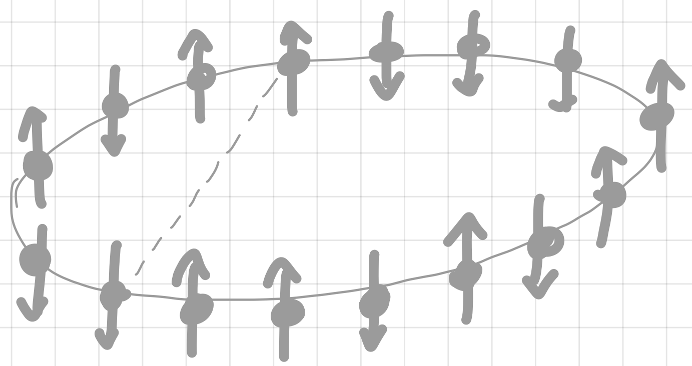
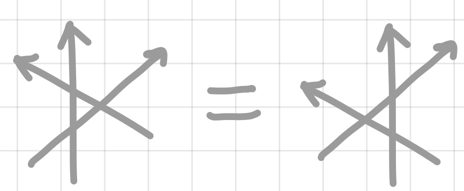


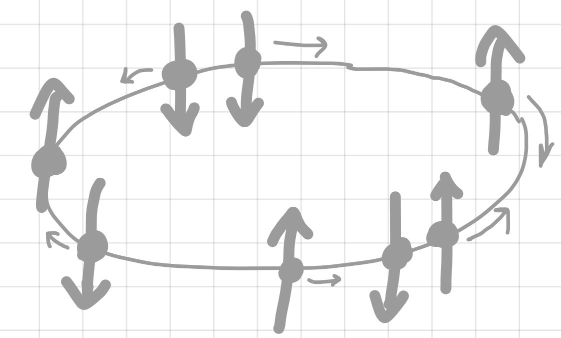
the q-deformed Inozemtsev chain

by



Jules Lamers

Institut de Physique Théorique



based on joint work with Rob Klabbers
to appear this week

Non-perturbative long-range integrability

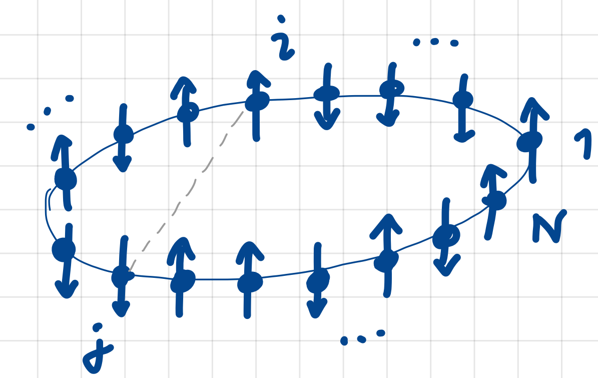
nature atomic, molecular and optical physics

naturally occur in AdS/CFT integrability

consider spin-$\frac{1}{2}$ chain

$$H = \sum_{i<j}^{N} V(i-j)\, \overbrace{(1-P_{ij})}^{(1-\vec{\sigma}_i\cdot\vec{\sigma}_j)/2}$$

pairwise · pair potential · spin interaction



$$\delta_{|x| \bmod N,1} \xleftarrow{\kappa\to\infty} \sim \wp(x) \text{ periods } N, \tfrac{i\pi}{\kappa} \xrightarrow{\kappa\to 0} \frac{(\pi/N)^2}{\sin(\frac{\pi}{N}x)^2} = \frac{1}{\mathrm{crd}^2}$$

guest role in AdS/CFT
Serban Staudacher 04

"SU(2)$_1$ WZW on the lattice"
Ha et al 92
Bernard et al 94
Bouwknegt et al 94 96

