Inversion relations in integrable QFTs

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IGST 2023

1. Key facts: Inversion relations

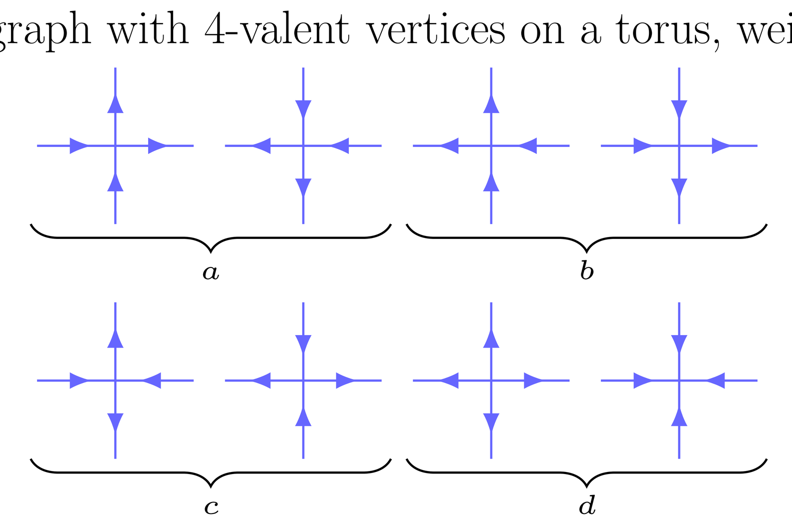
- The method of inversion relations was established for the 8-vertex model [Stroganov'79] and other statistical models [Baxter'82]
- Soon after, applied to fishing-net vacuum graphs, interpreting QFT as an integrable lattice model [Zamolodchikov '80, Bazhanov,Kels,Sergeev'16]
- Bi-scalar fishnet theory which generates these graphs was found [Gürdogan,Kazakov'15] as an integrable limit of $\mathcal{N}=4$ SYM, another such limit is the brick-wall theory [Caetano,Gürdogan,Kazakov'16, Kazakov,Olivucci,Preti'19].
- Integrability of brick-wall model provided by "spinning" star-triangle relations [Chicherin,Derkachov,Isaev,Olivucci '12-'23]
- Inversion relations can be used to calculate free energy in the thermodynamic limit in statistical models and the critical coupling in QFTs. This is the radius of convergence

$$\kappa = \lim_{M,N\to\infty} (Z_{MN})^{\frac{1}{MN}}$$

for the expansion of the free energy
$Z = \sum_{M,N=1}^{\infty} Z_{MN}\,(\xi^2)^{MN}$.

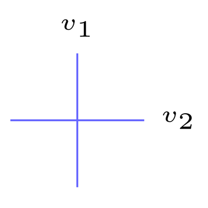
2. 8-vertex model

Calculate its free energy: Configuration corresponds to a rectangular graph with 4-valent vertices on a torus, weighted as

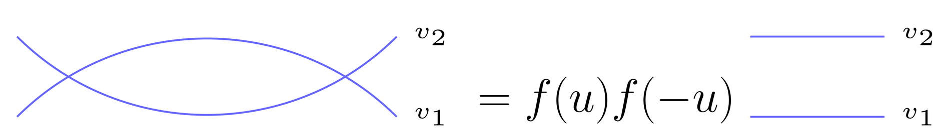


Encode general vertex in R-matrix ($u = v_1 - v_2$)

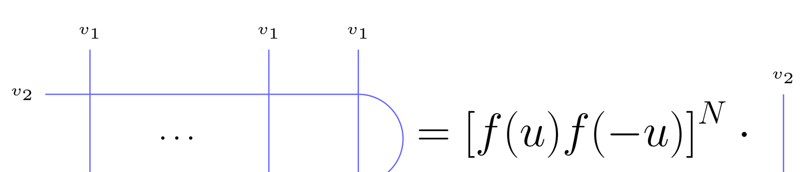
$$R(u,\eta) = \begin{pmatrix} a(u,\eta) & 0 & 0 & d(u,\eta) \\ 0 & b(u,\eta) & c(u,\eta) & 0 \\ 0 & c(u,\eta) & b(u,\eta) & 0 \\ d(u,\eta) & 0 & 0 & a(u,\eta) \end{pmatrix} =$$



satisfying Yang-Baxter eq., crossing and unitarity

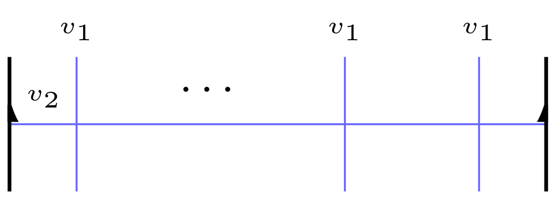


for $f(u) := -\mathrm{i}\vartheta_4(0|q)\vartheta_4(\frac{\mathrm{i}(\eta+u)}{\sqrt{\vartheta_3}}|q)\vartheta_1(\frac{\mathrm{i}(\eta+u)}{\sqrt{\vartheta_3}}|q)$. Using this, one can show that



