On loop corrections to integrable 2D sigma model backgrounds

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Abstract

We study regularization scheme dependence of β-function for sigma models with two-dimensional target space. Working within four-loop approximation, we conjecture the scheme in which the β-function retains only two tensor structures up to certain terms containing ζ_3. Using this scheme, we provide explicit solutions to RG flow equation corresponding to Yang-Baxter- and λ-deformed $SU(2)/U(1)$ sigma models, for which these terms disappear.

Motivation

- The β-functions in QFT are known to depend on the renormalization scheme.
- In QFT's with one coupling constant we can make the β-function 2-loop exact (for example, in φ^4 theory).
- In QFT's with two or more couplings it is not known in general, whether and how it is possible to achieve such a simple form.
- It is particularly interesting to study integrable deformations of 2-dimensional sigma models, for example, η-deformed $O(N)$ ones with two couplings.
- We know the β-function for 2-dimensional sigma models up to 4-loop order and how it varies under scheme changes.
- For $D = 2$ target space there are much less different tensor structures and we have a hope to obtain a particularly simple expression for the β-function in some scheme.
- We know some conjectured all-loop metrics in a certain scheme for η- and 2-loop ones for λ-deformed models (Hoare et al.'19), so it could be possible to find a simple expression for higher-loop β-functions.

Sigma models in 2 dimensions

- We study 2-dimensional sigma models

$$S[\boldsymbol{X}] = \frac{1}{4\pi}\int G_{ij}(\boldsymbol{X})\partial_a X^i \partial_a X^j \, d^2\sigma \,. \tag{1}$$

- The metric $G_{ij}(\boldsymbol{X})$ also depends on some parameters treated as coupling constants, which vary with the scale according to RG flow equation

$$\dot{G}_{ij} + \nabla_i V_j + \nabla_j V_i = -\beta_{ij}(G) \,. \tag{2}$$

- The metric β-function $\beta_{ij}(G)$ admits the covariant loop expansion

$$\beta_{ij}(G) = \beta^{(1)}_{ij}(G) + \beta^{(2)}_{ij}(G) + \beta^{(3)}_{ij}(G) + \ldots \,, \tag{3}$$

where L-th loop order β-function coefficient β^{L}_{ij} belongs to the finite dimensional space of tensors with given scaling properties.

- It is convenient to have in mind that the metric is proportional to the inverse of the Planck constant, which implies the following scaling for basic tensors

$$G_{ij} \sim \hbar^{-1} \to G^{ij} \sim \hbar \,,\ \Gamma^k_{ij} \sim \hbar^0 \,,\ \nabla_i \sim \hbar^0 \,,\ R_{ijk}{}^{l} \sim \hbar^0 \,,\ R_{ij} \sim \hbar^0 \,,\ R \sim \hbar \,. \tag{4}$$

β-function for $D = 2$ sigma models

- The β-function is known up to 4 loops (Friedan'80, Graham'87, Foakes et al.'87, Jack et al.'89) in the minimal subtraction scheme.
- The higher loop coefficients $\beta^{(L)}_{ij}$ for $L > 1$ are scheme dependent. They are related by covariant metric redefinitions

$$G_{ij} \to \tilde{G}_{ij} = G_{ij} + \sum_{k=0}^{\infty} G^{(k)}_{ij} \,, \tag{5}$$

where $G^{(k)}_{ij}$ is of the order $\hbar^k$.

