Spherical weight-shifting algebra

Ilija Burić, University of Pisa

joint works with Volker Schomerus and with Francesco Russo and Alessandro Vichi

Motivation and summary of results

Partial wave decompositions form a basic analytic tool in quantum field theory. They give a representation of amplitudes/correlation functions which clearly separates kinematical and dynamical aspects of the theory. The decompositions are used in 'bootstrap' studies in particular, including the CFT, S-matrix and EFT bootstrap.

Mathematically, partial waves can be understood in terms of representation theory of the symmetry group of the QFT, or some of its subgroups. In particular, they are expressed in terms of irreducible matrix elements. Explicit computation of these matrix elements is a classical question of representation theory.

In several setups, including $2 \to 2$ scattering in QFT, or four-point functions in CFT, the partial waves are identified with so-called spherical functions. The latter can also be interpreted as wavefunctions of spinning Calogero-Moser-Sutherland models.

We have developed a general method to compute spherical functions, constructing them via a novel set of weight-shifting operators that close into a certain 'algebra of exchange relations'. In field theory, this leads to

- New conformal blocks for the bulk two-point function of spinning fields in the presence of a co-dimension two defect, [1].
- General partial waves for $2 \to 2$ scattering of four particles of spins $J_1, \dots, J_4$ in arbitrary spacetime dimension, [2].
- A systematic implementation for the second problem is given in [3].

Harmonic analysis of spherical functions

Gelfand pair: G =Lie group, $K \subset G$ fixed by involutive automorphism

Cartan decomposition, KA_pK, is a generalisation of the Euler-angle factorisation in $SO(3)$. The dimension of the middle, abelian, factor A_p is called the split rank of (G, K). E.g.

$$G = SO(m+n), \quad K = SO(m) \times SO(n) \quad \Longrightarrow \quad \text{rank}(G, K) = \min(m, n) \ .$$

Spherical functions [Gelfand, Godement, Harish-Chandra] are K-K covariant vector-valued functions on G, specified by two finite-dimensional representations ρ, σ of K

$$\Gamma_{\rho,\sigma} = \{f : G \to \text{Hom}(W_r, W_l) \, | f(k_l g k_r) = \rho(k_l) f(g) \sigma(k_r)\} \ .$$

Examples are given by matrix elements $\pi^a{}_\alpha(g)$, where the index a transforms in ρ and α in σ. Spherical functions are determined by their values on A_p. Moreover, these values are highly constrained

$$f(a) = f(mam^{-1}) = \rho(m) f(a) \sigma(m^{-1}) \,, \tag{1}$$

where M is the centraliser of A_p in K.

The radial component map (Π) allows to express the action of differential operators on G on spherical functions in terms of their restrictions to A_p. Any $u \in U(\mathfrak{g})$ can be radially decomposed. We illustrate the concept on the example (relevant for QFT partial waves)

$$G = SO(d-1), \quad K = SO(d-2), \quad M = SO(d-3) \ .$$

The quadratic Casimir can be written as

$$C_2 = L_{12}^2 + (d-3) \cot\theta L_{12} + \frac{L'_{2i}L'_{2i} - 2\cos\theta L'_{2i}L_{2i} + L_{2i}L_{2i}}{\sin^2\theta} + \frac{1}{2} L^{ij} L_{ij} \ .$$

Rules: Fix an element $a \in A_p$ (here $a = e^{\theta L_{12}}$). Write $u \in U(\mathfrak{g})$ (C_2) in terms of generators of A_p (L_{12}), generators of K (L_{2i}, L_{ij}) and their conjugates ($k' = a^{-1}ka$). The process is formalised by the map

$$\Pi : U(\mathfrak{g}) \to \text{Fun}(A_p) \otimes U(\mathfrak{a}_p) \otimes U(\mathfrak{k}) \otimes_{U(\mathfrak{m})} U(\mathfrak{k}) \cong \mathcal{D}(A_p) \otimes U(\mathfrak{k}) \otimes_{U(\mathfrak{m})} U(\mathfrak{k}) \ .$$

Harish-Chandra's theorem: For $u \in U(\mathfrak{g})$, $f \in \Gamma_{\rho,\sigma}$

$$(uf)|_{A_p} = (\rho \otimes \sigma^*) \circ \Pi(u) \left(f|_{A_p}\right) \ .$$

Radial decompositions of the quadratic Casimir, and elements of $\mathfrak{g} \subset U(\mathfrak{g})$ are known in general. In the above example, invariant fields decompose as

$$\Pi(L_{1i}) = \cot\theta L_{2i} - \frac{L'_{2i}}{\sin\theta} \ . \tag{2}$$

Some known results

- Zonal spherical functions, $\rho = \sigma = 1$, any rank: Heckman-Opdam theory
- Various results in rank one: e.g. for $\rho = (l)$, $\sigma = (l')$, $\pi = (J)$ integral representations are known [Vilenkin, Klimyk...]
- Selected results in rank two from CFT literature [Costa et al; Karateev et al...]
- Matrix hypergeometric functions, $\rho = \sigma$ [Tirao et al; Koelink et al...]
- More abstract accounts [Reshetikhin, Stokman; Feher, Pusztai; Etingof, Frenkel, Kirillov...]