Heisenberg XXX ← **Inozemtsev** → **Haldane-Shastry**

Heisenberg 28
Bethe 31

Inozemtsev 90
Inozemtsev 90 95 00
Klabbers JL 22

Haldane 88 Shastry 88
Haldane 91 Bernard et al 93

note: all wrapping corrections are automatically included

Non-perturbative long-range integrability

nature atomic, molecular and optical physics

naturally occur in AdS/CFT integrability

	paradigm for q integrable models	challenges our understanding of q integrability	paradigm for long-range q integrab
Yangian	✓ Faddeev et al late 70s	? unknown	✓ different from Heis Ha et al 92 Bernard et al 93
commuting hamiltonians	✓ Sutherland 70	✓ conjectured partial proof Dittrich Inozemtsev 08	✓ Inozemtsev 90 Ha et al 92 Talstra Haldane 95
exactly solvable	✓ up to solving BAE Bethe 31	✓ up to solving BAE Inozemtsev 90 95 00 Klabbers JL 22	✓ explicit Haldane 91 Bernard et al 93
	Heisenberg XXX Heisenberg 28 Bethe 31	← **Inozemtsev** → Inozemtsev 90 Inozemtsev 90 95 00 Klabbers JL 22	**Haldane-Shastry** Haldane 88 Shastry 88 Haldane 91 Bernard et al 93

note: all wrapping corrections are automatically included

Lessons from controlled symmetry breaking

$$H_{Heis} = \frac{1}{2}\sum \left(\Delta - \sigma_i^x \sigma_{i+1}^x - \Gamma\, \sigma_i^y \sigma_{i+1}^y - \Delta\, \sigma_i^z \sigma_{i+1}^z\right)$$

↓ degree of spin symmetry

anisotropic — **Heisenberg XYZ**
Sutherland 70
Baxter 73

face/vertex transformation
Q-operator

↓

partially (an)isotr
S^z symm
— **Heisenberg XXZ**
Orbach 58
Yang Yang 66

↓

isotropic
SU(2) symm
— **Heisenberg XXX**
Heisenberg 28
Bethe 31

Bethe ansatz

New integrable unification of Ino and qHS

interaction range

degree of spin symmetry

	nearest neighbour	intermediate range	long range
anisotropic	**Heisenberg XYZ** Sutherland 70 Baxter 73		
partially (an)isotr S^z symm	**Heisenberg XXZ** Orbach 58 Yang Yang 66		
isotropic SU(2) symm	**Heisenberg XXX** Heisenberg 28 Bethe 31	**Inozemtsev** Inozemtsev 90 Inozemtsev 90 95 00 Klabbers JL 22	**Haldane-Shastry** Haldane 88 Shastry 88 Haldane 91 Bernard et al 93

New integrable unification of Ino and qHS

interaction range

degree of spin symmetry

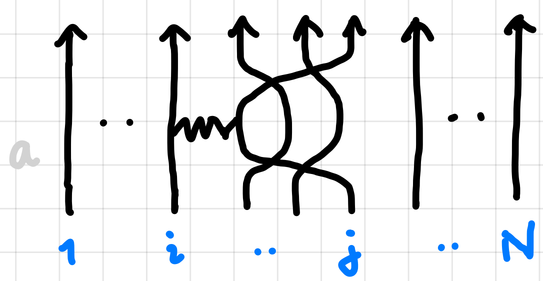
	nearest neighbour	intermediate range	long range
anisotropic	**Heisenberg XYZ** Sutherland 70 Baxter 73	?	?
partially (an)isotr S^z symm	**Heisenberg XXZ** Orbach 58 Yang Yang 66	**q-deformed Inozemtsev** Klabbers JL 2306.0xxxx	**q-deformed HS** Uglov 95 JL 18 JL Pasquier Serban 22
isotropic SU(2) symm	**Heisenberg XXX** Heisenberg 28 Bethe 31	**Inozemtsev** Inozemtsev 90 Inozemtsev 90 95 00 Klabbers JL 22	**Haldane-Shastry** Haldane 88 Shastry 88 Haldane 91 Bernard et al 93

