Spectral Curves from Inverse Scattering

Kunal Gupta [1,2] Niklas Beisert [2]

[1]Uppsala University [2]ETH Zurich

Abstract

Solutions of a classical integrable field theory with periodic boundary conditions are described by spectral curves, while those with infinite length asymptotically decaying boundary conditions are described by the classical inverse scattering method. Although the latter solutions can be obtained by decompactifying the periodic solutions, spectral curves and the inverse scattering method differ significantly. In this work we show how relevant structures in the inverse scattering method can be obtained in the decompactification limit of spectral curves. We will do this explicitly for the KdV equation and then motivate generalisation to other integrable field theories.

Inverse Scattering

The inverse scattering problem for an infinite length KdV solution $h(x,t)$ with periodic boundary conditions is a unitary scattering due to a potential problem, where the wavefunctions are known but the potential is not. Asymptotics of a generic right Jost solution (ie. $k \in \bar{\mathbb{C}}^{+}$) to this auxiliary linear problem are given in Fig 1.

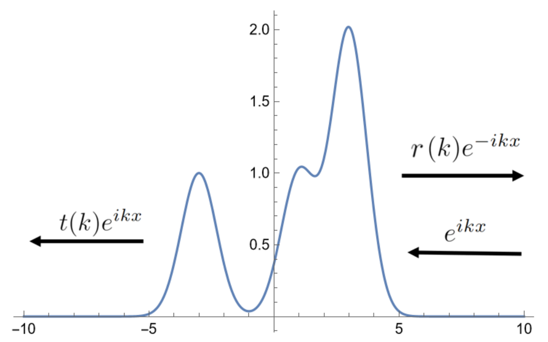

Figure 1. Scattering due to potential $V(x) = h(x,t)$ of right Jost solutions

Corresponding to this, we have the **Lax Scattering Matrix** that connects the asymptotic left to the asymptotic right in the (e^{ikx}, e^{-ikx}) basis as

$$S(k) := \lim_{L\to\infty} R_{L/2} \overleftarrow{\mathrm{P}} \exp\left(\int_{-L/2}^{L/2} dx A_x(k)\right) R_{L/2} = \begin{pmatrix} \frac{1}{t(k)} & \frac{r(-k)}{t(-k)} \\ \frac{r(k)}{t(k)} & \frac{1}{t(-k)} \end{pmatrix} \quad (1)$$

where A_x is the spatial Lax connection and $R_{L/2}$ is the "renormalisation" matrix defined as

$$R_{L/2} := \exp\left(-ikL\sigma_z/2\right) \quad (2)$$

The Lax formulation then implies that the trace of $S(k)$ given by

$$\mathrm{Tr}(S(k)) = \frac{1}{t(k)} + \frac{1}{t(-k)} \quad (3)$$

is preserved under time evolution, and hence it encodes information about the conserved charges associated to $h(x,t)$.

Spectrum

The spectrum of the auxiliary linear problem gets divided into two parts-

- $k \in \mathbb{R}$ correspond to **Scattering States** and satisfy the unitarity relationships
$$t(-k) = \bar{t}(k),\ r(-k) = \bar{r}(k),\ |t(k)|^2 + |r(k)|^2 = 1 \quad (4)$$
- $k \in \{i\kappa_{j=1,\ldots,N}\}$ for $\kappa_j \in \mathbb{R}^+$ correspond to **Bound States** appearing as poles in $r(k)$ and $t(k)$

The above properties along with Eq (3) imply that fixing $t(k \in \mathbb{R})$ and the location of the poles $i\kappa_j$'s fixes the conserved charges.

Cutting and Patching

Now, we construct a spatial window of length L centered appropriately so that most of the "activity" of the solution $h(x,t)$ at a particular time slice is in the window. Call this the cut solution $h^{(L)}_{cut}(x,t)$. Then, we patch the cut solution periodically to construct the periodic solution $h^{(L)}_{periodic}(x,t)$. This process is shown in Fig 2.

