M + N - 1$

[Duhr, Klemm, FL
Nega, Porkert, to appear]

$$\det \left[\theta_{\bar{z}}^{i-1}\theta_z^{j-1}\phi_W(z)\right]_{1\le i,j\le M}$$

$$\simeq \frac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar{z}}^{a_1-1}\theta_z^{b_1-1}\phi_W(z)\right]\cdots\left[\theta_{\bar{z}}^{a_M-1}\theta_z^{b_M-1}\phi_W(z)\right]$$

Basso–Dixon from Calabi–Yau

[Duhr, Klemm, FL
Nega, Porkert, to appear]

$W = M + N - 1$

$$\det\left[\theta_{\bar z}^{i-1}\theta_z^{j-1}\phi_W(z)\right]_{1\le i,j\le M}$$

$$\simeq \frac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\theta_z^{b_1-1}\phi_W(z)\right]\cdots\left[\theta_{\bar z}^{a_M-1}\theta_z^{b_M-1}\phi_W(z)\right]$$

$$\simeq \frac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\Pi_{W,i_1}(\bar z)\left(\Sigma_W\right)_{i_1j_1}\theta_z^{b_1-1}\Pi_{W,j_1}(z)\right]\times\ldots$$

$$\times\left[\theta_{\bar z}^{a_M-1}\Pi_{W,i_M}(\bar z)\left(\Sigma_W\right)_{i_Mj_M}\theta_z^{b_M-1}\Pi_{W,j_M}(z)\right]$$

Basso–Dixon from Calabi–Yau

$W = M + N - 1$

[Duhr, Klemm, FL Nega, Porkert, to appear]

$$\det\left[\theta_{\bar z}^{i-1}\theta_z^{j-1}\phi_W(z)\right]_{1\le i,j\le M}$$

$$\simeq \frac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\theta_z^{b_1-1}\phi_W(z)\right]\cdots\left[\theta_{\bar z}^{a_M-1}\theta_z^{b_M-1}\phi_W(z)\right]$$

$$\simeq \frac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\Pi_{W,i_1}(\bar z)\left(\Sigma_W\right)_{i_1 j_1}\theta_z^{b_1-1}\Pi_{W,j_1}(z)\right]\times\ldots$$

$$\times\left[\theta_{\bar z}^{a_M-1}\Pi_{W,i_M}(\bar z)\left(\Sigma_W\right)_{i_M j_M}\theta_z^{b_M-1}\Pi_{W,j_M}(z)\right]$$

Basso–Dixon from Calabi–Yau

[Duhr, Klemm, FL
Nega, Porkert, to appear]

$W = M+N-1$

$$\det\left[\theta_{\bar z}^{i-1}\theta_z^{j-1}\phi_W(z)\right]_{1\le i,j\le M}$$

$$\simeq \tfrac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\theta_z^{b_1-1}\phi_W(z)\right]\cdots\left[\theta_{\bar z}^{a_M-1}\theta_z^{b_M-1}\phi_W(z)\right]$$

$$\simeq \tfrac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\Pi_{W,i_1}(\bar z)\left(\Sigma_W\right)_{i_1j_1}\theta_z^{b_1-1}\Pi_{W,j_1}(z)\right]\times\ldots$$

$$\times\left[\theta_{\bar z}^{a_M-1}\Pi_{W,i_M}(\bar z)\left(\Sigma_W\right)_{i_Mj_M}\theta_z^{b_M-1}\Pi_{W,j_M}(z)\right]$$

$$\simeq D_I^{(W)}(\bar z)\left[\tfrac{1}{M!}\varepsilon_{i_1\cdots i_M}\varepsilon_{j_1\cdots j_M}\left(\Sigma_W\right)_{i_1j_1}\cdots\left(\Sigma_W\right)_{i_Mj_M}\right]D_J^{(W)}(z)$$

with $\begin{array}{c} I=(i_1,\ldots,i_M) \\ J=(j_1,\ldots,j_M)\end{array}$ and $D_I^{(W)}=\begin{vmatrix}\Pi_{W,i_1} & \cdots & \Pi_{W,i_M}\\ \theta_z\Pi_{W,i_1} & & \theta_z\Pi_{W,i_M}\\ \vdots & \ddots & \vdots \\ \theta_z^{M-1}\Pi_{W,i_1} & \cdots & \theta_z^{M-1}\Pi_{W,i_M}\end{vmatrix}$