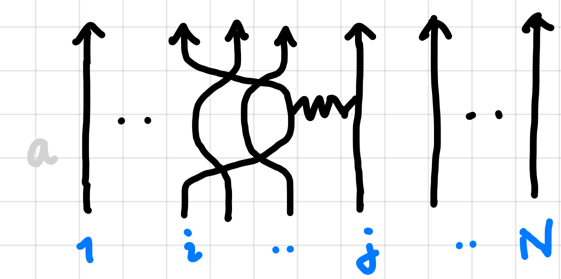
The q-deformed Inozemtsev chain: anatomy

isotropic: $H^{\mathrm{Ino}} = \sum_{i<j}^{N} \wp(i-j)\,\underbrace{(1-P_{ij})}$ 'chiral' decompositions

$$P_{j-1,j}\cdots P_{i+1,i+2}(1-P_{i,i+1})P_{i+1,i+2}\cdots P_{j-1,j} = P_{i\,i+1}\cdots P_{j-2,j-1}(1-P_{j-1,j})P_{j-2,j-1}\cdots P_{i,i+1}$$

deformed level: **'chiral' hamiltonians** (like for q-deformed Haldane-Shastry)

$H^L = \sum_{i<j}^{N} V(i-j) \times$ 

$H^R = \sum_{i<j}^{N} V(i-j) \times$ 

Klabbers JL

both with **potential**

$$V(x) = -\frac{\rho(x+\eta)-\rho(x-\eta)}{\theta(2\eta)}$$ periods $N, i\pi/\kappa$

η anisotropy

Klabbers JL 22

$\xrightarrow[\eta\to 0]{} \wp(x) + \mathrm{cst} \xrightarrow[N\to\infty]{} \frac{1}{\sinh^2(\kappa x)}$

$\kappa>0$ ~ interaction range

pre pot $\rho(x) = \frac{\theta'(x)}{\theta(x)}$

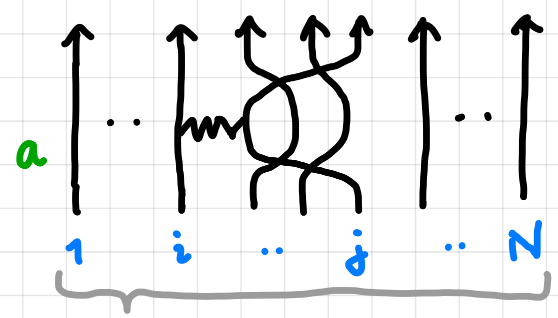
Jacobi $\theta(x)$ $\left(\xrightarrow{N\to\infty} \kappa^{-1}\sinh(\kappa x)\right)$ entire, odd, $\theta'(1)=1$, $\begin{cases} \theta(x+i\pi/\kappa) = -\theta(x) \\ \theta(x+N) = -e^{\kappa(N+2x)}\theta(x) \end{cases}$

The q-deformed Inozemtsev chain: anatomy

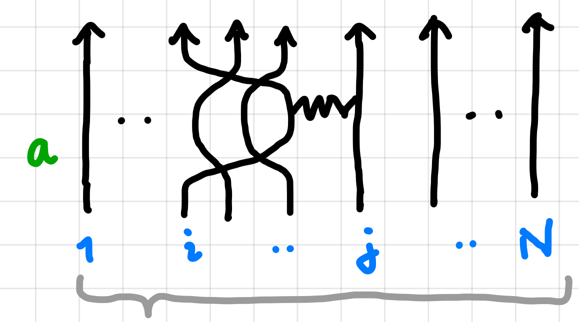
isotropic: $$H^{\mathrm{Ino}} = \sum_{i<j}^{N} \wp(i-j)\,\underbrace{(1-P_{ij})}$$ 'chiral' decompositions

$$P_{j-1,j}\cdots P_{i+1,i+2}(1-P_{i,i+1})P_{i+1,i+2}\cdots P_{j-1,j} = P_{i\,i+1}\cdots P_{j-2,j-1}(1-P_{j-1,j})P_{j-2,j-1}\cdots P_{i,i+1}$$

