Heating up the AdS_4 Quantum Spectral Curve

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2306.09883, 2306XX.XXXX w J. Minahan & C. Thull

(2211.07810 w A.Cavaglià, N. Gromov & P. Ryan)

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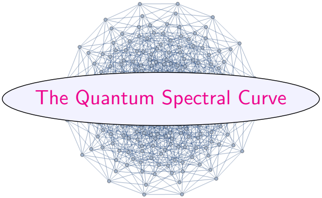
Introduction

- Main topic of this talk: [Gromov, Kazakov,Leurent,Volin '13'14]



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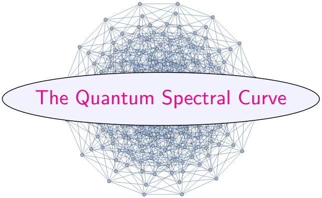

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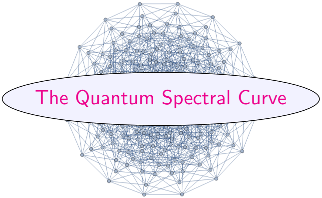

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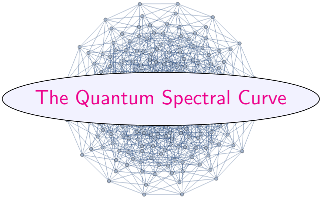

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- The QSC also has applications beyond $\mathcal{N} = 4$ and the spectral problem.
- Focus of today: The Hagedorn temperature in AdS_4/CFT_3 using the QSC.

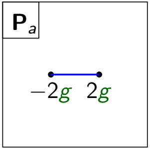
Outline of the Talk

1. QSC for AdS_5, AdS_4 and AdS_3
2. The Hagedorn temperature from the AdS_4 QSC
3. Technical details or Solving the AdS_4 QSC
4. Conclusions and outlook

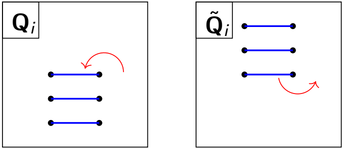
QSC for AdS_5, AdS_4 and AdS_3

What is the QSC?

- The $\mathcal{N} = 4$ QSC is based on $\mathfrak{psu}_{2,2|4}$. It is a collection of 256 Q-functions, functions of 1 complex parameter u. Among them: $\mathbf{P}_a(u)$ and $\mathbf{Q}_i(u)$



$g = \frac{\sqrt{\lambda}}{4\pi}$



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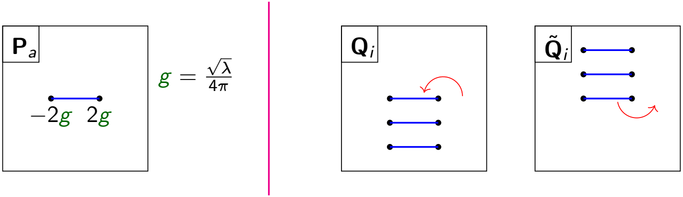

- Other Q-functions are obtained from QQ-relations. Example:

$$Q^+_{a|i} - Q^-_{a|i} = \mathbf{P}_a \mathbf{Q}_i\,, \qquad f^{[n]} = f(u + \mathrm{i}\frac{n}{2}), \quad f^{\pm} = f^{[\pm 1]}$$

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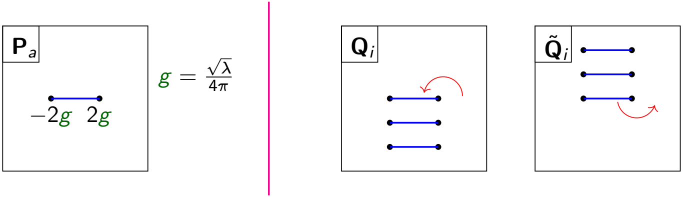

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- Large u encode quantum numbers.