- The β-function for the SM with *two-dimensional* target space is significantly simplified

$$\beta^{(1)}_{ij} = \frac{1}{2}RG_{ij} \,, \tag{6}$$

$$\beta^{(2)}_{ij} = \frac{1}{4}R^2 G_{ij} \,, \tag{7}$$

$$\beta^{(3)}_{ij} = \left(\frac{5}{32}R^3 + \frac{1}{16}(\nabla R)^2\right) G_{ij} - \frac{1}{16}\nabla_i R \nabla_j R \,, \tag{8}$$

$$\beta^{(4)}_{ij} = \left(\frac{23}{192}R^4 + \frac{2+\zeta(3)}{32}R^2\nabla^2 R + \frac{41+12\zeta(3)}{192}R(\nabla R)^2 + \frac{1}{192}(\nabla^2 R)^2 + \right. \tag{9}$$

$$\left. + \frac{1}{192}(\nabla_i\nabla_j R)^2\right) G_{ij} - \frac{\zeta(3)}{48}R^2\nabla_i\nabla_j R - \frac{25+8\zeta(3)}{192}R\nabla_i R\nabla_j R - \frac{1}{96}(\nabla^2 R)\nabla_i\nabla_j R \,. \tag{10}$$

- Covariant metric redefinition is determined by several tensor structures at every order of $\hbar$

$$G^{(0)}_{ij} = c_1 R G_{ij} \,, \tag{11}$$

$$G^{(1)}_{ij} = \left(c_2 R^2 + c_3 \nabla^2 R\right) G_{ij} + c_4 \nabla_i \nabla_j R \,, \tag{12}$$

$$G^{(2)}_{ij} = \left(c_5 R^3 + c_6 (\nabla R)^2 + c_7 R \nabla^2 R + c_8 \nabla^2\nabla^2 R\right) G_{ij} + \tag{13}$$

$$+ c_9 \nabla_i R \nabla_j R + c_{10} R \nabla_i \nabla_j R + c_{11} \nabla_i \nabla_j \nabla^2 R \tag{14}$$

and so on.

- 1-loop and 2-loop β-functions $\beta^{(1)}_{ij}$ and $\beta^{(2)}_{ij}$ are scheme-independent.
- Higher order contributions to the β-function depend on the scheme, starting from the 3-loop order

$$\beta^{(3)}_{ij} = \left[\left(\frac{5}{32} + \frac{c_1 - 2c_2}{4}\right) R^3 + \left(\frac{1}{16} - \frac{c_1 - 2c_2}{2} - (c_1^2 + c_3)\right)(\nabla R)^2 - \right.$$
$$\left. -(c_1^2 + c_3) R \nabla^2 R\right] G_{ij} - \frac{1}{16}\nabla_i R \nabla_j R - \frac{c_4}{4}\nabla_i \nabla_j \left(3R^2 + 2\nabla^2 R\right) \,. \tag{15}$$

- Let us choose the covariant redefinition parameters to be

$$c_2 = -\frac{1}{16} + \frac{c_1}{2} \,, \quad c_3 = -\frac{c_1^2}{2} \,. \tag{16}$$

- We found the combination of the scheme change parameters, for which the β-functionup to the 4-loop order is given by

$$\beta_{ij} = \left(\frac{R}{2} + \frac{R^2}{4} + \frac{3R^3}{16} + \frac{5R^4}{32} + \frac{2+\zeta_3}{64}\nabla^2\left(R^3 + 2R\nabla^2 R - \frac{1}{2}\nabla^2 R^2\right) + \right.$$
$$\left. + \ldots\right) G_{ij} - \left(\frac{1}{16} + \frac{5R}{32} + \ldots\right)\nabla_i R \nabla_j R + \ldots \tag{17}$$

- One can notice that parts of this expression without ζ_3 are the expansion of

$$\frac{RG_{ij}}{2(1-R)^{\frac{1}{2}}} - \frac{1}{16(1-R)^{\frac{5}{2}}}\nabla_i R \nabla_j R \,. \tag{18}$$