deformed level: 'chiral' hamiltonians

$$H^{L} = \sum_{i<j}^{N} V(i-j) \times$$ 

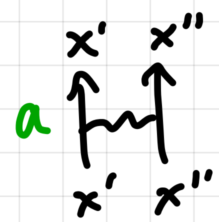
$$\prod_{j>k>i}^{\leftarrow} P_{k\,k+1}(j-k)\cdot E_{i\,i+1}(i-j)\cdot \prod_{i<k<j}^{\rightarrow} P_{k\,k+1}(k-j)$$

$$H^{R} = \sum_{i<j}^{N} V(i-j) \times$$ 

$$\prod_{i<k<j}^{\rightarrow} P_{k\,k+1}(i-k)\cdot E_{i\,i+1}(i-j)\cdot \prod_{j>k>i}^{\leftarrow} P_{k\,k+1}(k-i)$$

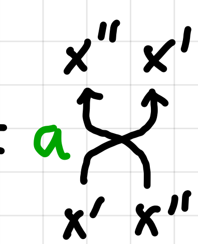
Klabbers JL

with

$$E_{i\,i+1}(x) = \frac{P_{i\,i+1}(-x)P'_{i\,i+1}(x)}{\theta(x)\,V(x)} = $$ 

$$P_{i\,i+1}(x) = \check{R}_{i\,i+1}(x; a-(\sigma_1^z+\cdots+\sigma_{i-1}^z))$$

$$P_{i\,i+1}(x-y)P_{i+1\,i+2}(x)P_{i\,i+1}(y)$$
YBE $$= P_{i+1\,i+2}(y)P_{i\,i+1}(x)P_{i+1\,i+2}(x-y)$$ & (braid) unitarity

elliptic **dynamical** R-matrix $$\check{R}(x;a) = \begin{pmatrix} 1 & & & \\ & f(x,\eta;\eta a) & f(x,\eta;\eta a) & \\ & f(\eta,x;-\eta a) & f(\eta,x;-\eta a) & \\ & & & 1 \end{pmatrix} = $$ 

$$f(x,\eta;a) = \frac{\theta(\eta+a)\,\theta(x)}{\theta(a)\,\theta(x+\eta)}$$

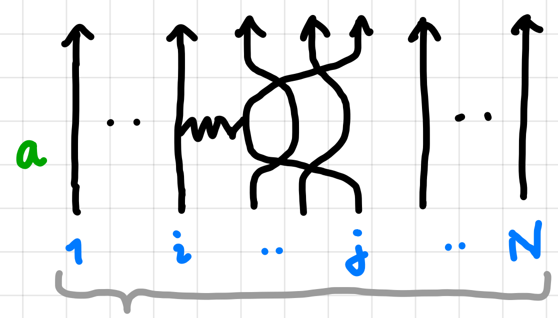
Felder 94

e.g. $P_{23}(x)\,|s_1 s_2 s_3\rangle = |s_1\rangle \otimes \check{R}(x, a-s_1)\,|s_2 s_3\rangle$ $\qquad x = x'-x''$

The q-deformed Inozemtsev chain: properties

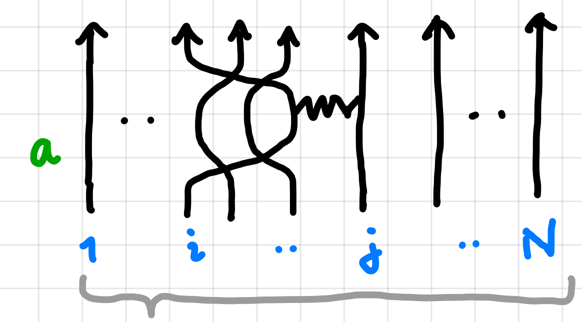
key limits	Inozemtsev	q-def Haldane-Shastry
	$\eta \to 0$ & $a \to -i\infty$	$\kappa \to 0^+$ & $a \to -i\infty$
$V(x) = -\dfrac{\rho(x+\eta)-\rho(x-\eta)}{\theta(2\eta)}$	$\wp(x) + \text{cst}$	$\dfrac{1}{\sin\frac{\pi}{N}(x+\gamma)\,\sin\frac{\pi}{N}(x-\gamma)}$
$E_{i\,i+1}(x) = \dfrac{P_{i\,i+1}(-x)\,P'_{i\,i+1}(x)}{\theta(x)\,V(x)}$	$1 - P_{i\,i+1}$	$e_{i\,i+1}$ Temperley-Lieb
$P_{i\,i+1}(x) = \check{R}_{i\,i+1}(x; a-(\sigma^z_1+\dots+\sigma^z_{i-1}))$	$P_{i\,i+1}$	$\check{R}_{i\,i+1}(x)$ Jimbo $U_q\widehat{\mathfrak{sl}}_2$

