Full analytic transseries of IQFTs

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Chemical potential [1]

- 1+1 dim. $O(N)$ NLSM

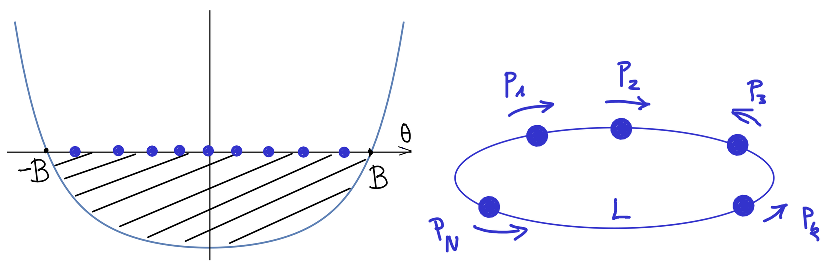
$$\mathcal{L}=\frac{1}{2g_0^2}\sum_{\alpha=1}^{N}\left(\partial_\mu\phi_\alpha(x)\right)^2,\qquad \sum_{\alpha=1}^{N}\phi_\alpha^2=1$$

- chemical potential h for **conserved charge** Q_3:

$$H=H_0-hQ_3,\quad Q_3=\int \mathrm{d}x\,(\phi_1\partial_t\phi_2-\phi_2\partial_t\phi_1)\,.$$

- $h>m$: **particles** charged under Q_3 **condense**
- knowing their S-matrix, **linear TBA** for rapidity distribution $\chi(\theta)$ in the condensate:

$$\chi(\theta)-\int_{-B}^{B}\mathrm{d}\theta' K(\theta-\theta')\chi(\theta')=m\cosh(\theta)-h,\quad |\theta|<B$$



- at Fermi surface $\chi(\theta=\pm B)=0\ \Rightarrow$ gives relation $\frac{h}{m}\Leftrightarrow B$.
- free energy density's change

$$\delta f(h)=f(h)-f(0)=\int_{-B}^{B}\frac{\mathrm{d}\theta}{2\pi}m\cosh(\theta)\chi(\theta).$$

- h **introduces an energy scale** - for large h we use standard PT and comparison gives the **mass-gap relation**
- other models: **SUSY NLSM, PCF, Gross-Neveu, chiral GN**, etc.

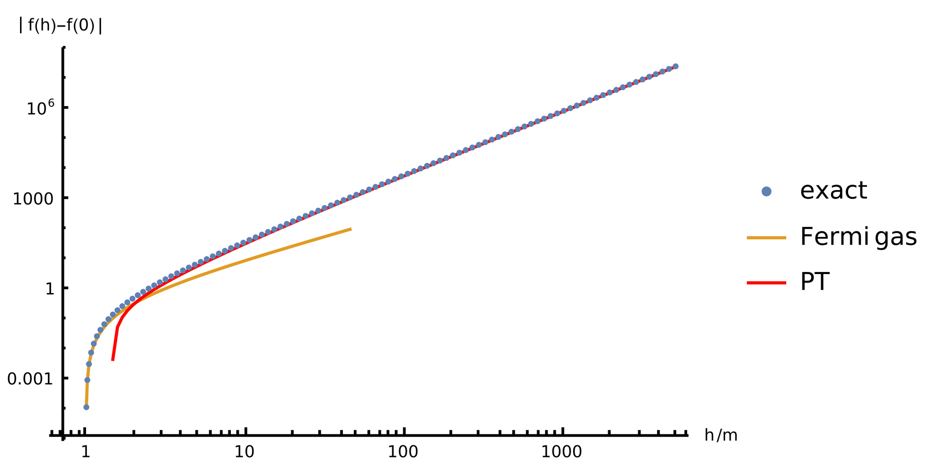


Figure 1. Free energy density change of the $O(3)$ NLSM. [1]

Non-perturbative physics and resurgence [2]

- large B expansion of $\delta f(h)$ is an **asymptotic series** [3]
- Reconstructing exact value from PT expansion?
- causes of **factorially growing** coefficients:
  - **instantons** - $n!$ number of diagrams
  - **renormalons** - contribution of specific graphs [4]
- after **Borel-transformation** it is a convergent series:

$$\varphi(x)\sim\sum_{n=0}^{\infty}a_n x^{n+1}\quad\Rightarrow\quad \mathcal{B}\varphi(s)=\sum_{n=0}^{\infty}\frac{a_n}{n!}s^n$$