Basso–Dixon from Calabi–Yau

$W = M+N-1$

[Duhr, Klemm, FL
Nega, Porkert, to appear]

$$
\begin{aligned}
\det\left[\theta_{\bar z}^{i-1}\theta_z^{j-1}\phi_W(z)\right]_{1\le i,j\le M}
&\simeq \tfrac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\theta_z^{b_1-1}\phi_W(z)\right]\cdots\left[\theta_{\bar z}^{a_M-1}\theta_z^{b_M-1}\phi_W(z)\right]\\
&\simeq \tfrac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\Pi_{W,i_1}(\bar z)\left(\Sigma_W\right)_{i_1j_1}\theta_z^{b_1-1}\Pi_{W,j_1}(z)\right]\times\ldots\\
&\qquad\times\left[\theta_{\bar z}^{a_M-1}\Pi_{W,i_M}(\bar z)\left(\Sigma_W\right)_{i_Mj_M}\theta_z^{b_M-1}\Pi_{W,j_M}(z)\right]\\
&\simeq D_I^{(W)}(\bar z)\left[\tfrac{1}{M!}\varepsilon_{i_1\cdots i_M}\varepsilon_{j_1\cdots j_M}\left(\Sigma_W\right)_{i_1j_1}\cdots\left(\Sigma_W\right)_{i_Mj_M}\right]D_J^{(W)}(z)
\end{aligned}
$$

with $\begin{matrix} I = (i_1,\ldots,i_M) \\ J=(j_1,\ldots,j_M)\end{matrix}$ and $D_I^{(W)} = \begin{vmatrix} \Pi_{W,i_1} & \cdots & \Pi_{W,i_M} \\ \theta_z\Pi_{W,i_1} & & \theta_z\Pi_{W,i_M} \\ \vdots & \ddots & \vdots \\ \theta_z^{M-1}\Pi_{W,i_1} & \cdots & \theta_z^{M-1}\Pi_{W,i_M}\end{vmatrix}$

Basso–Dixon from Calabi–Yau

$W = M+N-1$

[Duhr, Klemm, FL
Nega, Porkert, to appear]

$$
\begin{aligned}
\det\left[\theta_{\bar z}^{i-1}\theta_z^{j-1}\phi_W(z)\right]_{1\le i,j\le M}
&\simeq \tfrac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\theta_z^{b_1-1}\phi_W(z)\right]\cdots\left[\theta_{\bar z}^{a_M-1}\theta_z^{b_M-1}\phi_W(z)\right]\\
&\simeq \tfrac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\Pi_{W,i_1}(\bar z)\left(\Sigma_W\right)_{i_1j_1}\theta_z^{b_1-1}\Pi_{W,j_1}(z)\right]\times\ldots\\
&\qquad\times\left[\theta_{\bar z}^{a_M-1}\Pi_{W,i_M}(\bar z)\left(\Sigma_W\right)_{i_Mj_M}\theta_z^{b_M-1}\Pi_{W,j_M}(z)\right]\\
&\simeq D_I^{(W)}(\bar z)\left[\tfrac{1}{M!}\varepsilon_{i_1\cdots i_M}\varepsilon_{j_1\cdots j_M}\left(\Sigma_W\right)_{i_1j_1}\cdots\left(\Sigma_W\right)_{i_Mj_M}\right]D_J^{(W)}(z)\\
&\simeq D_I^{(W)}(\bar z)\,\det\left[\left(\Sigma_W\right)_{ij}\right]_{i\in I,j\in J}D_J^{(W)}(z)
\end{aligned}
$$

with $I=(i_1,\ldots,i_M)$, $J=(j_1,\ldots,j_M)$ and

$$
D_I^{(W)} = \begin{vmatrix} \Pi_{W,i_1} & \cdots & \Pi_{W,i_M} \\ \theta_z\Pi_{W,i_1} & & \theta_z\Pi_{W,i_M} \\ \vdots & \ddots & \vdots \\ \theta_z^{M-1}\Pi_{W,i_1} & \cdots & \theta_z^{M-1}\Pi_{W,i_M} \end{vmatrix}
$$