$$H^L = \sum_{i<j}^{N} V(i-j) \times$$



$$\prod_{j>k>i}^{\leftarrow} P_{k\,k+1}(j-k) \cdot E_{i\,i+1}(i-j) \cdot \prod_{i<k<j}^{\leftarrow} P_{k\,k+1}(k-j)$$

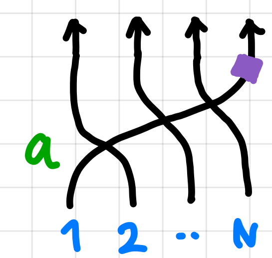
$$H^R = \sum_{i<j}^{N} V(i-j) \times$$



$$\prod_{i<k<j}^{\rightarrow} P_{k\,k+1}(i-k) \cdot E_{i\,i+1}(i-j) \cdot \prod_{j>k>i}^{\leftarrow} P_{k\,k+1}(k-i)$$

Klabbers JL

integrability belong to hierarchy of commuting ham's that also includes twisted translation



$= K_N^{-1}\, P_{N-1,1}(1-N) \cdots P_{12}(1-2)$

(diagonal) twist

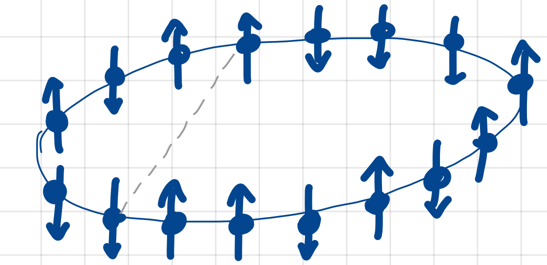
Klabbers JL

new limits short range: dynamical twisted Heisenberg XXZ

related to affine Temperley-Lieb

intermediate isotropic: a-dependent generalisation of H^{Ino}

Summary: integrable long-range spin chains



interaction range