- Borel-resummable case $\Rightarrow$ apply Laplace transformation, e.g.:

$$a_n\sim(-1)^n n!\ \Rightarrow\ \mathcal{B}\varphi(s)=\frac{1}{s+1}\ \Rightarrow\ \mathcal{S}\varphi(x)=\int_0^\infty \mathrm{d}s\,e^{-s/x}\,\mathcal{B}\varphi(s)$$

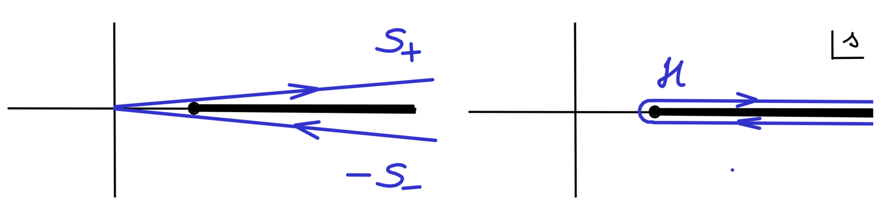
- Non Borel-resummable case $\Rightarrow$ singularities along real line

$$a_n\sim n!\Bigg(A_0+\overbrace{\frac{A_1}{n}+\frac{A_2}{n(n-1)}+\frac{A_3}{n(n-1)(n-2)}+\dots}^{1/n\text{ corrections in asymptotics}}\Bigg)$$

$$\Rightarrow\ \mathcal{B}\varphi(s)=\text{reg.}(s)-\frac{A_0}{s-1}-\ln(1-s)\sum_{j=0}^{\infty}\frac{A_{j+1}}{j!}(s-1)^j$$

- **resurgence**: A_k-s are also expansion coefficients around $s=1$!
- introduce **lateral Borel-resummation** $\Rightarrow$ imaginary part:

$$\mathcal{S}_\pm\varphi(x)=\int_0^{e^{\pm i0}\infty}\mathrm{d}s\,e^{-s/x}\mathcal{B}\varphi(s),$$



- the **ambiguity** (Hankel-contour above) contains the Δ_1 **alien derivative** of $\varphi(x)$ at $x=1$:

$$(\mathcal{S}_+-\mathcal{S}_-)\varphi(x)=2\pi i\underbrace{e^{-1/x}}_{\text{NP!}}\mathcal{S}_-\left(\Delta_1\varphi(x)\right),\quad \Delta_1\varphi(x)=\sum_{j=0}^{\infty}A_j x^j$$

- **Stokes-automorphism** relates lateral resums:

$$\mathfrak{S}=\exp\left(\sum_{k=1}^{\infty}e^{-k/x}\Delta_k\right),\qquad S_-(\varphi)=S_+(\mathfrak{S}^{-1}\varphi)$$

- **median resummation**: its square-root generates a **trans-series**, after lateral resummation it is **real**

$$\mathfrak{S}^{1/2}\varphi=\varphi+\frac{1}{2}e^{-1/x}\Delta_1\varphi+\frac{1}{2}e^{-2/x}\left(\frac{1}{4}\Delta_1^2\varphi+\Delta_2\varphi\right)+\dots$$

- **strong resurgence**: median resummation gives the physical answer

$$\varphi_{\text{phys}}=\mathcal{S}_{\text{med}}(\varphi)=S_+(\mathfrak{S}^{1/2}\varphi)$$

Wiener-Hopf method

- finite domain convolution $\Rightarrow$ **extend** with unknown $X(\theta\le 0)=0$

$$\chi(\theta)-\int_{-\infty}^{\infty}\mathrm{d}\theta' K(\theta-\theta')\chi(\theta')=r(\theta)+X(\theta-B)+X(-B-\theta)$$

- **generic source**:

$$r(\theta)=\cosh(n\theta),\quad |\theta|<B$$

- **Fourier-space**:

$$\left(1-\tilde K(\omega)\right)\tilde\chi(\omega)=\tilde r(\omega)+e^{iB\omega}\tilde X(\omega)+e^{-iB\omega}\tilde X(-\omega)$$