Basso–Dixon from Calabi–Yau

[Duhr, Klemm, FL Nega, Porkert, to appear]

$W = M + N - 1$

$$
\begin{aligned}
&\det\left[\theta_{\bar z}^{i-1}\theta_z^{j-1}\phi_W(z)\right]_{1\le i,j\le M}\\
&\simeq \tfrac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\theta_z^{b_1-1}\phi_W(z)\right]\cdots\left[\theta_{\bar z}^{a_M-1}\theta_z^{b_M-1}\phi_W(z)\right]\\
&\simeq \tfrac{1}{M!}\varepsilon_{a_1\cdots a_M}\varepsilon_{b_1\cdots b_M}\left[\theta_{\bar z}^{a_1-1}\Pi_{W,i_1}(\bar z)\left(\Sigma_W\right)_{i_1j_1}\theta_z^{b_1-1}\Pi_{W,j_1}(z)\right]\times\dots\\
&\qquad\times\left[\theta_{\bar z}^{a_M-1}\Pi_{W,i_M}(\bar z)\left(\Sigma_W\right)_{i_Mj_M}\theta_z^{b_M-1}\Pi_{W,j_M}(z)\right]\\
&\simeq D_I^{(W)}(\bar z)\left[\tfrac{1}{M!}\varepsilon_{i_1\cdots i_M}\varepsilon_{j_1\cdots j_M}\left(\Sigma_W\right)_{i_1j_1}\cdots\left(\Sigma_W\right)_{i_Mj_M}\right]D_J^{(W)}(z)\\
&\simeq D_I^{(W)}(\bar z)\,\det\left[\left(\Sigma_W\right)_{ij}\right]_{i\in I,j\in J}D_J^{(W)}(z)\\
&\simeq D_I^{(W)}(\bar z)\left(\Sigma_{M,N}\right)_{IJ}D_J^{(W)}(z)\\
&\simeq \phi_{M,N}(z)
\end{aligned}
$$

with $\begin{matrix} I = (i_1,\dots,i_M) \\ J = (j_1,\dots,j_M)\end{matrix}$ and $D_I^{(W)} = \begin{vmatrix} \Pi_{W,i_1} & \cdots & \Pi_{W,i_M} \\ \theta_z\Pi_{W,i_1} & & \theta_z\Pi_{W,i_M} \\ \vdots & \ddots & \vdots \\ \theta_z^{M-1}\Pi_{W,i_1} & \cdots & \theta_z^{M-1}\Pi_{W,i_M}\end{vmatrix}$

Last equality if BD periods are $D_I^{(W)} \Leftrightarrow L_{M,N} = \wedge^M L_W \Leftarrow$ [Derkachov, Kazakov Olivucci 2018]

Volume Interpretation

Mirror Symmetry

Calabi-Yau comes with natural partner related by mirror symmetry

$$M_G \overset{\text{m.s.}}{\longleftrightarrow} W_G$$

Mirror symmetry exchanges complex and Kähler structure: $\Omega_G \leftrightarrow \omega_G$

Hence, Ω_G on M_G provides Kähler structure ω_G on W_G which yields the classical volume of W_G:

$$\begin{aligned}\mathrm{Vol}_{\mathrm{cl}}(W_G) &= \int_{W_G} \frac{\omega_G^\ell}{\ell!} \\ &= \frac{1}{\ell!} \sum_{i_1,\cdots,i_\ell} C^{\mathrm{cl}}_{i_1,\cdots,i_\ell}\, t^{\mathbb{R}}_{i_1}(z) \cdots t^{\mathbb{R}}_{i_\ell}(z)\end{aligned}$$

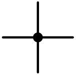
Here

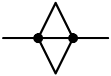
- $t_j^{\mathbb{R}} = \mathrm{Im}(t_j)$ from mirror map: $t_j(z) = \frac{\Pi_j(z)}{\Pi_0(z)} \simeq \log(z_j) + \mathcal{O}(z^2)$
- $C^{\mathrm{cl}}_{i_1,\cdots,i_\ell}$ are explicitly computable intersection numbers of M_G.