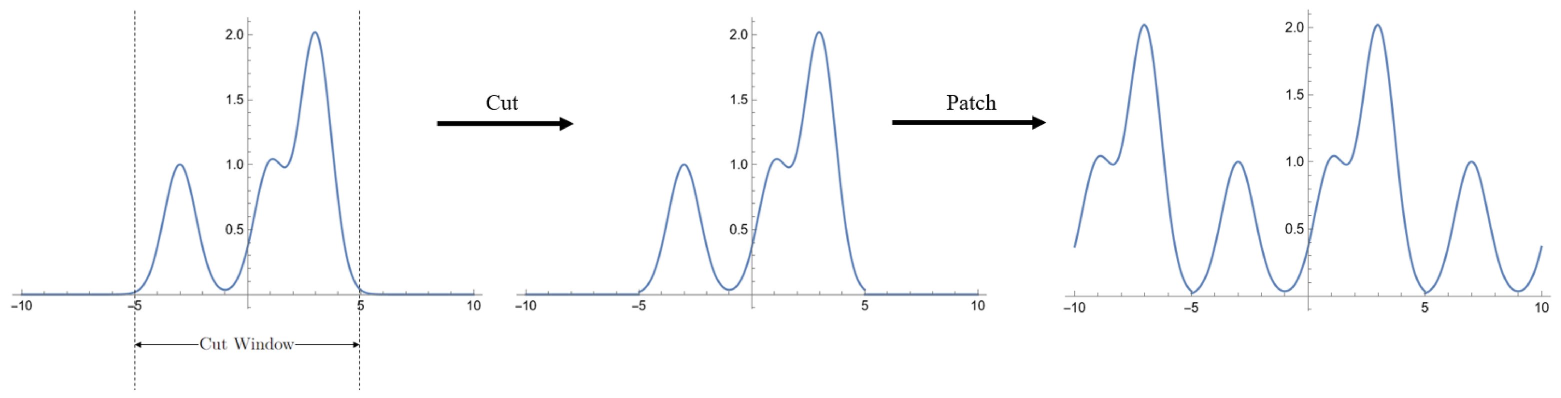

Figure 2. Cutting and Patching

In this way, we have constructed two series of solutions - a series of asymptotic solutions $h^{(L)}_{cut}$ and a series of periodic solutions $h^{(L)}_{periodic}$ that under $L \to \infty$ converge to $h(x,t)$. We now assume that the scattering matrix for $h^{(L)}_{cut}$ is given by $S_L(k)$.

Monodromy and Quasi-Momentum

To describe $h^{(L)}_{periodic}$, we need to construct the **Monodromy** given by

$$T_L(k) := \overleftarrow{\mathrm{P}} \exp\left(\int_{-L/2}^{L/2} dx A_x(k)\right) = R^{-1}_{L/2} S_L(k) R^{-1}_{L/2} \quad (5)$$

Its eigenvalues are given by $e^{\pm i q_L(k)}$, where $q_L(k)$ is referred to as the **Quasi-Momentum**. More explicitly, it is given by

$$q_L(k) = \cos^{-1}\left(\frac{1}{2}\left(\frac{e^{ikL}}{t_L(k)} + \frac{e^{-ikL}}{t_L(-k)}\right)\right) \quad (6)$$

where $t_L(k)$ are the transmission coefficients in $S_L(k)$. Again, the Lax formulation implies $q_L(k)$ encodes conserved charges.

Branch Points

To compute the spectral curve, we need to find the branch points of $q_L(k)$ which are extracted by solving $\cos q_L(k) = \pm 1$ for non-degenerate k. These branch points get divided into two sectors -

- We define the **Scattering States Sector** as the collection of real branch points, that is $k \in \mathbb{R}$, and they are given by
$$k^{(S)}_{n\pm} = \frac{n\pi}{L} + \frac{\arg t_L\left(\frac{n\pi}{L}\right) \pm \cos^{-1}\left|t_L\left(\frac{n\pi}{L}\right)\right|}{L} + \mathcal{O}\left(\frac{1}{L^2}\right) \quad (7)$$
- We define the **Bound States Sector** as the collection of completely imaginary branch points, that is $k \in i\mathbb{R}^+$, and they are given by
$$k^{(B)}_{n\pm} = i\kappa_n\left(1 \pm 2e^{-\kappa_n L} \lim_{k\to i\kappa_n} \sqrt{\left|\frac{t_L(k)}{t_L(-k)}\right|}\right) + \mathcal{O}\left(e^{-2\kappa_n L}\right) \quad (8)$$

This is in direct correspondence with the spectrum of the auxiliary linear problem.

Spectral Curves

A qualitative description of KdV spectral curves under decompactification is shown in Fig 3.