- decomposing inverse to upper-/lower-half plane analytic $G_\pm(\omega)$

$$\frac{1}{1-\tilde K(\omega)}=G_+(\omega)G_-(\omega)$$

- inverting $G_+(\omega)$ only, then projecting the equation
  - "+" part $\Rightarrow$ integral equation for X
  - "−" part $\Rightarrow$ solution as moment of X

$$X_n(im)-\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi i}\frac{e^{2Bi\omega}\sigma(\omega)X_n(\omega)}{\omega+im}=\frac{1}{n-m}\qquad(1)$$

$$\chi_{n,+}(im)-\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi i}\frac{e^{2Bi\omega}\sigma(\omega)X_n(\omega)}{\omega-im}=\frac{1}{n+m}\qquad(2)$$

- $\chi_{n,+}(\omega)$ analytic for upper half plane, thus $m>0$
- $X_n(\omega)$ has single pole at $\omega=in$
- **Wiener-Hopf kernel** - has **cuts/poles** along imaginary line

$$\sigma(\omega)=\frac{G_-(\omega)}{G_+(\omega)},\quad G_+(\omega)=G_-(-\omega)$$

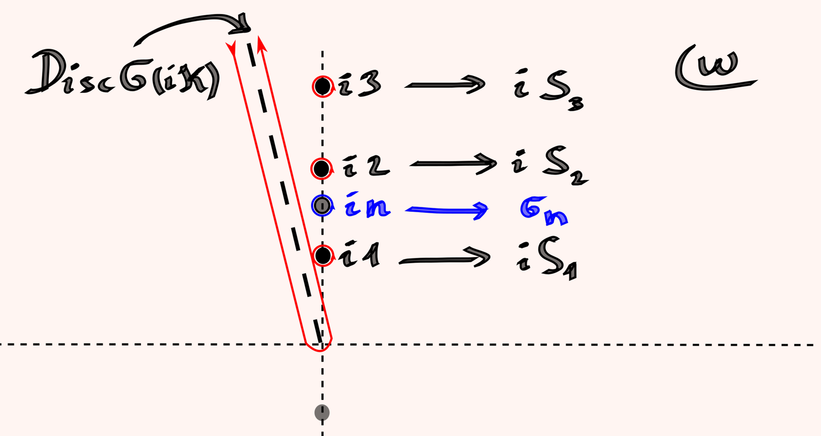
- $\chi_{n,+}(im)$ encodes **generic moments**, is $n\leftrightarrow m$ **symmetric**:

$$\int_{-B}^{B}\frac{\mathrm{d}\theta}{2\pi}\chi_n(\theta)\cosh(m\theta)=\frac{e^{(n+m)B}}{4\pi}G_+(in)G_+(im)\,\chi_{n,+}(im)$$

Source of non-perturbative terms and the transseries Ansatz [5]

Contour deformation

- $B>0$: contour in (1),(2) closed from above



- shrink to **Hankel-contour** around cut of $\sigma(\omega)$
- sources of e^{-2B} NP corrections: poles along the cut in $\sigma(\omega)$ [6]
- rotating Hankel-contour and decomposing into **discontinuity** and pole contributions
- discontinuity is **integrated laterally**
- **residues**: residue of $X_n(\omega)$ at $\omega=in$ is unity + sum over residues of $\sigma(\omega)$ along the cut at $\omega=ik$-s

$$X_n(im)-\int_0^{e^{i0}\infty}\frac{\mathrm{d}\kappa}{2\pi}\frac{e^{-2B\kappa}\mathrm{Disc}\sigma(\kappa)X_n(i\kappa)}{\kappa+m}$$

$$+\frac{e^{-2Bn}\sigma_n}{n+m}+\sum_k\frac{e^{-2Bk}iS_kX_n(ik)}{k+m}=\frac{1}{n-m}$$

where

$$S_k=\mathrm{Res}_{\kappa=k}i\sigma(i\kappa+0)\qquad \sigma_n=\sigma(in+0)$$

- NP terms look like PT source term

$$\frac{1}{n-m}\quad\to\quad\frac{1}{(-n)-m},\ \frac{1}{(-k)-m}$$

- PT quantities will be building blocks:

$$X_n(im)=P_n(m)+\text{NP}\quad P_n(m)=A_{n,-m}=\frac{1}{n-m}+\mathcal{O}(1/B)$$

$$\chi_{n,+}(im)=A_{n,m}+\text{NP},\quad A_{n,m}=A_{m,n}=\frac{1}{n+m}+\mathcal{O}(1/B),$$