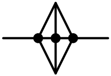
Fishnets and Classical Volumes

In 2D:

[Duhr, Klemm, FL Nega, Porkert 2022]

1 loop:  $\simeq |K|^2 \, \mathsf{Vol}_{\mathsf{cl}}(W_{G^1_{1,1}})$

2 loops:  $\simeq |K_-|^4 \, \mathsf{Vol}_{\mathsf{cl}}(W_{G^1_{1,2}})$

3 loops:  $\neq |\Pi_0|^2 \, \mathsf{Vol}_{\mathsf{cl}}(W_{G^1_{1,3}})$,

Quantum Volume

$\ell \geq 3$ volume gets instanton corrections:

$$\phi(z) = \Pi^\dagger \cdot \Sigma \cdot \Pi = |\Pi_0|^2 \mathsf{Vol}_\mathsf{q}(W_G)$$
$$= |\Pi_0|^2 \mathsf{Vol}_\mathsf{cl}(W_G) + \mathcal{O}(e^{-t_j}(z))$$

cf. [Greene Kanter 1996] [Lee,Lerche Weigand 2019]

No instanton corrections for $\ell = 1, 2$

Relation to Geometry:

Fishnet integrals compute quantum volumes of Calabi–Yau ℓ-folds!

Basso-Dixon Formula for Quantum Volume

Understand solutions of CY-operators as iterated integrals [Duhr, Klemm Nega, Tancredi 2022]

$$I(f_1,\dots,f_k;q) := \int_0^q \frac{\mathrm{d}q'}{q'}\, f_1(q')\, I(f_2,\dots,f_k;q)\,, \qquad I(;q) = 1\,.$$

Here role of f's taken by Y-invariants or structure series of the CY, which allow to write differential operator in canonical form.

Leads to BD formula for CY quantum volume: [Duhr, Klemm, FL Nega, Porkert, to appear]

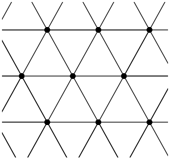
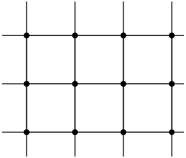
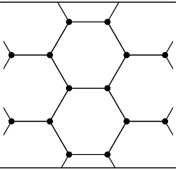
$$\boxed{\mathrm{Vol_q}(\mathcal{W}_{M,N}) = \det\left[\bar{\vartheta}^{i-1}\vartheta^{j-1}\mathrm{Vol_q}(\mathcal{W}_{1,M+N-1})\right]_{0\le i,j<M}}$$

The derivation ϑ clips off letters from the left:

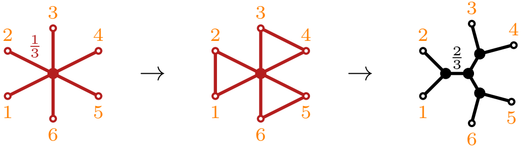
$$\vartheta I(f_1,\dots,f_k;q) := I(f_2,\dots,f_k;q) \quad \text{and} \quad \vartheta(1) = 0\,.$$

Beyond Square Fishnets

Isotropic Fishnets in 2D

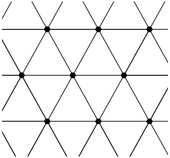
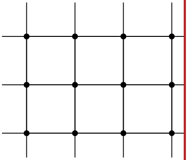
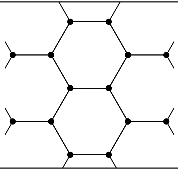
Propagator	$\lvert x_{ij}\rvert^{-2\frac{1}{3}}$	$\lvert x_{ij}\rvert^{-2\frac{1}{2}}$	$\lvert x_{ij}\rvert^{-2\frac{2}{3}}$
Isotropic Fishnet			

Conformal: $\sum_{j\in\text{vertex}} a_j = D$

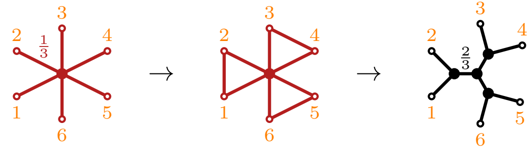
Star-triangle identity: 

The graph associated to an integral is not unique.

