INTEGRABILITY OF JORDANIAN DEFORMATIONS

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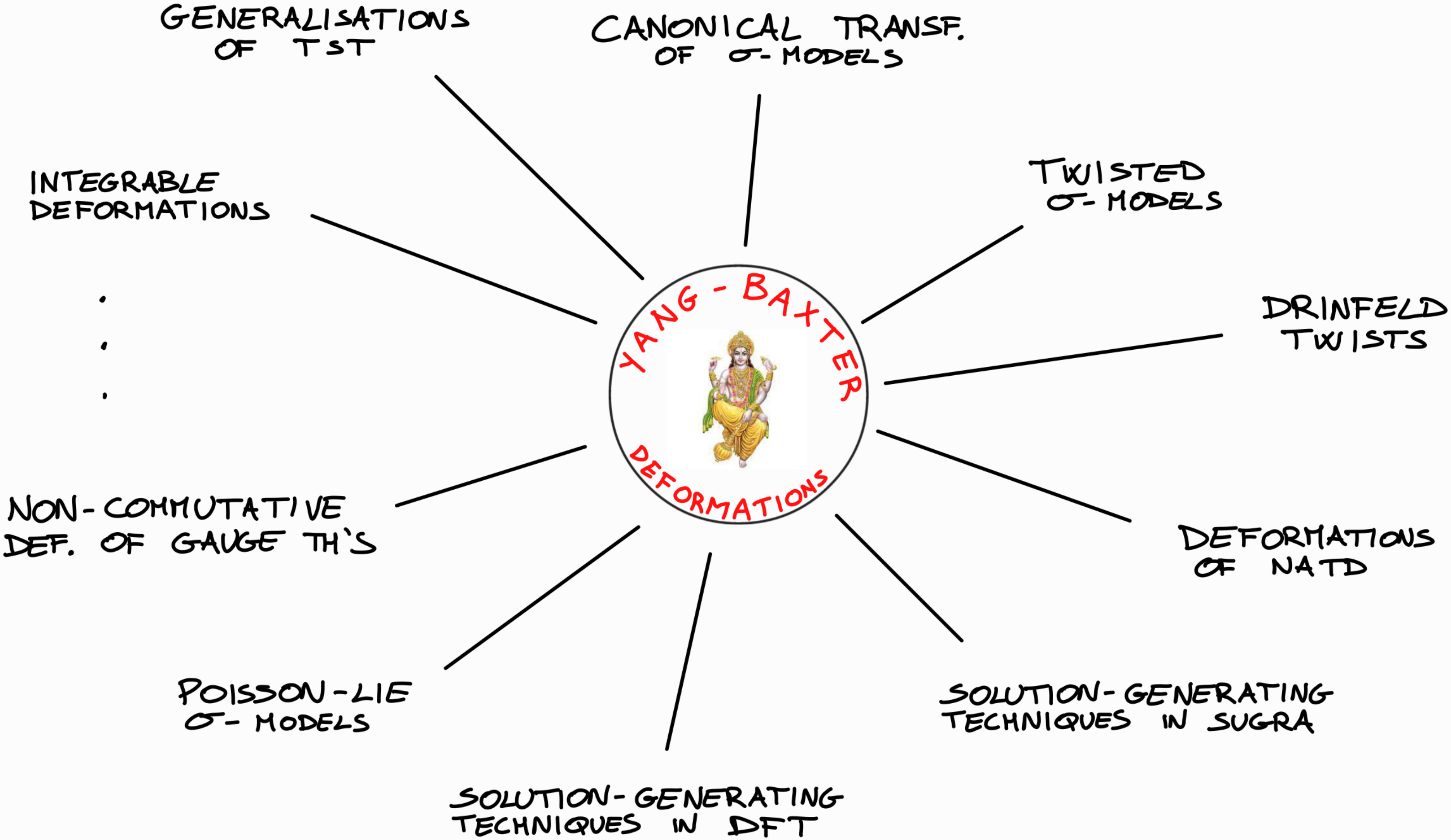
THIS TALK:

- INTRODUCTION ON YANG-BAXTER DEFORMATIONS
- INTEGRABILITY OF JORDANIAN DEFORMATIONS

MANY INCARNATIONS



YANG-BAXTER

DEFORMATIONS ARE...