$\mathfrak{so}_6$ quantum numbers

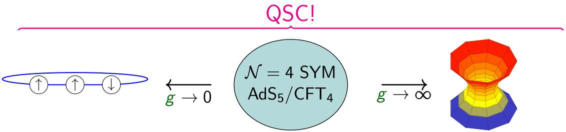
$$\mathbf{P}_a \simeq_{u\to\infty} A_a u^{-\tilde{M}_a}\qquad \Delta=\Delta^{(0)}+\gamma$$

$$\mathbf{Q}_i \simeq_{u\to\infty} B_i u^{\hat{M}_i-1}$$

$\mathfrak{su}_{2,2}$ quantum numbers

The spectral problem

- The QSC allows computations both at strong and weak coupling
  [Marboe,Volin'14,'18,Gromov,Levkovich-Maslyuk,Sizov'15,]



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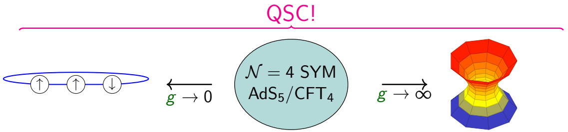
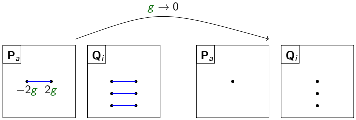


- Weak coupling: Perturbation around a non-compact spin chain.



Functions: $u, \frac{1}{u}, \eta_s = \sum_{n=0}^{\infty} \frac{1}{(u+\mathrm{i}\,n)^s}, \ldots$

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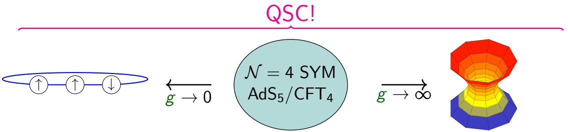

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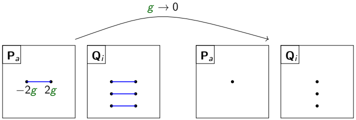


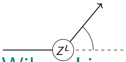
Functions: $u, \frac{1}{u}, \eta_s = \sum_{n=0}^{\infty} \frac{1}{(u+\mathrm{i}n)^s}, \ldots$

- Strong coupling: Have to resort to numerical methods!

Variations of QSC

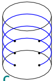
- There exist a plethora of deformations of the $\mathcal{N} = 4$ QSC.

[Gromov,Levkovich-Maslyuk '15]



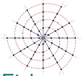
Wilson Lines

[Klabbers,van Tongeren '17]



η- deformations

[Gromov, Kazakov,Korchemsky,Negro,Sizov '17]



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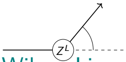

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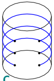

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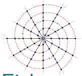


Fishnets

- For AdS/CFT currently only two other curves on the market

ABJM
$AdS_4 \times \mathbb{CP}^3$

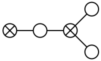
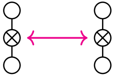
[Cavaglià,Fioravanti,Gromov,Tateo 14']

AdS_3/CFT_2

[SE,Volin'21]
[Cavaglià,Gromov,Stefański,Torrielli,21']

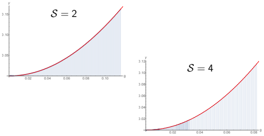
Conjecture: $AdS_3 \times S^3 \times T^4$ with RR-flux
(TBA was constructed [Frolov,Sfondrini '21])
Agree? Open question.

Status of low-dimensional QSC

	AdS_4	AdS_3
Status:	Derived from TBA	Conjectured
Algebraic Structure:	 $\mathfrak{osp}_{6\|4}$	 $\mathfrak{psu}_{1,1\|2} \oplus \mathfrak{psu}_{1,1\|2}$
Analytic Structure	Quadratic cuts	No quadratic cuts
"$\mathfrak{sl}_2$" Weak Coupling	✓ [Bombardelli,Cavaglià,Conti,Tateo '18] [Anselmetti,Bombardelli,Cavaglià,Tateo '15]	✓ [Cavaglià,SE,Gromov,Ryan '22]
Strong Coupling Numerics	✓ [Bombardelli,Cavaglià,Conti,Tateo '18]	?