For a transfer matrix

$T_N(u) :=$



one can derive a inversion relation

$$T_N(u) \circ T_N(-u) = [f(u)f(-u)]^N \cdot \mathbb{1}^{\otimes N}. \tag{1}$$

For a $M \times N$ toroidal lattice, the free energy in the thermodynamic limit is defined as

$$\kappa(u) := \lim_{M,N\to\infty} \mathrm{tr}\left[T_N(u)^M\right]^{\frac{1}{MN}}. \tag{2}$$

Eq. (1) and crossing of the R-matrix implies

$$\kappa(u)\kappa(-u) = f(u)f(-u) \tag{3a}$$
$$\kappa(u) = \kappa(\eta - u) \tag{3b}$$

One finds the solution to be

$$\log\kappa(u) = -\log c(u,\eta) + \log\left[\frac{1}{\Gamma^{(1)}(px|q)\Gamma^{(1)}(p^2x^{-1}|q)}\frac{\Gamma^{(2)}(p^4x^{-1}|q,p^2)\Gamma^{(2)}(p^3x|q,p^2)}{\Gamma^{(2)}(p^2x|q,p^2)\Gamma^{(2)}(p^3x^{-1}|q,p^2)}\right]$$

with q the elliptic nome, $x = e^{-\frac{2u}{\sqrt{\vartheta_3}}}$ and $p = e^{-\frac{4\eta}{\sqrt{\vartheta_3}}}$. The function $\Gamma^{(r)}(z|q_1,\cdots,q_r)$ is the order-r elliptic gamma function [Felder,Varchenko'99].

3. Integrable fishnet QFTs from $\mathcal{N}=4$ SYM theory

- Starting point: γ-deformed $\mathcal{N}=4$ SU(N) SYM theory
  For the sake of breaking supersymmetry, replace all products of two fields $A\cdot B$ in the $\mathcal{N}=4$ SYM action by $\mathrm{e}^{-\frac{\mathrm{i}}{2}\det(\mathbf{q}_A|\mathbf{q}_B|\gamma)}A\cdot B$, where $\mathbf{q}_A$ and $\mathbf{q}_B$ are the $\mathfrak{su}(4)$ R-symmetry weight vectors of A and B, respectively, and $\gamma = (\gamma_1,\gamma_2,\gamma_3)$ are the deformation parameters. Thus, they appear as powers of $q_i := \mathrm{e}^{-\frac{\mathrm{i}}{2}\gamma_i}$ in the Lagrangian.
- Double-scaling limit:
  γ-deformed $\mathcal{N}=4$ SU(N) SYM theory with $\gamma_i \longrightarrow \mathrm{i}\infty$ ($\Rightarrow q_i \to \infty$) while the 't Hooft coupling simultaneously $\lambda \longrightarrow 0$ such that $\xi_1 := q_1\cdot\lambda$, $\xi_2 := q_2\cdot\lambda$, $\xi_3 := q_3\cdot\lambda$ stay fixed at a finite value.
  This yield the so-called dynamical fishnet theory.
- Bi-scalar fishnet theory: Switch off $\xi_1 = 0$, $\xi_2 = 0$, rename $\xi := \xi_3$.

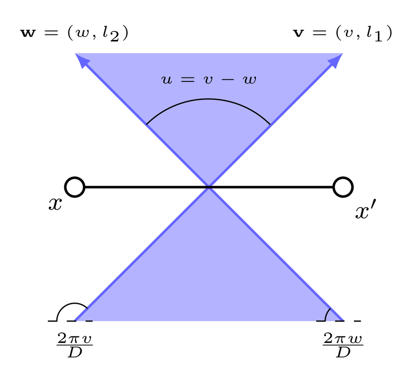
$$\mathcal{L}^{\text{fishnet}} = \frac{\mathrm{N}}{2}\cdot\mathrm{tr}\left[\partial^\mu\phi_1^\dagger\partial_\mu\phi_1 + \partial^\mu\phi_2^\dagger\partial_\mu\phi_2\right] + \mathrm{N}(4\pi)^2\xi^2\cdot\mathrm{tr}\left[\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2\right]$$

- Brick-wall model: Switch off $\xi_1 = 0$.