Weight-shifting operators

We illustrate the construction on the example $\rho = (l)$, $\sigma = (l')$ and $\pi = (J, q)$. Symmetric traceless tensors as polynomials

$$\rho(L_{23}) = -i(x_a\partial_a - l), \quad \rho(L_{2a}) = -\frac{1}{2}\Big((1-x^2)\partial_a + 2x_a(x^b\partial_b - l)\Big),$$
$$\rho(L_{3a}) = \frac{i}{2}\Big((1+x^2)\partial_a - 2x_a(x^b\partial_b - l)\Big), \quad \rho(L_{ab}) = x_a\partial_b - x_b\partial_a \ .$$

Invariant vectors, (1), take the form

$$F = (1-X)^l (1-X')^{l'} f(y), \qquad y = \frac{(X+1)(X'+1) - 4W}{(X-1)(X'-1)} \ .$$

They are polynomials in y of degree $\le \min(l, l')$. The Laplacian becomes

$$\Delta^{(d)}_{l,l'} = \partial_\theta^2 + (d-3)\cot\theta\, \partial_\theta - \mathcal{D}^{(d)}_y + \frac{2\mathcal{D}^{(d)}_y - l(l+d-4) - l'(l'+d-4)}{\sin^2\theta}$$
$$- 2\cos\theta \, \frac{y\mathcal{D}^{(d)}_y - (l+l'+d-5)(y^2-1)\partial_y + ll'y}{\sin^2\theta} \,,$$

where we made use of the Gegenbauer differential operator

$$\mathcal{D}^{(d)}_y = (y^2-1)\partial_y^2 + (d-4) y \partial_y \ .$$

One may an eigenfunction $f(\theta, y)$ of $\Delta^{(d)}_{l,l'}$ as the generating function of *independent* matrix elements. There is a simple map which turns $f(\theta, y)$ to the corresponding set of matrix elements in the Gelfand-Tsetlin basis. The upshot of using generating functions is that it considerably simplifies the solution theory. We have the external weight-shifting operators

$$q_{l,l'} = \partial_\theta - l\cot\theta + \frac{(y^2-1)\partial_y - l'y}{\sin\theta}, \quad \bar{q}_{l,l'} = \partial_\theta - l'\cot\theta + \frac{(y^2-1)\partial_y - ly}{\sin\theta} \ .$$

The name comes from the exchange relations

$$\Delta^{(d)}_{l+1,l'} q_{l,l'} = q_{l,l'} \Delta^{(d)}_{l,l'}, \qquad \Delta^{(d)}_{l,l'+1} \bar{q}_{l,l'} = \bar{q}_{l,l'} \Delta^{(d)}_{l,l'} \ .$$

These operators arise by taking radial components of invariant vector fields (2). They allow to shift l and l' by one unit, but cannot change π (i.e. the eigenvalue). Therefore, the full solution theory requires another type of shifting, the internal one. We illustrate their construction with an example. Suppose we wish to compute $f^{J,q}_{l,l'} = f^{3,1}_{2,1}$. Then

$$f^{3,0}_{2,1}(\theta) = \bar{q}_{2,0} \cdot q_{1,0} \cdot q_{0,0} \cdot f^{3,0}_{0,0}(\theta) \,,$$
$$f^{3,1}_{2,1}(\theta, y) = (\Delta^{(d)}_{2,1} - C_2(2,0))(\Delta^{(d)}_{2,1} - C_2(4,0)) \left(f^{3,0}_{2,1}(\theta, y) f^{1,0}_{0,0}(\theta)\right) \ .$$

Last step performs the projection using

$$(3,0) \otimes (1,0) = (2,0) \oplus (3,1) \oplus (4,0) \ .$$

We illustrated the theory for rank one and *spin rank* one, but it applies much more generally. Explicit extension to rank two ($G = SO(d+1,1)$, $K = SO(1,1) \times SO(d)$) was done in [1] and to spin rank two in [2]. The former case involves further weight-shifting operators $p, \bar{p}$ subject to additional exchange relations. The latter has two spin variables, x, y, and makes use of the two-variable generalisation of the Gegenbauer operator

$$\mathcal{D}^{(d)}_{x,y} = \mathcal{D}^{(d-2)}_x + \mathcal{D}^{(d+2\ell')}_y + 2(x^2-1)\partial_x\partial_y - 2\ell x \partial_y + \ell(\ell+d-5) \ .$$

Perspectives

- Go ahead and bootstrap
- Analytic study of spinning processes, inversion formulae
- Computations of matrix hypergeometric functions (in particular, higher rank)
- Tentative Ruijsenaars-Schneider dual and improved internal weight-shifting
- Higher-point functions, cosmological correlators...

References

[1] I. Buric and V. Schomerus, *"Universal spinning Casimir equations and their solutions"*, JHEP **03** (2023), 133, arxiv:2211.14340.

[2] I. Buric, F. Russo and A. Vichi, *"Spinning Partial Waves for Scattering Amplitudes in d Dimensions"*, arxiv:2305.18523.

[3] https://gitlab.com/russofrancesco1995/partial_waves