Example of explicit results

- The AdS_3 QSC was solved in an "$\mathfrak{sl}_2$"-sector, AdS_3 analogue of $\mathrm{tr}\, Z\mathcal{D}^{\mathcal{S}} Z$ [Cavaglià,SE,Gromov,Ryan'22]



$$
\begin{aligned}
\gamma_{\mathcal{S}=2} = {} & 12\, g^2 + \frac{864}{35\pi} g^3 + \left(-48 - \frac{576}{7\pi^2}\right) g^4 \\
& + \left(-\frac{405504}{875\pi^3} - \frac{51552}{143\pi}\right) g^5 \\
& + \left(444 - \frac{70665216}{4375\pi^4} + \frac{230121984}{175175\pi^2}\right) g^6 \\
& + \left(-\frac{16896}{35\pi}\zeta_3 - \frac{4965482496}{21875\pi^5} \right. \\
& \qquad \left. + \frac{6791453184}{875875\pi^3} + \frac{1102677696}{146965\pi} \right) g^7 + \mathcal{O}(g^8)
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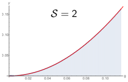

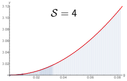

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- Fitting $\mathcal{S} = 2, 4, 6, 8$ it was found that

$$
\gamma_{\mathcal{S}} = 8 S_1(\mathcal{S})\, g^2 + \frac{384}{35\pi} S_1(\mathcal{S})^2\, g^3 + \left(\gamma_{(4)}^{\mathcal{N}=4} - \frac{512}{21\pi^2} S_1(\mathcal{S})^3\right) g^4 + \mathcal{O}(g^5)
$$

$$
S_1(\mathcal{S}) = \sum_{n=1}^{\mathcal{S}} \frac{1}{n}
$$

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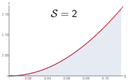

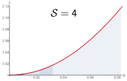

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$$S_1(\mathcal{S}) = \sum_{n=1}^{\mathcal{S}} \frac{1}{n}$$

- Features

g^{odd} $\qquad$ $\underbrace{\frac{1}{\pi^a}}_{\text{Unexpected}}$ $\qquad$ $\underbrace{\frac{384}{35\pi} S_1(\mathcal{S})^2 g^3}_{\text{Early wrapping!}}$

The Hagedorn temperature from the AdS_4 QSC

Beyond the spectral problem: Hagedorn

- The Hagedorn temperature, T_H, is the temperature for which the partition function diverges:

$$\lim_{T \to T_H} e^{-\beta\, \mathrm{D}} = \infty\,, \qquad \beta = \frac{1}{T}\,.$$

The dilation operator

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The dilation operator

- Goal: Compute the Hagedorn temperature for ABJM (in the large N limit) using QSC following the $\mathcal{N} = 4$ computation of Harmark and Wilhelm [Harmark,Wilhelm 17',18',21'].

Hagedorn temperature for free $\mathcal{N} = 4$ SYM

- We start from $g = 0$ [Sundborg '99].

$$\mathcal{N} = 4 \text{ Fields: } \{\mathcal{F}_{\mu\nu}, \psi_i, \overline{\psi}^i, \phi_I\} \qquad z_{\mathcal{O}} = \text{tr}_{\mathcal{O}}\, e^{-\beta D_0} = \text{tr}_{\mathcal{O}}\, y^{D_0}\,,$$

$z = z_{\mathcal{F}} + z_{\psi} + z_{\phi}$ is the singleton partition function

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- Let us calculate z_ϕ:

$$z_\Phi(y) = \frac{6\, y}{(1-y)^4} \left(1 - y^2\right) = \frac{6y(1+y)}{(1-y)^3}$$

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- Single-trace partition function, (Euler's ϕ-function, $\omega = e^{2\pi \mathrm{i}}$)

$$Z = \sum_{\text{single-trace } \mathcal{O}} y^{D_0(\mathcal{O})} = \underbrace{-\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log\left(1 - z(\omega^{n+1} y^n)\right)}_{\text{diverges for } z=1} - z\,.$$

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$$T_H^{(0)} = \frac{1}{2\log\left(2+\sqrt{3}\right)}\,.$$

Hagedorn from QSC

- Finite coupling using TBA and QSC explained in [Harmark,Wilhelm 17',18',21']. They gave the following recipe:

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- Twisted asymptotics

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$$\tilde{\mathbf{Q}}_i(u) = (-1)^i \overline{\mathbf{Q}}_i(u)\,.$$

- Done!