"All-loop" "sausage" metric

- In (Fateev et al.'93) there was obtained the solution of 1-loop RG flow equation, which was later identified as semiclassical η-deformed $O(3)$ metric (Hoare et al.'14) (also classically integrable (Lukyanov'12)).
- All-loop metric, however, in different scheme, was conjectured in (Hoare et al.'19).
- The metric takes the form

$$ds^2 = \frac{2\kappa}{\hbar}\frac{\left(1 - \frac{\hbar\kappa\cos^2\theta}{1-\kappa^2\sin^2\theta}\right) d\theta^2 + \cos^2\theta \, d\chi^2}{1 - \kappa^2\sin^2\theta} \,, \tag{19}$$

where the new couplings $\hbar$ and κ satisfy the following flow equations

$$\dot{\hbar} = 0, \quad \dot{\kappa} = \frac{\hbar(\kappa^2 - 1)}{2\left((1-\hbar\kappa)(1-\hbar\kappa^{-1})\right)^{\frac{1}{2}}} \,, \tag{20}$$

and vector field has the form

$$V = \frac{\hbar}{\rho}\left\{\frac{\kappa(\kappa^2-1)\sin 2\theta}{4(1-\kappa^2\sin^2\theta)^2}, \frac{\cos^2\theta}{1-\kappa^2\sin^2\theta}\right\}, \quad \rho \overset{\text{def}}{=} \sqrt{(1-\hbar\kappa)(1-\hbar\kappa^{-1})} \,. \tag{21}$$

- We note that differential equation for κ is uniformized by ρ

$$\left(\frac{1-\rho-\hbar}{1+\rho-\hbar}\right)^{1-\hbar}\left(\frac{1+\rho+\hbar}{1-\rho+\hbar}\right)^{1+\hbar} = e^{2\hbar(t-t_0)} \,, \tag{22}$$

which resembles the integral equation from (Fateev'19).

"All-loop" λ model metric

- There exists a solution to the 1-loop RG flow equation without any isometries

$$ds^2 = \frac{2}{\hbar}\frac{\kappa dp^2 + \kappa^{-1}dq^2}{1 - p^2 - q^2} \,, \quad \text{where} \quad \kappa = \frac{1-\lambda}{1+\lambda} \,. \tag{23}$$

- This metric is one-loop renormalizable with κ running according to the leading in $\hbar$ order of and the vector field given by

$$V_p = \frac{p}{1-p^2-q^2} \,, \quad V_q = \frac{q}{1-p^2-q^2} \,. \tag{24}$$

- We propose an $\hbar$ completion which is also two-loop exact similar to the all-loop "sausage" action

$$ds^2 = \frac{2}{\hbar}\left(\frac{(\kappa-\hbar)dp^2 + (\kappa^{-1}-\hbar)dq^2}{1-p^2-q^2} - \hbar\frac{(pdp+qdq)^2}{(1-p^2-q^2)^2}\right) \,. \tag{25}$$

supplemented by the following vector field

$$V_p = \frac{p\left(\frac{1-\hbar\kappa}{1-\hbar\kappa^{-1}}\right)^{\frac{1}{2}}}{1-p^2-q^2}\left(1 - \frac{\hbar}{2\kappa}\frac{1 - \left(\frac{1-\kappa^2}{1-\hbar\kappa}\right)q}{1-p^2-q^2}\right) \,, \quad V_q = \{p \leftrightarrow q, \kappa \to \kappa^{-1}\} \,. \tag{26}$$

- Surprisingly, the parameter κ satisfies the same RG flow differential equation as for the η-deformed model.

Conclusions and open problems

- We found the renormalization scheme, in which the expression for the 4-loop β-function for $D = 2$ sigma models is particularly simple.
- It was shown to be connected to the β-function in the minimal subtraction scheme in the first 4 loop orders by some covariant metric redefinition.
- We found the 4-loop solution to RG flow equation, corresponding to the η- and λ-deformed $O(3)$ sigma model, which was also shown to be consistent with the screening charges defining this theory.
- The renormalization scheme in question possesses an interesting property that the screenings do not receive counterterm corrections, which requires further investigation.
- Found the "cigar" metric with one exponent solves the RG flow with some certain dilaton field.
- Generalize the obtained result for higher dimensional sigma model target spaces.