... INTEGRABLE DEFORMATIONS

[KLIMČIK '09 – DELDUC, MAGRO, VICEDO '13
KAWAGUCHI, MATSUMOTO, YOSHIDA '14 – ...]

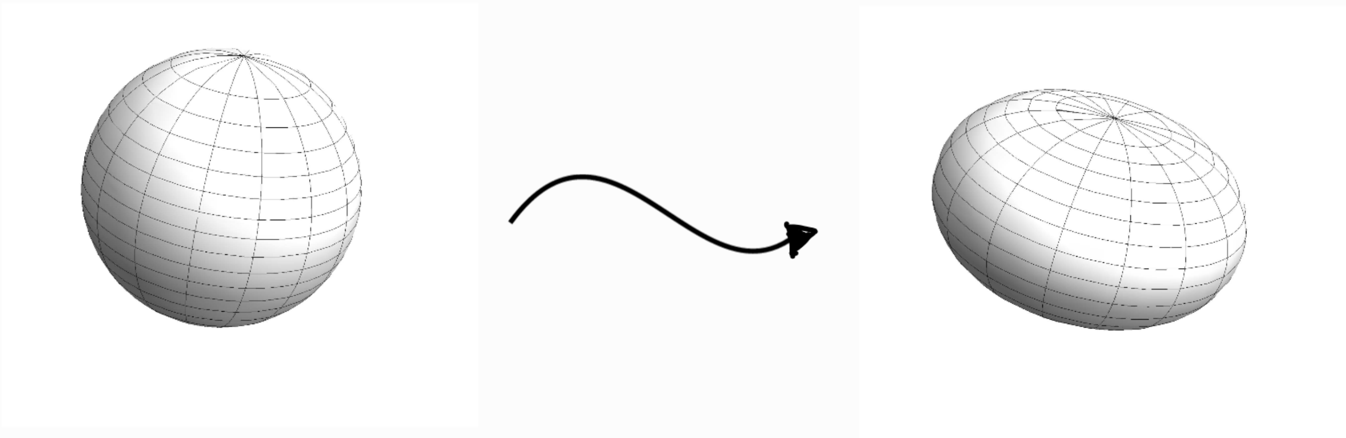
INTEGRABLE σ- MODEL
WITH ISOMETRY GROUP G