AdS$_4$ in a nutshell

- We now turn to ABJM. Symmetry algebra $\mathfrak{osp}_{6|4}$. Two basic representations

$$\underbrace{(\phi_i, \psi^i)}_{\text{Particle A } (N, \bar{N})} \qquad \underbrace{(\overline{\phi}^i, \overline{\psi}_i)}_{\text{Particle B } (\bar{N}, N)}$$

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- Single-trace operators:

$$\mathcal{O} = \operatorname{tr} W_A W_B W_A \ldots \qquad W_A \in \{\phi_i, \psi^i\}, W_B \in \{\overline{\phi}^i, \overline{\psi}_i\}\,.$$

AdS$_4$ in a nutshell

- We now turn to ABJM. Symmetry algebra $\mathfrak{osp}_{6|4}$. Two basic representations

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- Singelton partition functions

$$z_A = z_B = \frac{4\sqrt{y}}{(1-\sqrt{y})^2}\,,$$

$$z_A(y_H^{(0)})\, z_B(y_H^{(0)}) = 1 \implies T_H^{(0)} = \frac{1}{4\log\big(1+\sqrt{2}\big)}\,.$$

What I will explain

- To Do:
  - Identify z in the QSC and twist the curve appropriately.
  - Solve analytically at weak coupling.
  - Go to strong coupling using numerics.
  - Additional exercise: Twist the R-symmetry.

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Weak coupling results

- Use h for integrability coupling constant. Conjecture [Gromov,Sizov '14]

$$\lambda = \frac{\sinh(2\pi h)}{2\pi} \, {}_3F_2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\sinh^2(2\pi h) \right)$$

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- At weak coupling we computed up to $\mathcal{O}(h^8)$. Write

$$T_H = T_H^{(0)} + T_H^{(1)} h^2 + T_H^{(2)} h^4 + \mathcal{O}(h^6) \,, \quad h = \lambda - \frac{\pi^2 \lambda^3}{3} + \mathcal{O}(\lambda^5) \,.$$

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$$T_H^{(2)} = 7\sqrt{2} - 8 - 4(1+2\sqrt{2})\mathrm{Li}_1\left(\frac{1}{(1+\sqrt{2})^2}\right)$$

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- Agrees with tree-level and, up to a factor 2, with h^2 calculation in [Papathanasiou,Spradlin 09']

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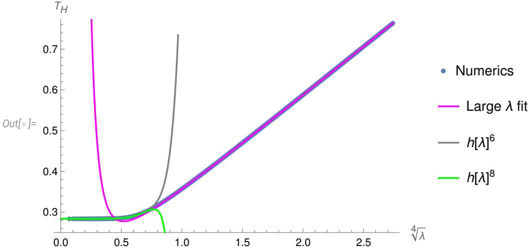

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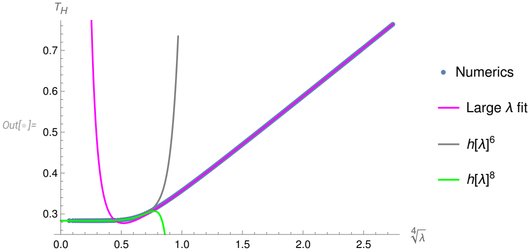

- Fitting the curve:

$$T_H = \frac{\lambda^{\frac{1}{4}}}{2^{\frac{5}{4}}\sqrt{\pi}} + \frac{3}{8\pi} - \frac{(0.0308 \pm 0.0004)}{\lambda^{\frac{1}{4}}} + \frac{0.046 \pm 0.002}{\lambda^{\frac{1}{2}}} + \dots$$

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$$T_H^{\mathrm{AdS}_{d+1}} = \frac{1}{2\pi\sqrt{2\alpha'}} + \frac{d}{8\pi}$$

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- First principle derivation? Validity beyond $d = 3, 4$?

