QSC Solver for Excited States

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1. Historical

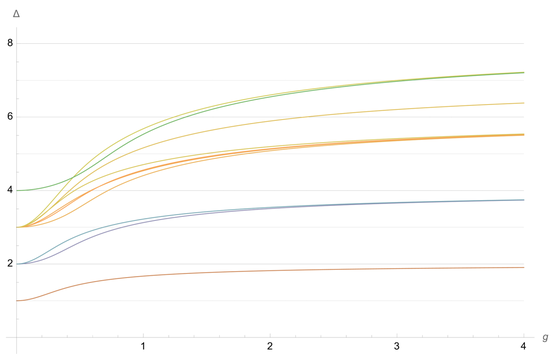
First non-perturbative numerical data: Thermodynamic Bethe Ansatz for the Konishi operator [Gromov, Kazakov, Vieira '08]. Linear convergence rate.

Quantum Spectral Curve [Gromov, Kazakov, Leurent, Volin '14]: the numerical algorithm with quadratic convergence rate [Gromov, Levkovich-Maslyuk, Sizov '15]. Hard to go beyond ground state.

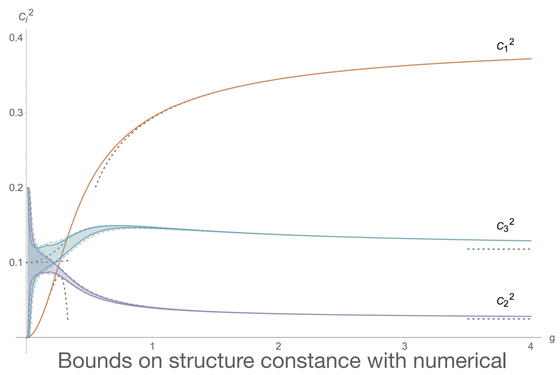
2. Motivation: Bootstrability for $\mathcal{N}$=4 SYM

Idea
Non-perturbative spectral data + conformal bootstrap: tight bounds on structure constants!

Bootstrability program has shown incredible bounds in for the 1d defect in $\mathcal{N}=4$ SYM:



Spectrum from integrability (QSC-based numerics) [Grabner, Gromov, Julius '20] [Julius '21] [Cavaglia, Gromov, Julius, Preti '21]



Bounds on structure constance with numerical conformal bootstrap (SDPB) [Cavaglia, Preti, Gromov, Julius '21 '22]

How do we test the concept for $\mathcal{N}=4$ SYM?

1. The first step is to obtain the spectrum of local operators in the "bootstrability" spirit: all states up to some $\Delta_{\texttt{cutoff}}$, in a wide range of coupling. Then, to use numerical conformal bootstrap.
2. How to tame contribution from double-trace operators? [Caron-Huot, Coronado, Trinh, Zahraee '22] [Alday, Hansen, Silva '22]
3. We can continue the spectrum from weak to strong coupling. At strong coupling, spectrum can be used to extract OPE coefficients with the dispersive sum rules [Alday, Hansen, Silva '22 '23]
4. We can provide building blocks for resumming wrapping corrections [Basso, Georgoudis, Klemenchuk Sueiro '22] in Hexagon approach [Basso, Komatsu, Vieira '15]

3. Objective: Fast QSC Solver

To build a Fast QSC Solver: to reinvent the numerical algorithm so that it allows to compute many states in a reasonable timescale.

1. We wanted an algorithm which goes beyond the ground state, and can initialise any state, given the weak coupling solution.
2. We wanted to initialise many states starting from the weak coupling QSC solver of [Marboe, Volin '17 '18]. We needed the solver to perform well at weak coupling (g ~ 1/1000) and for highly degenerate states which present at higher bare dimensions.
3. We wanted an implementation with high efficiency.

4. Technicalities: Quantum Spectral Curve

The main spirit:

Quantum numbers (including Δ) of the state define the asymptotics of distinguished functions of "spectral parameter" u: $\{\mathbf{P}_a(u), \mathbf{P}^a(u), \mathbf{Q}(u)_i, \mathbf{Q}^i(u)\}$ where $a,i=1,\dots,4$. Distinguished functions are connected by functional relations, and have a specific analytic structure; solving for them allows to extract non-perturbative spectral information.

Main idea for new algorithm

1. $\{\mathbf{P}_a, \mathbf{P}^a\}$ can be parametrised by the Zhukovsky variable $x(u)$ which gives a set of "starting" parameters $\{\Delta_{\texttt{weak coupling}}, c_{a,n}, c^{a,n}\}$.
Use functional relations to obtain $\{\mathbf{Q}_i, \mathbf{Q}^i\}$.

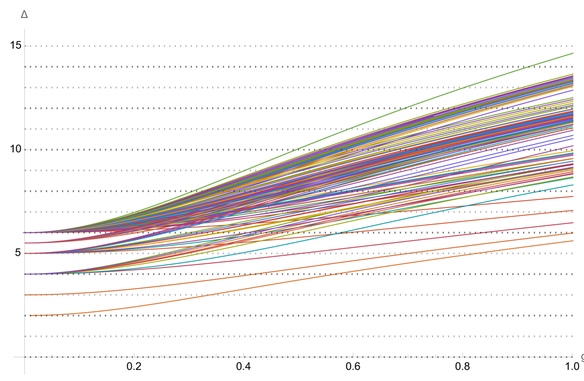
2. $\{\mathbf{Q}_i, \mathbf{Q}^i\}$ have a branch cut between $[-2g, 2g]$. Sample them at a set of problem points $\{u_0\}$, close to the branch cut. Impose a set of "gluing" conditions on $\{\mathbf{Q}_i, \mathbf{Q}^i\}$ and their analytic continuations $\{\tilde{\mathbf{Q}}_i, \tilde{\mathbf{Q}}^i\}$, which ensure correct analytic properties. Use functional relations to compute $\{\mathbf{P}_a, \mathbf{P}^a, \tilde{\mathbf{P}}_a, \tilde{\mathbf{P}}^a\}$ on the probe points $\{u_0\}$. Perform a Fourier transform to get the updated parameters $\{\Delta, c_{a,n}^{\texttt{Updated}}, c^{a,n,\texttt{Updated}}\}$