Isotropic Fishnets in 2D

Propagator	$\lvert x_{ij}\rvert^{-2\frac{1}{3}}$	$\lvert x_{ij}\rvert^{-2\frac{1}{2}}$	$\lvert x_{ij}\rvert^{-2\frac{2}{3}}$
Isotropic Fishnet			

Conformal: $\sum_{j\in\text{vertex}} a_j = D$

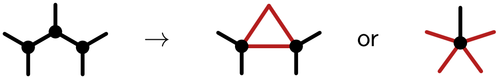
Star-triangle identity: 

The graph associated to an integral is not unique.

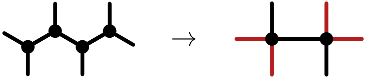
Symmetries and Graph Representation

Consider examples with propagator powers $\frac{2}{3}$ and $\frac{1}{3}$

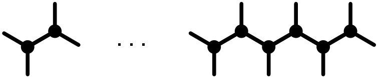
Example 1: Three-loop zigzag



Example 2: Four-loop zigzag



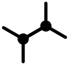
- Preferred graph representation from symmetry perspective.
- Yangian gives Picard–Fuchs ideal at least up to 6 loops [Duhr, Klemm, FL Nega, Porkert, tbp]



Geometry and Graph Representation

Example: Two-loop zig-zag graph in 2D: $x_{jk}^2 = (w_j - w_k)(\bar{w}_j - \bar{w}_k)$
[Duhr, Klemm, FL Nega, Porkert, to appear]

$$\int \frac{\mathrm{d}^2 x_0 \mathrm{d}^2 x_{\bar{0}}}{x_{10}^{2\frac{2}{3}} x_{20}^{2\frac{2}{3}} x_{0\bar{0}}^{2\frac{2}{3}} x_{3\bar{0}}^{2\frac{2}{3}} x_{4\bar{0}}^{2\frac{2}{3}}} =$$



$$\simeq$$

$$= \int \frac{\mathrm{d}^2 x_0}{x_{10}^{2\frac{2}{3}} x_{20}^{2\frac{1}{3}} x_{30}^{2\frac{2}{3}} x_{40}^{2\frac{1}{3}}}$$

LHS: Natural geometry is (singular) K3

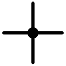
$$y^3 = (w_1 - w_0)(w_2 - w_0)(w_0 - w_{\bar{0}})(w_3 - w_{\bar{0}})(w_4 - w_{\bar{0}})$$

RHS: Natural geometry is Picard-curve (genus 2):

$$y^3 = (w_1 - w_0)^2 (w_2 - w_0)(w_3 - w_0)^2 (w_4 - w_0) \quad \overset{\text{conf.}}{\rightarrow} \quad (z - w)(1 - w) w^2$$

Different geometries realize Calabi–Yau motive, which is characterized by Picard–Fuchs ideal and intersection form Σ, cf. [Bönisch, Duhr, Fischbach Klemm, Nega '21]

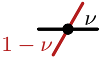
Geometries and 2D Cross Integrals

1) from four-point fishnet:  powers $\frac{1}{2}$ (elliptic curve)

2) from three-point fishnet:  powers $\frac{2}{3}$ and $\frac{1}{3}$ (Picard curve)

Two geometries in one-parameter family (cf. Zamolodchikov's graphs):

$$\phi_\nu(z) = \Pi_\nu^\dagger(z)\Sigma_\nu\Pi_\nu(z)$$



built from Yangian-invariant Legendre functions [Corcoran, FL Miczajka '21]

$$\vec{\Pi}_\nu(z) = (P_{\nu-1}(2z-1), Q_{\nu-1}(2z-1)), \qquad \Sigma_\nu = \begin{pmatrix} -2\pi\cot(\pi\nu) & 2 \\ 2 & 0 \end{pmatrix}.$$

Natural ν-deformation in 2D fishnet theory [Kazakov Olivucci 2018] [Derkachov, Kazakov Olivucci 2018]

Conclusions

Fishnet integrals are rich topic with connections to

- AdS/CFT integrability
- Feynman integrals
- geometry

Many directions to explore:

- more loops, legs, masses, dimensions
- geometry vs hypergeometry