Technical details or Solving the AdS_4 QSC

Basics of $\mathfrak{osp}_{6|4}$ Q-systems

- Intuition: The structure of the Q-system should reflect the underlying $\mathfrak{osp}_{6|4}$ symmetry algebra.

$\det Q_{a|i} = \det Q^{a}{}_{i} = -1$

$$\mathbf{P}_A = \mathcal{Q}_{A|\emptyset}\,, \qquad \mathbf{Q}_I = \mathcal{Q}_{\emptyset|I}\,, \qquad Q_{a|i} \quad Q^{a}{}_{|i}\,.$$

$$A = 1,\dots,6 \qquad I = 1,\dots,5 \qquad a, i = 1,\dots,4$$

$\mathfrak{so}_6$ vector — $\mathfrak{so}_5$ vector — Spinors

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- $Q_{a|i}$ and $Q^a|i$ are basic. Construct other functions from them

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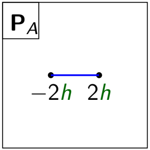
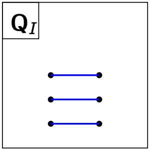
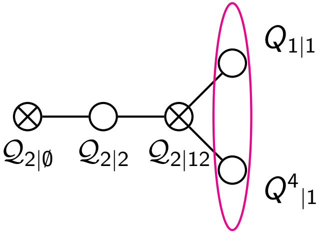
- $\mathbf{P}_A$ and $\mathbf{Q}_I$ have the same structure as in AdS_5





Bethe equations

- Bethe equations follows naturally from the bilinear expressions.
- Example:



$$\left.\frac{\mathcal{Q}_{2|2}^{+}}{\mathcal{Q}_{2|2}}\right|_{\mathbf{P}_2=0} = 1\,,$$

$$\left.\frac{\mathcal{Q}_{2|2}^{[2]}}{\mathcal{Q}_{2|2}^{[-2]}}\frac{\mathbf{P}_2^{-}\,\mathcal{Q}_{2|12}^{-}}{\mathbf{P}_2^{+}\,\mathcal{Q}_{2|12}^{+}}\right|_{\mathcal{Q}_{2|2}=0} = -1\,,$$

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Twisting the curve

- Recall: Hagedorn at $h = 0$ is controlled by the partition function, character,
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- We need to find characters from QSC, this means twisting.

Twisting the curve

- Let us turn off the coupling $h = 0$ and take a twisted ansatz

$$\begin{cases} Q_{a|i} = A_{a|i}\,\mathbf{x}^{\mathrm{i} u \omega_a}\,\mathbf{y}^{-\mathrm{i} u \nu_i}\,, \\ Q^{a}{}_{|i} = A^{a}{}_{|i}\,\mathbf{x}^{-\mathrm{i} u \omega_a}\,\mathbf{y}^{-\mathrm{i} u \nu_i}\,, \end{cases} \qquad \omega = \frac{1}{2}\begin{pmatrix} +++ \\ +-- \\ -+- \\ --+ \end{pmatrix}, \quad \nu = \frac{1}{2}\begin{pmatrix} ++ \\ +- \\ -+ \\ -- \end{pmatrix}.$$

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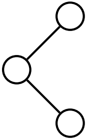
- Trick to solve the Q-system: Consistency equations



$$\mathbf{4} \wedge \mathbf{4} = \mathbf{6}$$
$$\overline{\mathbf{4}} \wedge \overline{\mathbf{4}} = \mathbf{6}$$

$$\mathcal{Q}_{A|IJ} = \Sigma_{IJ}^{ij}\, \overline{\sigma}_A^{ab}\, Q^{+}_{a|i}\, Q^{-}_{b|j}$$
$$= -\Sigma_{IJ}^{ij}\, (\sigma_A)_{ab}\, (Q^{a}{}_{|i})^{+}\, (Q^{b}{}_{|j})^{-}$$