$+$

$R : Lie(G) \longrightarrow Lie(G)$

$\Longrightarrow$

INTEGRABLE σ- MODEL
WITH DEFORMATION PARAMETER η
AND G BROKEN



$$\boxed{R(T_a) = R_a{}^b \, T_b}$$

- **ANTISYMMETRY** $\qquad R^T = -R$

- **CLASSICAL YANG-BAXTER EQ.** (CYBE) ON $Lie(G)$

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

$$\boxed{r = R^{ab} \, T_a \wedge T_b}$$

EXAMPLE : DEFORMATION OF PCM

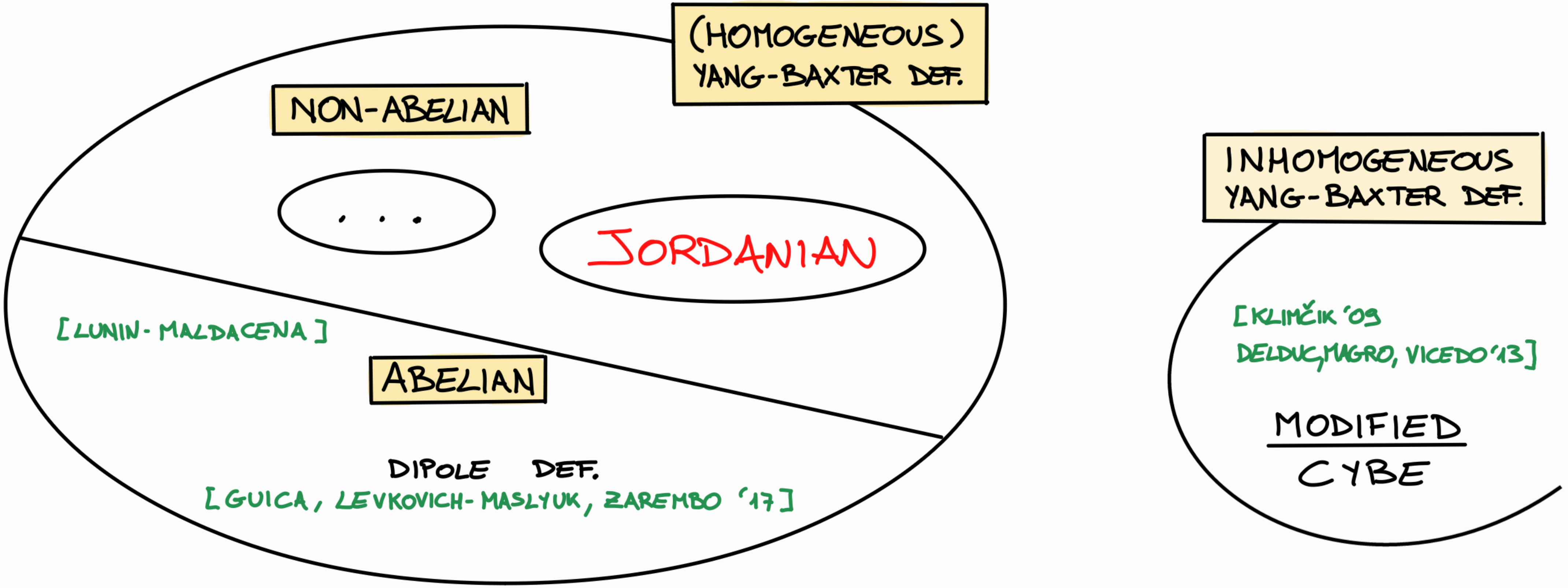
$$S = \int d^2\sigma \, \mathrm{Tr}\left[\partial_+ g g^{-1} \frac{1}{1-\eta R} (\partial_- g g^{-1})\right]$$

...GENERALISATIONS OF TsT DEF.'s

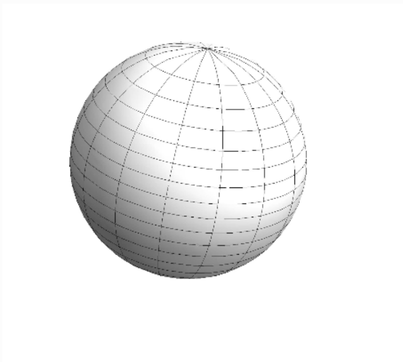
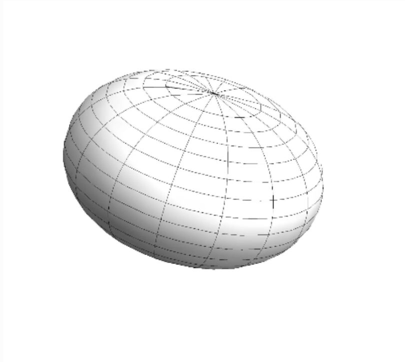
$$r = T_1 \wedge T_2 , \qquad [T_1, T_2] = 0$$

$\Longrightarrow$ TsT = T-DUALITY, SHIFT, T-DUALITY

[OSTEN, VAN TONGEREN '16]



... CANONICAL TRANSFORMATIONS OF σ- MODELS

 $\simeq$ 

<u>TsT</u> : $\phi'^{\mu} = \tilde{\phi}'^{\mu} + \eta\, r^{\mu\nu}\, \tilde{p}_{\nu}$, $\quad p_{\mu} = \tilde{p}_{\mu}$