3. Numerical search: Newton's method to solve
$\vec{F}(\{c\},\Delta)=\left\{c_{a,n}^{\texttt{Updated}}-c_{a,n},\ c^{a,n\ \texttt{Updated}}-c^{a,n}\right\}=0\,.$

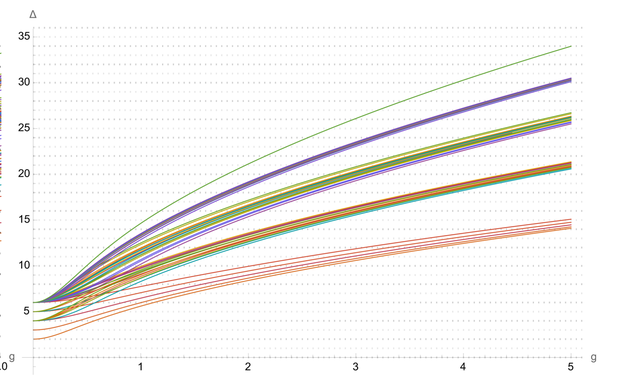
4. Precision control is achieved by seeing how close to zero, certain parameters, which are "gauge fixed" to be zero, are.
Once the desired precision is reached, read off the value of Δ obtained.

5. Results

Numerical spectrum of all states of $\mathcal{N}=4$ SYM with bare dimension ≤ 6



Spectrum of all states with bare dimension ≤ 6



Spectrum of "left-right" and parity symmetric states with bare dimension ≤ 6

The running time of states depends on symmetries of the state, which results in symmetries of $\mathbf{P}_a$ and $\mathbf{P}^a$. We have data up to the following coupling:

"Left-right" and parity symmetric: $g\in[0,5]$; "Left-right" symmetric and general parity: $g\in[0,2]$; General and parity symmetric: $g\in[0,2]$; General: $g\in[0,1]$; Konishi: $g\in[0,13]$

Numerical fits at strong coupling

At strong coupling, we do numerical fits for the spectrum of computed states. The expansion is expected to be of the form

$$\Delta\simeq 2\delta\,\lambda^{1/4}-\Delta_{\texttt{const}}+\frac{d_1}{\sqrt{\delta}\,\lambda^{1/4}}$$

where δ is the string mass level [Gubser, Klebanov, Polyakov '98]. We provide a heuristic argument that $\Delta_{\texttt{const}}=-2$ for all states in planar $\mathcal{N}=4$ SYM and give credence to it by fitting out data. For most of the states we have a prediction for the subleading coefficient d_1, which turns out to be a simple rational number.

Bootstrability for $\mathcal{N}=4$ SYM? Breaking degeneracies at strong coupling

Consider the four-point function of 20' operators: operators with different twists $\tau\equiv\Delta-\ell$ are exchanged in the operator product expansion. Here ℓ is the Lorentz spin label.

$$\langle\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle=\sum_{\tau,\ell}C^2_{\tau,\ell}G_{\tau+4,\ell}(u,v)$$

At strong coupling, under a simple change of variables, $C^2_{\tau,\ell}\to f_{\delta,\ell}$, [Alday, Hansen, Silva '22] derived conformal bootstrap constraints on CFT-data combinations of type $\sum_k f^{(n)}_{\delta,\ell;\,k}$ and $\sum_k f^{(n)}_{\delta,\ell;\,k}\,\tau^{(m)}_{\delta,\ell;\,k}$. The superscript refers to order in the strong coupling expansion. These sums are over over all operators with same δ and ℓ, and need to be "unmixed" in order to extract individual predictions.

For $\delta=2$ and $\ell=0$, there are two such operators, which we can unmix using our spectral data. Therefore, we obtain

$$f^{(0)}_{2,0;\,1}=0\,,\qquad f^{(0)}_{2,0;\,2}=\frac{1}{4}\,,$$

which are the first Bootsrability predictions at strong coupling for $\mathcal{N}=4$ SYM!

6. Open Source

How can the community use the Solver?

We plan to make the code open-source and user-friendly. The Solver will be available on GitHub and will include:

1. C++ core code
2. Auxiliary packages which allow to initialise from perturbative QSC solver of [Marboe, Volin '17 '18].
3. Python script which manages the parameters of the run and does not require babysitting from the user
4. Mathematica notebook with the code prototype
5. Numerical data for all already computed states

so that it is possible for a user to run own states and continue those already computed.

7. Future directions

1. The algorithm can be modified to be many other problems: integrable boundaries, ABJM, AdS_3, building Regge trajectories for non-integer spin, β- and γ-deformations, etc.
2. Construct systematic expansion of QSC at strong coupling.
3. Combine with numerical conformal bootstrap to get bounds on structure constants. Can we dream of tight bounds as of 1d CFT?
4. Combine with analytical conformal bootstrap at strong coupling.
5. Possible insights to fixing of SoV measure.