Characters from Q-functions

- To find the partition functions we construct bilinears again!
  - Compact spin chains:

$$\frac{1}{2}\kappa^{ij}Q_{a|i}^{[2]}\,(Q^{a}{}_{|j})^{[-2]} = x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2} + x_3 + \frac{1}{x_3} - y_1 - \frac{1}{y_1} - y_2 - \frac{1}{y_2}$$

  - Non-compact: $(y_1 = y_2)$

$$Q_{a|4}^{[1]}\,(Q^{a}{}_{|1})^{[-1]} = \frac{1}{(1-y^2)^3}(y(1-y^2)\chi_{\mathbf{4}} - 2y^2(1-y^2)\chi_{\bar{\mathbf{4}}}) \quad (3.1)$$

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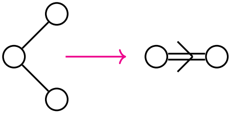
- Simplest case: Set $y_1^{-\mathrm{i}u} = y_2^{-\mathrm{i}u} = e^{\pi u} y^{-\mathrm{i}u}, x_a = 1$.

$$Q_{a|i} \simeq \begin{pmatrix} e^{-\pi u} y^{\mathrm{i}u} & u & 1 & e^{\pi u} y^{-\mathrm{i}u} \\ e^{-\pi u} y^{\mathrm{i}u} u^2 & u^3 & u^2 & e^{\pi u} y^{-\mathrm{i}u} u^2 \\ e^{-\pi u} y^{\mathrm{i}u} u & u^2 & u & e^{\pi u} y^{-\mathrm{i}u} u \\ e^{-\pi u} y^{\mathrm{i}u} u^3 & u^4 & u^3 & e^{\pi u} y^{-\mathrm{i}u} u^3 \end{pmatrix}, \qquad y = e^{-\frac{1}{2T}}.$$

Turning on coupling

- We are now in the symmetric sector

$$Q^{a}{}_{|i} = -\kappa^{ab}\, Q_{b|j}\, \overset{\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}}{\mathbb{K}_i^j}$$



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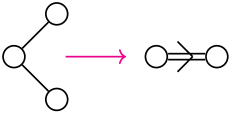
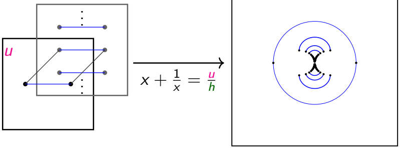


- While $Q_{a|i}$ are "basic" $\mathbf{P}_A$ have simpler analytic properties. Can parameterise $\mathbf{P}_A$ using Zhukovsky $x + \frac{1}{x} = \frac{u}{h}$

Parameters to fix!

$$\mathbf{P}_A \propto \sum_{n=-M_a}^{\infty} \frac{c_{A,n}}{x^n}$$



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- Parameterise

$$Q_{a|i} = Q^{(0)}_{a|i} + h^2 Q^{(0)}_{a|j}(b^j{}_i)^{+} + \mathcal{O}(h^4)\,,$$

Then one finds

$$b^j{}_i - \underbrace{(-1)^{\frac{i(i-1)}{2}+\frac{j(j-2)}{2}}}_{\text{Different from } \mathcal{N}=4}(b^j{}_i)^{[2]} = -\underbrace{\mathbb{K}^j_k\kappa^{kl}((Q^{(0)})^a{}_{|l})^{+}\mathbf{P}^{(1)}_{ab}\kappa^{bc}(Q^{(0)}_{c|i})^{-}}_{\text{Known}}$$

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- b^i_j will be given in terms of

$$u\,, \qquad \frac{1}{u}\,, \qquad \eta^t_s = \sum_{n=0}^{\infty}\frac{t^n}{(u+\mathtt{i}n)^s}\,, \qquad t = 1, y^{\pm 1}, y^{\pm 2} \qquad (3.2)$$

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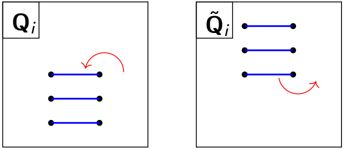
- Still many free parameters around, we need to fix them!