$\Rightarrow$ LAX CONNECTION $\quad [\partial_+ + \mathcal{L}_+ , \partial_- + \mathcal{L}_-] = 0$

<u>EXAMPLE : YANG-BAXTER DEF. OF PCM</u>

$$\mathcal{L}_{\pm}(z) = \frac{A_{\pm}}{1 \mp z} , \qquad g A_{\pm} g^{-1} = \frac{1}{1 \pm \eta R} \left(\partial_{\pm} g g^{-1} \right)$$

... TWISTED MODELS

[VICEDO '15 – VAN TONGEREN '18 – RB, DRIEZEN, MIRAMONTES '21]

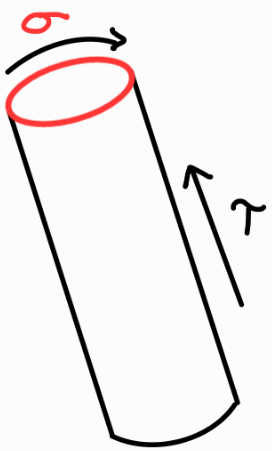
$$\mathcal{L}_\pm = \frac{A_\pm}{1 \mp z}$$

$$A_\pm = \tilde{J}_\pm = \tilde{g}^{-1} \partial_\pm \tilde{g}$$ UNDEFORMED PCM

IF PERIODIC BOUNDARY CONDITIONS FOR YANG–BAXTER

$$g(2\pi) = g(0)$$

$$\Downarrow$$



TWISTED BOUNDARY CONDITIONS FOR PCM

$$\boxed{\tilde{g}(2\pi) = W \tilde{g}(0)}$$

TWIST $W \neq 1$

$$W(\eta) \xrightarrow{\eta \to 0} 1$$

... DRINFELD TWISTS

$\mathcal{F}$ = DRINFELD TWIST

$$\tilde{\Delta}(x) = \mathcal{F}\Delta(x)\mathcal{F}^{-1}$$ COPRODUCT

$$\tilde{\mathcal{R}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$$ R-MATRIX

DRINFELD: TWISTS CONTINUOUSLY CONNECTED TO THE IDENTITY

$$\mathcal{F} = 1 + \eta\,\mathcal{F}^{(1)} + \mathcal{O}(\eta^2)$$

ARE IN 1 TO 1 CORRESPONDENCE WITH

ANTISYMMETRIC SOLUTIONS OF THE CYBE

MORE IN STIJN'S TALK AND [VAN TONGEREN '15, '16]

$\neq$ q-DEFORMATION FOR INHOMOGENEOUS YB

... σ-MODELS WITH POISSON-LIE SYMMETRIES

$$\boxed{[K_a, K_b] = f_{ab}{}^c K_c}$$

$E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$ $\qquad \mathcal{L}_{K_a} E_{\mu\nu} \neq 0$ NOT KILLING VECTORS

$$\boxed{\mathcal{L}_{K_a} E_{\mu\nu} = \tilde{f}^{bc}{}_a \, K_b{}^\rho K_c{}^\sigma E_{\mu\rho} E_{\nu\sigma}}$$

$$[T_a, T_b] = f_{ab}{}^c T_c \qquad [\tilde{T}^a, \tilde{T}^b] = \tilde{f}^{ab}{}_c \tilde{T}^c$$

$$[T_a, \tilde{T}^b] = \tilde{f}^{bc}{}_a T_c - f_{ac}{}^b \tilde{T}^c$$

DRINFELD DOUBLE

... DEFORMATIONS OF NON-ABELIAN T-DUALITY

[HOARE, TSEYTLIN '16 — RB, WULFF '16]

DUALISATION OF $F \subset G$

~ GENERALISATION OF BUSCHER

DEFORMATION BY PARAMETER ζ AND 2-COCYCLE $\hat{\omega}$ ON $\mathrm{Lie}(F)$

$$\hat{\omega}(x,[y,z]) + \hat{\omega}(y,[z,x]) + \hat{\omega}(z,[x,y]) = 0$$

YANG-BAXTER DEF.