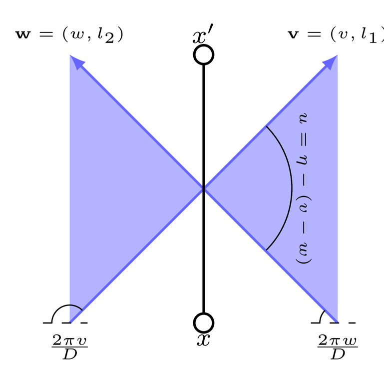
$$\mathcal{L}^{\text{brickwall}}_{\text{int}} = \mathrm{N}\cdot\mathrm{tr}\left[(4\pi)^2\xi_3^2\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2 + (4\pi)^2\xi_2^2\phi_3^\dagger\phi_1^\dagger\phi_3\phi_1 + (4\pi)\mathrm{i}\sqrt{\xi_2\xi_3}\left(\psi_2\phi_1\psi_3 + \bar\psi_2\phi_1^\dagger\bar\psi_3\right)\right]$$

4. Integrable QFTs from a lattice model perspective

Encode the 4-valent vertices of the medial lattice as weights ($\eta = \frac{D}{2}$). A medial line carries a spectral parameter u and a spin-label l. The weights are the deformed propagators of the Feynman diagram

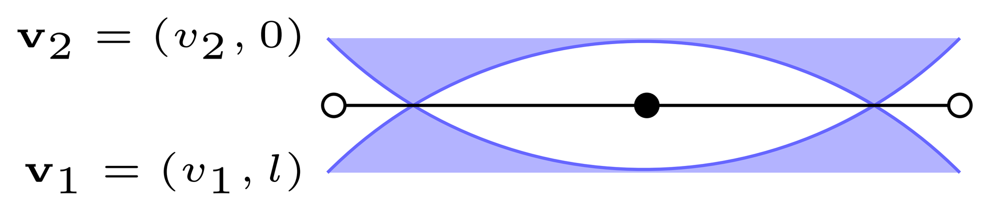


$$= \frac{1}{\left[(x-x')^2\right]^u}\left[\frac{\not{x} - \not{x}'}{(x-x')^{2\cdot\frac{1}{2}}}\right]^{|l_1-l_2|},$$



$$= \frac{1}{\left[(x-x')^2\right]^{\frac{D}{2}-u}}\left[\frac{\not{x} - \not{x}'}{(x-x')^{2\cdot\frac{1}{2}}}\right]^{|l_1-l_2|}$$

and they satisfy a Yang-Baxter equation on the medial lattice (which is the star-triangle relation). Unitarity of these weights is obtained by Feynman bubble integrals



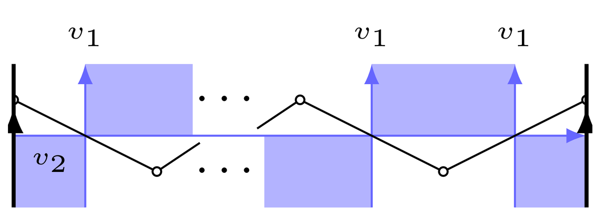
$$= f_l(u)f_l(-u)\cdot\delta^D(x-x')$$

with $f_l(u) := \pi^{\frac{D}{2}}\frac{\Gamma(\frac{D}{2}-u+\frac{l}{2})}{\Gamma(u+\frac{l}{2})}$. Similar to the 8-vertex case, using the unitarity, one can derive inversion relations for the transfer matrices. These are taylored to match the models above.