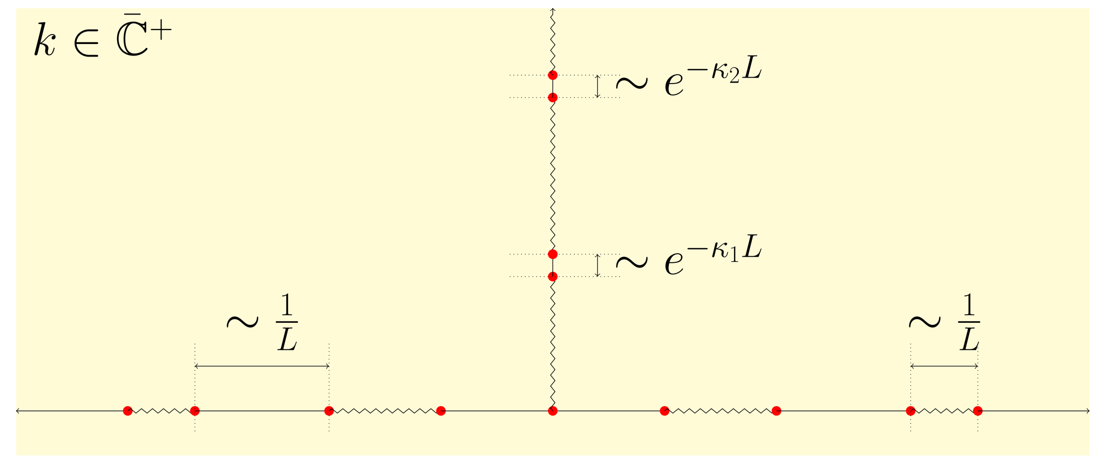

Figure 3. Spectral Curves for KdV parameterised by the window length L

Reality of KdV solutions imply symmetry about the imaginary axis. If we included left Jost solutions, then we will have symmetry about the real axis. Further, this degeneration can not arbitrary because physical spectral curves are constrained by period integrals.

Other Integrable Field Theories

This analysis can be easily done for other integrable field theories. On doing it for the Continuous Heisenberg Magnet (CHM) model, we get Fig 4.

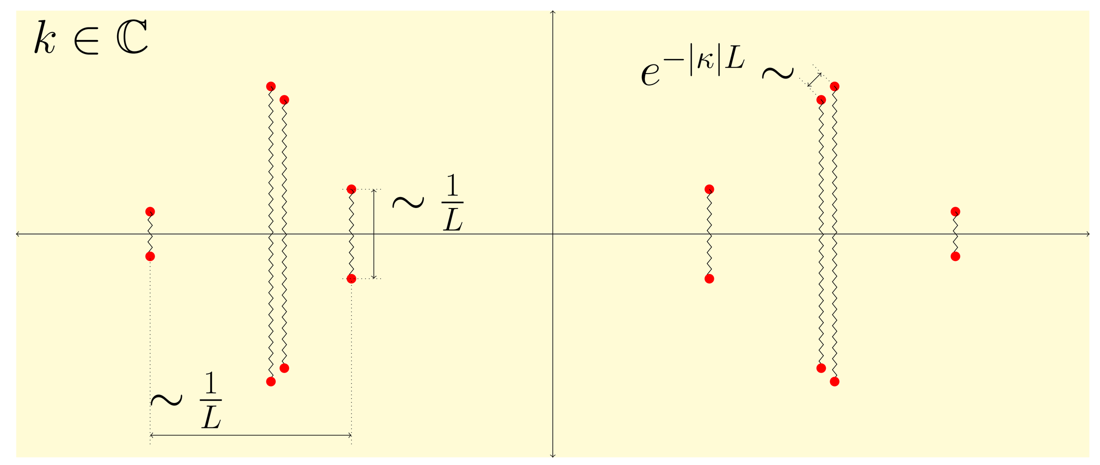

Figure 4. Spectral Curves for CHM parameterised by the window length L

As compared to Fig 3, the cuts in Fig 4 are rotated by a $\pi/2$ angle because the CHM auxiliary linear problem is anti-unitary.

Conclusion

Given an integrable field theory, we have presented a procedure to start from some asymptotic solution and obtain a series of spectral curves that degenerate to the spectrum of the auxiliary linear problem under decompactification. We showed this explicitly for the KdV equation, but this can also be done for other integrable field theories.

References

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