$\zeta = \eta^{-1}$ $\quad \mathrm{Lie}(F) = \mathrm{Im}(R)$

$\omega = R^{-1}$ $\quad$ 2-COCYCLE $\Leftrightarrow$ CYBE

UP TO COORDINATE REDEFINITIONS AND B-FIELD GAUGE TRANSF.

<u>SINGULAR</u> IN $\eta \rightarrow 0$ LIMIT

...SOLUTION-GENERATING TECHNIQUES IN SUGRA

[RB, WULFF '16,'18 – HRONEK, WULFF '20]

SUGRA SOLUTION + $R^T = -R$ + CYBE + **UNIMODULARITY** $\boxed{R^{ab} f_{ab}{}^{c} = 0}$

$\Downarrow$

SUGRA SOLUTION

$\Downarrow$

CAN GENERATE DEF.'S OF E.G. $AdS_5 \times S^5$

$\Downarrow$

DEF.'S OF AdS_5/CFT_4
WITH LESS SYMMETRIES

... SOLUTION-GENERATING TECHNIQUES IN DOUBLE FIELD THEORY

[RB, CATAL OZER, DRIEZEN, HASSLER, SAKAMOTO, SAKATANI, WULFF, YOSHIDA, ...]

DOUBLE FIELD THEORY [...]

α' - CORRECTIONS

[RB, VILAR LÓPEZ, WULFF '20]

CLASSIFICATION OF SOLUTION-GENERATING TECHNIQUES INITIATED IN

[RB, DRIEZEN, HASSLER '21]

... NON-COMMUTATIVE GAUGE THEORIES

SEE STIJN'S TALK

JORDANIAN DEFORMATIONS

JORDANIAN DEFORMATIONS

NON-ABELIAN r-MATRICES
⇒ CYBE ON NON-COMPACT ALGEBRA
⇒ DEF.'S OF AdS

JORDANIAN $r = h \wedge e$, $[h, e] = e$

EXTENDED JORDANIAN $r = h \wedge e + \underbrace{\sum_{i=1}^{N} e_{+i} \wedge e_{-i}}$

$N = N_0 + N_1$

N_0: # EVEN (BOSONIC)

N_1: # ODD (FERMIONIC)

[RB, WULFF '16]

UNIMODULAR
$\Updownarrow$
$N_1 = N_0 + 1$

UNIMODULAR JORDANIAN DEF.'S FROM CONTRACTIONS OF INHOMOGENEOUS DEF.

[VAN TONGEREN '19]

JORDANIAN DEF.'S OF $AdS_5 \times S^5$

CLASSIFICATION OF (EXTENDED) JORDANIAN r-MATRICES IN $psu(2,2|4)$ [RB, DRIEZEN '22]

UP TO INNER AUTOMORPHISMS ⇒ GENUINELY DISTINCT

CAN HAVE $N_0 = 0, 1, 2$

CAN I MAKE r UNIMODULAR ADDING FERMIONS?

IT DEPENDS... E.G. $r = D \wedge p_0$ YES, $r = D \wedge p_1$ NO

SEVERAL UNIMODULAR r-MATRICES, SOME CONTINUOUS PARAM.

E.G. $r = [(1+a)D + a J_{03} + b J_{12}] \wedge (p_0 + p_3)$

$\boxed{r = (D - J_{03}) \wedge (p_0 + p_3) + \text{FERMIONIC}}$ PRESERVES 12 SUPERCHARGES

OTHERS HAVE 8, 6, 4, 0 SUPERISOMETRIES

THE TWIST FOR YANG-BAXTER MODELS

$$\boxed{\tilde{g}(2\pi) = W\,\tilde{g}(0)} \qquad \partial_\pm W = 0$$

$$\boxed{W = \exp\left[-\eta\, R(\underbrace{Q}_{\text{CONSERVED CHARGE}})\right] \in F}$$