- Ansatz: linear combination of PT solutions:

$$X_n(im)=P_n(m)+\sigma_n e^{-2Bn}P_{-n}(m)+\sum_k iS_k e^{-2Bk}X_n(ik)P_{-k}(m)$$

- **consistency equation** for $X_n(ik)$, i.e. X_n evaluated at poles
- **recursive solution** in terms of $A_{n,m}$

Solution as a transseries

$$\chi_{n,+}(im)=\mathcal{A}_{n,m}+\sigma_n e^{-2Bn}\mathcal{A}_{-n,m}+\sigma_m e^{-2Bm}\mathcal{A}_{n,-m}+\sigma_n\sigma_m e^{-2B(n+m)}\mathcal{A}_{-n,-m}$$

where $\mathcal{A}_{n,m}$ contains every possible NP "insertion":

$$\mathcal{A}_{n,m}=A_{n,m}+\sum_{N=1}^{\infty}\sum_{k_1,k_2,\dots,k_N}A_{n,-k_1}iS_{k_1}e^{-2Bk_1}A_{-k_1,-k_2}iS_{k_2}e^{-2Bk_2}\dots iS_{k_N}e^{-2Bk_N}A_{-k_N,m}$$

Non-trivial $n\to m$ and $n=0$ or $m=0$ limits!

Resurgence and median resummation [5]

- $\chi_{n,+}(im)$ and thus $\mathcal{A}_{n,m}$ is real $\Rightarrow$ **ambiguity cancellation**:

$$\mathrm{Im}\,\mathcal{S}_+(\mathcal{A}_{n,m})=\frac{1}{2i}(\mathcal{S}_+-\overbrace{\mathcal{S}_-}^{\mathcal{S}_+(\mathfrak{S}^{-1})})\mathcal{A}_{n,m}+\mathrm{Im}\,\mathcal{S}_+(\mathcal{A}_{n,m}-A_{n,m})\overset{!}{=}0$$

- expanding in e^{-2B} orders, if S_k is **real** this leads to:

$$\Delta_k A_{n,m}=2iS_kA_{n,-k}A_{-k,m}$$

- easy to see that **Stokes-automorphism generates the insertions**:

$$\mathfrak{S}^{1/2}A_{n,m}=\mathcal{A}_{n,m}$$

Energy density

- (dimensionless) energy density of condensing particles is the $n,m=1$ moment:

$$\epsilon=\int_{-B}^{B}\mathrm{d}\theta\cosh(\theta)\chi_1(\theta)=\frac{e^{2B}G_+^2(i)}{4\pi}\chi_{1,+}(i).$$

- $\lim_{n\to m}$ differentiates $e^{-2Bn}\sigma_n$ due to pole $1/(n-m)$ in $\mathcal{A}_{n,-m}$

$$\hat\epsilon\equiv\chi_{1,+}(i)=\mathcal{A}_{1,1}+e^{-2B}\left(2\sigma_1\mathcal{A}_{1,-1}^{\text{reg.}}+\sigma_1 2B-\sigma'_1\right)+e^{-4B}\sigma_1^2\mathcal{A}_{-1,-1},$$

where $\sigma'_1=\partial_n\sigma(in+0)|_{n=1}$, and for $\mathcal{A}_{1,-1}^{\text{reg.}}$ we drop the pole part in $\mathcal{A}_{n,-m}$.