Gluing conditions

- To find T_H we need to also consider $\tilde{\mathbf{Q}}_I$. Recall:



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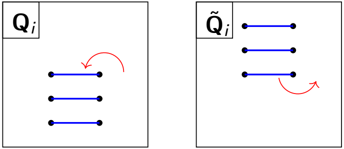

- Construct a lower-halfplane analytic $\mathbf{Q}_I$ using parity:

$$\tilde{\mathbf{Q}}_I(u) = \begin{pmatrix} e^{2\pi u} & 0 & \bullet & 0 & 0 \\ 0 & -e^{2\pi u} & 0 & \bullet & 0 \\ 0 & 0 & -e^{-2\pi u} & 0 & 0 \\ 0 & 0 & 0 & e^{-2\pi u} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}_I{}^J \mathbf{Q}_J(-u) \quad (3.5)$$

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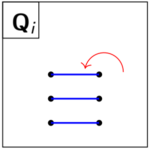

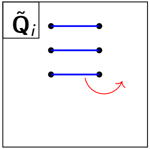

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- Zeros are fixed by asymptotics and parity. Example:

$$\mathbf{Q}_3 \simeq_{u \to i\infty} e^{-\frac{|u|}{2T}} \qquad \tilde{\mathbf{Q}}_3 \simeq_{u \to -i\infty} e^{-\frac{|u|}{2T}} \quad (3.6)$$

Quantisation and results

- Finally we can now fix all coefficients by demanding that

$$\left(\mathbf{Q}_I+\tilde{\mathbf{Q}}_I\right)\Big|_{u\simeq 0}=\text{regular}\,,\quad \left(\frac{\mathbf{Q}_I-\tilde{\mathbf{Q}}_I}{\sqrt{u-2h}\sqrt{u+2h}}\right)\Bigg|_{u\simeq 0}=\text{regular}\,.$$

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- This fixes T_H! For example

$$T_H^{(0)} = \frac{1}{4\log\left(1+\sqrt{2}\right)} \simeq 0.2836481643\ldots$$

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- We computed up to h^8, rather long expressions, numerically

$$T_H^{(2)} = -2.542811207\ldots \qquad T_H^{(3)} = 21.77821058\ldots$$

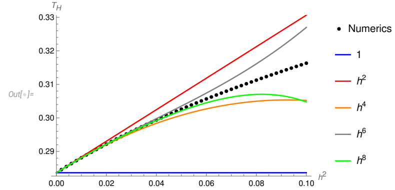
$$T_H^{(4)} = -222.2996920\ldots$$

Explicit T_H^4

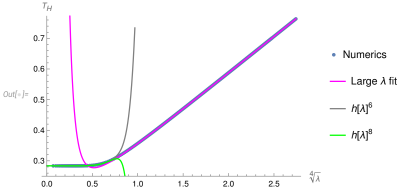
$$
\begin{aligned}
T_H^{(3)} =& \frac{4}{3}\Bigg(48\mathrm{Li}_{1,1}\left(\frac{1}{(1+\sqrt{2})^2}, \frac{1}{(1+\sqrt{2})^2}\right) - 48\mathrm{Li}_{1,1}\left(3+2\sqrt{2}, \frac{1}{(1+\sqrt{2})^2}\right) \\
&+ 12\sqrt{2}\mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^4}\right) - 45\sqrt{2}\mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right) + 84\mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right) \\
&+ 36\sqrt{2}\mathrm{Li}_3\left(\frac{1}{(1+\sqrt{2})^2}\right) + 24\mathrm{Li}_3\left(\frac{1}{(1+\sqrt{2})^2}\right) \\
&+ 48\sqrt{2}\log\left(1+\sqrt{2}\right)\left(\mathrm{Li}_1\left(\frac{1}{(1+\sqrt{2})^2}\right)\right)^2 \\
&+ \mathrm{Li}_1\left(\frac{1}{(1+\sqrt{2})^2}\right)\left(48\sqrt{2}\mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right) + 18\left(4-5\sqrt{2}\right)\log\left(1+\sqrt{2}\right)\right) \\
&+ \frac{12\sqrt{2}\left(\mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right)\right)^2}{\log\left(1+\sqrt{2}\right)} + 48\sqrt{2}\log\left(1+\sqrt{2}\right)\mathrm{Li}_1\left(\frac{1}{(1+\sqrt{2})^4}\right) \\
&+ 24\sqrt{2}\log\left(1+\sqrt{2}\right)\mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right) + 16\log\left(1+\sqrt{2}\right)\mathrm{Li}_2\left(\frac{1}{(1+\sqrt{2})^2}\right) \\
&+ \frac{18\sqrt{2}\mathrm{Li}_4\left(\frac{1}{(1+\sqrt{2})^2}\right)}{\log\left(1+\sqrt{2}\right)} + \frac{12\mathrm{Li}_4\left(\frac{1}{(1+\sqrt{2})^2}\right)}{\log\left(1+\sqrt{2}\right)} + 45\sqrt{2} - 66 + 35\sqrt{2}\log\left(1+\sqrt{2}\right) \\
&-52\log\left(1+\sqrt{2}\right)\Bigg)
\end{aligned}
$$