degree of spin symmetry

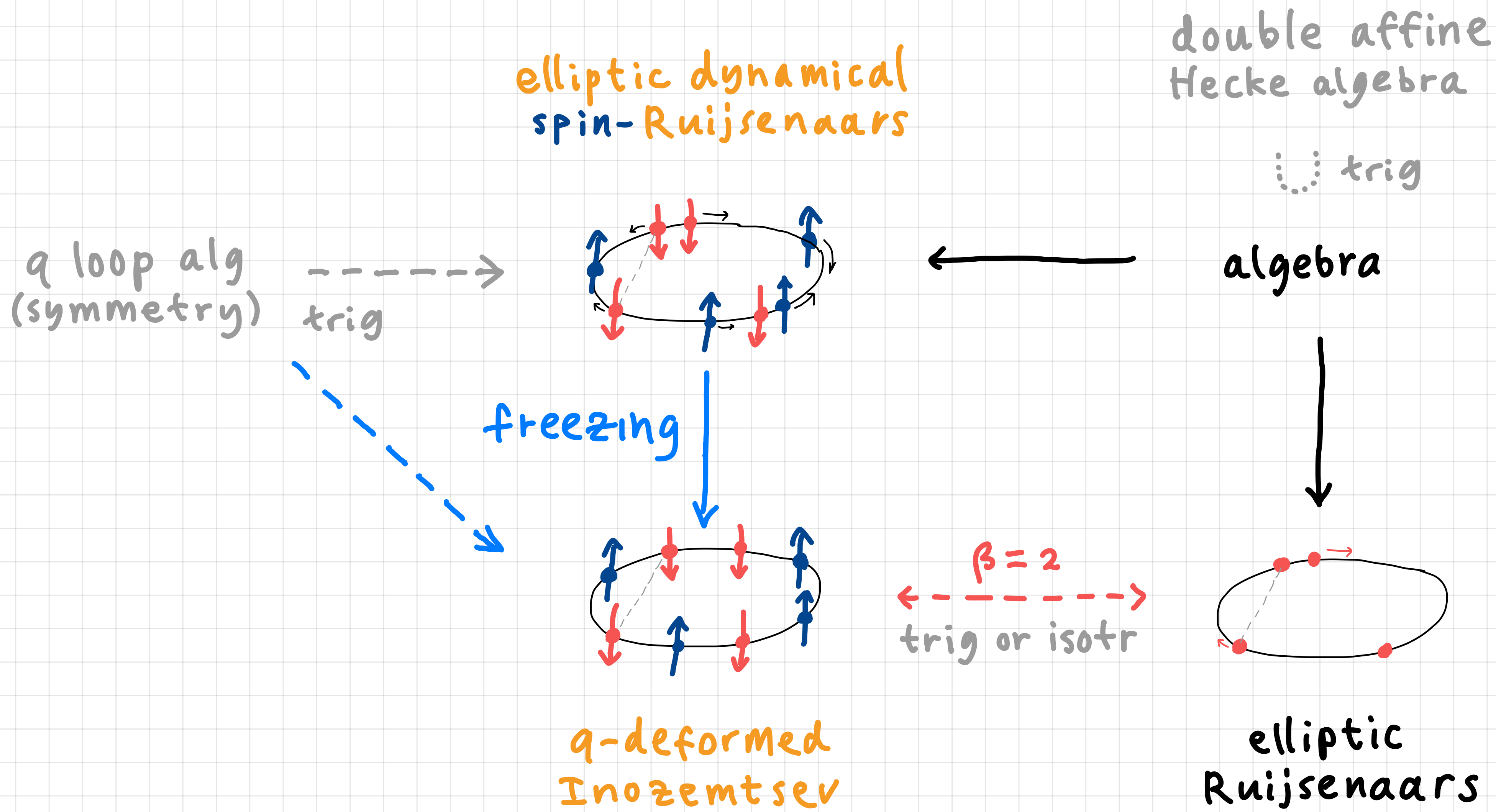
	nearest neighbour	intermediate range	long range
anisotropic	Heisenberg XYZ Sutherland 70 Baxter 73	?	?
partially (an)isotr S^z symm	Heis XXZ Orbach 58 Yang Yang 66	q-deformed Inozemtsev Klabbers JL	q-deformed HS Uglov 95 JL 18 JL Pasquier Serban 22
isotropic SU(2) symm	Heisenberg XXX Heisenberg 28 Bethe 31	Inozemtsev Inozemtsev 90 Inozemtsev 90 95 00 Klabbers JL 22	Haldane-Shastry Haldane 88 Shastry 88 Haldane 91 Bernard et al 93

dynamical twisted XXZ

a-dep Inozemtsev

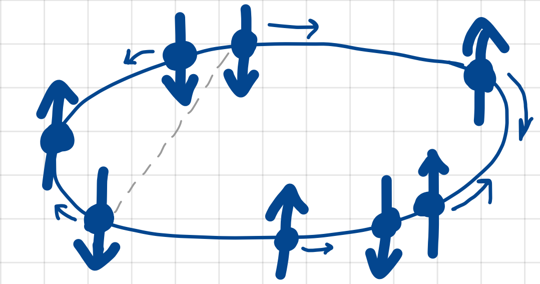
Behind the scenes: quantum many-body systems

Key to long-range integrable spin chains:
connection to QMBS



Polychronakos 92
Bernard et al 93
Talstra Haldane 94
Uglov 95
Inozemtsev 95
Klabbers JL 22
JL Pasquier Serban 22
Matushko Zotov 23
Klabbers JL