Comparison of models

The $O(4)$ NLSM case

- $\sigma_1=0,\ \sigma'_1=2i$, set of poles $k=2,4,6,\dots$

$$\hat\epsilon=\mathcal{A}_{1,1}-2ie^{-2B}$$

- $A_{1,1}$ has asymptotical expansion in small variable $v=1/(2B)$, **thousands of coefficients by Volin**'s method [3]
- **strong resurgence holds** up to simple pole term:

$$\hat\epsilon_{\text{phys}}=\mathcal{S}_{\text{med}}(A_{1,1})-2ie^{-2B}$$

The $O(3)$ NLSM case

- $\sigma_1=1/e,\ \sigma'_1=1/(2e)$

$$\hat\epsilon=\mathcal{A}_{1,1}+e^{-2B}\underbrace{\left(\frac{2}{e}\mathcal{A}_{1,-1}^{\text{reg.}}+\frac{1}{e}2B-\frac{i}{2e}\right)}_{\text{1 inst.}}+\underbrace{\frac{1}{e^2}e^{-4B}\mathcal{A}_{-1,-1}}_{\text{2 inst.}},$$

- the $A_{1,1}$ expansion contains $\ln B$ terms, we introduce **"running coupling"** $v\ \Rightarrow$ power series in v

$$2B=1/v+\ln(v)+\text{const.}$$

- physical value **not reconstructible from PT** sector alone!

$$\hat\epsilon_{\text{phys}}=\mathcal{S}_{\text{med}}(A_{1,1}+1\text{ inst.}+2.\text{ inst})-\frac{i\pi}{2}e^{-2B}$$

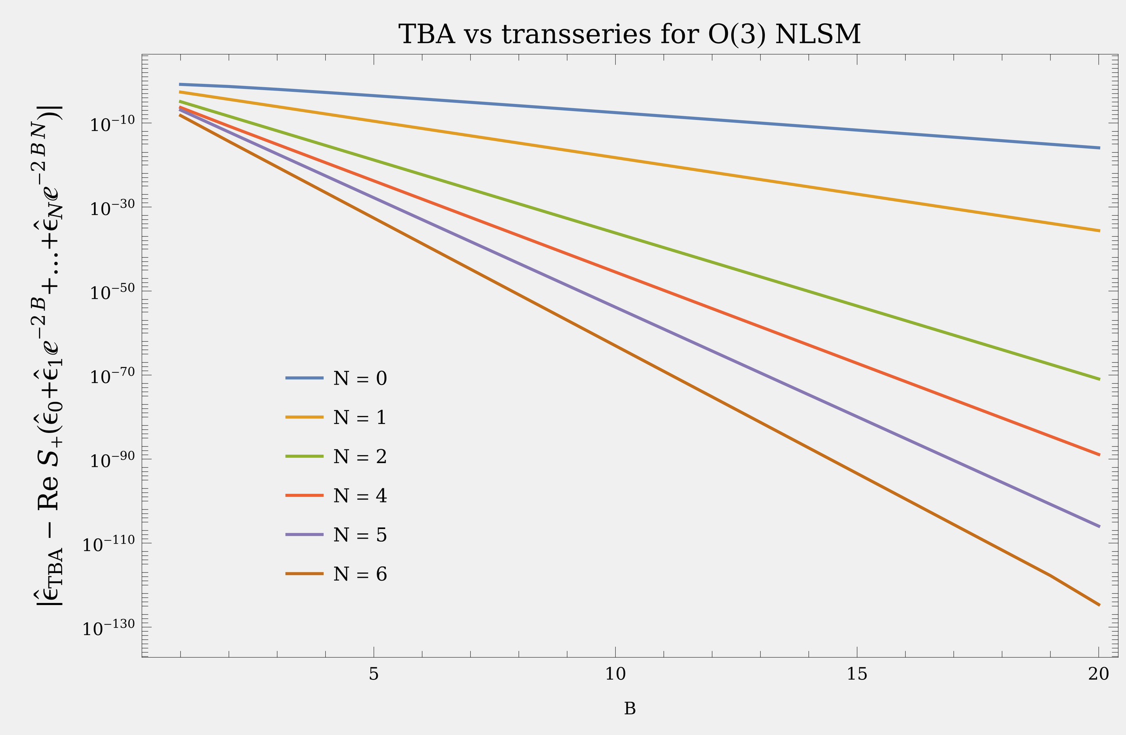


Figure 2. Comparison of numerical TBA solution and lateral Borel-resummation of transseries.

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