Numerics

- Numerics: Use the $\mathcal{N} = 4$ algorithm of [Gromov,Levkovich-Maslyuk,Sizov '15].
- Procedure: Minimise the gluing condition.
- We can verify weak-coupling



- And go to strong coupling



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- Computation of $\langle D_2 \rangle$

$$\langle D_2 \rangle = \sum_{j=0}^{\infty} M_j \times \chi_j\,, \qquad \mathcal{V}_A \otimes \mathcal{V}_B \otimes \mathcal{V}_A = \oplus_{j=0}^{\infty} \mathcal{V}_j\,. \tag{3.8}$$

χ_j obtainable from [Dolan '08] and M_j from [Papathanasiou ,Spradlin '09]

Turning on fugacities, Part 2

- Result:

$$\frac{T_H^{(1)}}{T_H^{(0)}} = \frac{4(y_H^{(0)})^2}{(1+y_H^{(0)})^2(1-y_H^{(0)})^5} \prod_{a=1}^{2} \frac{(1+x_a)(y_H^{(0)}+x_a)(1+y_H^{(0)}x_a)}{x_a^{\frac{3}{2}}}$$

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- Twisting R-symmetry doesn't change the gluing matrix. Everything keeps working!
- Slow to compute with undetermined fugacities... For numerical values we find a perfect match!

Conclusions and outlook

Conclusions

- The Quantum Spectral Curve is useful not only for AdS_5 but beyond.
- Obtained weak coupling expansion for T_H in AdS_4 up to $\mathcal{O}(h^8)$
- Numerical prediction for strong coupling expansion + Conjecture for exact expression:

$$T_H = \left(\frac{\lambda}{2}\right)^{\frac{1}{4}} \frac{1}{2\sqrt{\pi}} + \frac{3}{8\pi} + \frac{3 - 6\log(2)}{8\pi^{3/2}} \left(\frac{\lambda}{2}\right)^{-\frac{1}{4}} + \frac{165}{512\pi^2} \left(\frac{\lambda}{2}\right)^{-\frac{1}{2}} + \mathcal{O}(\lambda^{-\frac{3}{4}})$$

- Included R-symmetry fugacities and matched to order $\mathcal{O}(h^2)$ (Also works in AdS_5)

Outlook

- AdS_3 using QSC or TBA? Inclusion of NSNS-flux?
- Strong coupling calculations with additional fugacities. (Work in progress)
- Twisting the ABJM curve should be useful for
  - Wilson lines [Correa, Giraldo-Rivera,Lagares,'23]
  - Study various deformations $\gamma, \beta \ldots$ [Chen,Liu,Wu,'16]
- More general ABJM questions: Structure constants?

  [Basso,Georgoudis,Klemenchuk Sueiro '22;Bercini,Homrich,Vieira '22]

Thank you

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