Outlook: integrable quantum many-body systems



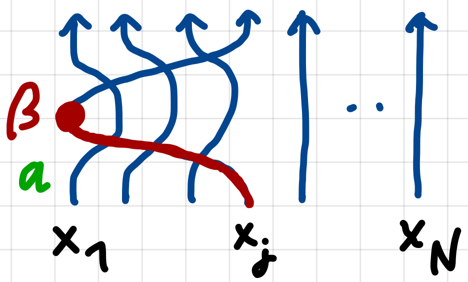
interaction range

	nearest neighbour contact (positions)	intermediate range elliptic (positions)	long range trig (positions)
(?) elliptic (momenta)	?	'DELL'	?
relativistic trig (momenta) difference op's	?	ell Ruijsenaars dynamical spin Klabbers JL 2306.0xxxx a-dep ell Cal-Sut	trig Ruijsenaars-Macdonald
non-rlt rational (momenta) differential op's	~Lieb-Liniger?	ell Cal-Sut	trig Cal-Sut

towards **general theory** for
long-range quantum integrability

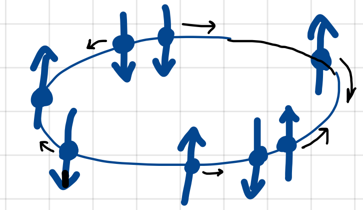
Bonus: dynamical spin-Ruijsenaars and freezing

hierarchy of commuting difference operators, including

$$\widetilde{D}_1 = \sum_{j=1}^{N} A_j \times$$ 

$$= A_1 \Gamma_1 + A_2 \, P_{12}(x_2-x_1)\,\Gamma_2\, P_{12}(x_1-x_2) + A_3 \, P_{23}(x_3-x_2)\,P_{12}(x_3-x_1) \times \Gamma_3 \, P_{12}(x_1-x_3)\,P_{23}(x_2-x_3) + \dots$$

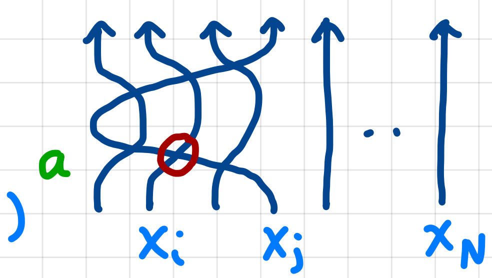
Klabbers JL



$$A_j = \prod_{k(\neq j)}^{N} \frac{\theta(x_k - x_j + \eta)}{\theta(x_k - x_j)} \qquad \Gamma_j : x_i \mapsto x_i - \delta_{ij}\,\beta$$

semiclass limit $\beta \to 0$

$$\Gamma_j = 1 + \frac{i\kappa}{\pi}\,\beta\,\partial_{x_j} + O(\beta)^2$$

$$\widetilde{D}_1 = 1 + \frac{i\kappa}{\pi}\,\beta \sum_{j=1}^{N} A_j \Big(\partial_{x_j} + \sum_{i(<j)}$$ 

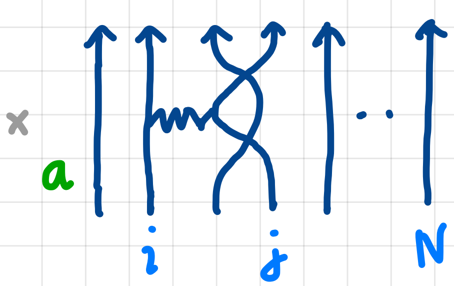
$$\Big) + O(\beta)^2$$

class equilib $x_k \mapsto k$

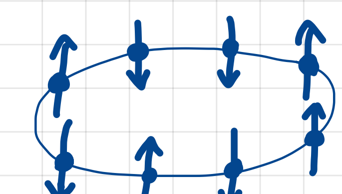
remove consistently

~ spin chain

Polychronakos 92
Talstra Haldane 94
Uglov 95
JL Pasquier Serban 22
Matushko Zotov 23
Klabbers JL

yields $\quad H^L = \sum_{i<j}^{N} V(i-j) \times$ 

$$E_{i\,i+1}(x) = \frac{P_{i\,i+1}(-x)\,P'_{i\,i+1}(x)}{\theta(\eta)\,V(x)}$$



Klabbers JL