5. Bi-scalar fishnet theory

To model the fishnet theory, the 4-valent medial lattice consists purely of scalar rapidity lines, i.e. $\forall i : l_i = 0$. At the leading order in N, its vacuum diagrams wrap a torus. One can build a transfer matrix (N even)

$T_N(u) :=$



which should be understood as an integral kernel. Using the unitarity of the propagators gives the inversion relation

$$T_N(u)\circ T_N(-u) = [f_0(u)f_0(-u)]^N\cdot\prod_{i=1}^{N}\delta^D(x_i - x_i').$$

The edge free energy is related to eq. (2) by $\kappa^B(u) =: \kappa_e^B(u)\kappa_e^B(D/2-u)$. Together with crossing this yields

$$\kappa_e^B(u)\kappa_e^B(-u) = 1 \tag{4a}$$
$$\kappa_e^B(D/2-u) = \kappa_e^B(u)f_0(u) \tag{4b}$$

A solution is

$$\kappa_e^B(u) = \pi^u\frac{\Gamma\left(\frac{D}{2}-\frac{D\alpha}{2\pi}\right)}{\Gamma\left(\frac{D}{2}\right)}\cdot\prod_{l=1}^{\infty}\frac{\Gamma\left(Dl+\frac{D}{2}-u\right)\Gamma\left(Dl+u\right)\Gamma\left(Dl-\frac{D}{2}\right)}{\Gamma\left(Dl-\frac{D}{2}+u\right)\Gamma\left(Dl-u\right)\Gamma\left(Dl+\frac{D}{2}\right)}.$$

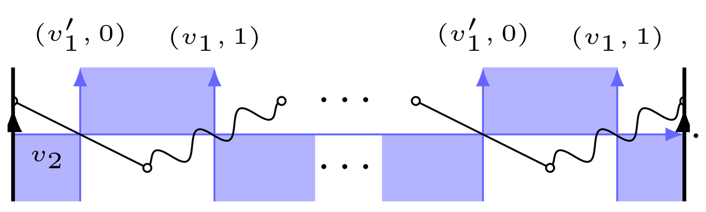
At $u = 1$ and $D = 4$ one recovers scalar propagators and the medial lattice is rectangular. One obtains for the bi-scalar fishnet

$$\kappa_e^B(1) = \frac{1}{4}\sqrt{\pi/2}\,\Gamma(1/4)^2.$$

6. New result: Brick-wall model

The vacuum diagrams can have rectangle, bi-scalar and brick-wall, fermionic regions wrapping a cycle of the torus. The transfer matrix is thus inhomogeneous. However, the free energy factorizes. For the brick-wall free energy one has itself an inhomogeneous, stacked transfer matrix $T_N^2(u) := T_N^+(u)\circ T_N^-(u)$ with

$T_N^+(u) :=$



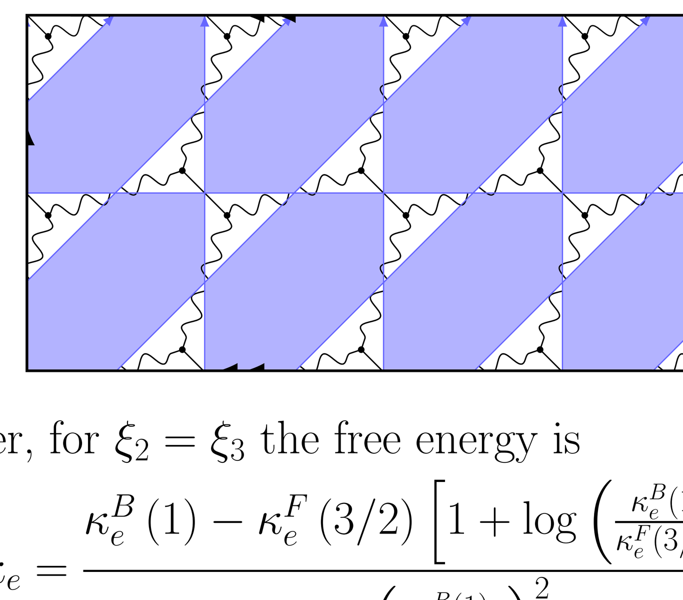
and $T_N^-(u)$ the same with inverted shading. The inversion relation can be obtained by unitarity. Finally, solve

$$\kappa_e^F(u)\kappa_e^F(-u) = 1 \tag{5a}$$
$$\kappa_e^F(D/2-u) = \kappa_e^F(u)f_1(u) \tag{5b}$$

which can be done simultaneously as in the fishnet case. Notably, in $D = 4$ and $u = 3/2$ one finds

$$\kappa_e^F(3/2) = \frac{\pi^2}{2}.$$

This corresponds to a brick-wall graph



Altogether, for $\xi_2 = \xi_3$ the free energy is

$$\kappa_e = \frac{\kappa_e^B(1) - \kappa_e^F(3/2)\left[1+\log\left(\frac{\kappa_e^B(1)}{\kappa_e^F(3/2)}\right)\right]}{\log\left(\frac{\kappa_e^B(1)}{\kappa_e^F(3/2)}\right)^2}.$$