C.F. TsT : $W = \exp\left[\eta\left(Q_1 T_2 - Q_2 T_1\right)\right]$

$$\underset{\downarrow\ \text{YB}}{g} = F\,\underset{\downarrow\ \text{PCM}}{\tilde{g}}$$

$$\tilde{g}(2\pi) = F^{-1}(2\pi)\, g(2\pi) = F^{-1}(2\pi)\, F(0)\, \tilde{g}(0)$$

$$\boxed{W = F^{-1}(2\pi)\, F(0)}$$

[RB, DRIEZEN, MIRAMONTES '21]

$$\underbrace{g = \mathcal{F}\tilde{g} \quad , \quad A_\pm = \tilde{J}_\pm}$$

$$\downarrow$$

$$\partial_\pm \mathcal{F}\mathcal{F}^{-1} = \pm\eta\, R\left(\mathcal{F}\, \partial_\pm \tilde{g}\tilde{g}^{-1}\, \mathcal{F}^{-1}\right)$$

$$\downarrow$$

$$\boxed{\eta^{-1} P^T\left(\mathcal{F}^{-1}\, \omega(\partial_\pm \mathcal{F}\mathcal{F}^{-1})\, \mathcal{F}\right) = P^T\left(\partial_\pm \tilde{g}\tilde{g}^{-1}\right)}$$

"s.o.v."

$\mathcal{F} = \exp(RX) \in F$

$$\boxed{Y = \frac{1}{\eta} P^T \frac{1 - e^{-\mathrm{ad}_{RX}}}{\mathrm{ad}_{RX}} X}$$

$$e^{-z} de^{z} = \frac{1 - e^{-\mathrm{ad}_z}}{\mathrm{ad}_z} dz$$

$$\partial_\pm Y = \eta^{-1} P^T\left(\mathcal{F}^{-1}\, \omega(\partial_\pm \mathcal{F}\mathcal{F}^{-1})\, \mathcal{F}\right) \quad \rightsquigarrow \quad Y = \int P^T\left(* d\tilde{g}\tilde{g}^{-1}\right)$$

JORDANIAN TWIST

[RB, DRIEZEN, NIETO GARCIA, WYSS '22]

$$\boxed{W = \exp\left[\mathbf{Q}(h + \mathbf{q}\, e)\right]}$$

IF $\mathbf{Q} \neq 0$

$$W = e^{-\mathbf{q} e}\, e^{\mathbf{Q} h}\, e^{\mathbf{q} e}$$

$$\tilde{g}' = e^{\mathbf{q} e}\, \tilde{g}$$

$$\Downarrow$$

$$\boxed{W' = e^{\mathbf{Q} h}}$$

DIAGONALISABLE SECTOR

IF $\mathbf{Q} = 0$ BUT $\mathbf{Q}\mathbf{q} \neq 0$

$$\boxed{W = e^{\mathbf{Q}\mathbf{q} e}}$$

NON-DIAGONALISABLE SECTOR

JORDAN FORM WITH GENERALISED EIGENVALUES $= 1$

CLASSICAL SPECTRAL CURVE

$$r = (D - J_{03}) \wedge (p_0 + p_3) + \text{FERMIONS}$$

DIAGONALISATION OF <u>TWISTED</u> MONODROMY MATRIX

AdS QUASIMOMENTA

$$\begin{cases} \hat{p}_1 = -\frac{2\pi}{x}(E + S) + O(1/x^2) \\ \hat{p}_2 = -\frac{i}{2} Q + \frac{2\pi}{x} S + O(1/x^2) \\ \hat{p}_3 = +\frac{i}{2} Q + \frac{2\pi}{x} S + O(1/x^2) \\ \hat{p}_4 = +\frac{2\pi}{x}(E - S) + O(1/x^2) \end{cases}$$

$$\hat{p}_1 \sim \frac{2\pi}{x}(+E - S_1 + S_2)$$

$$\hat{p}_2 \sim \frac{2\pi}{x}(+E + S_1 - S_2)$$

$$\hat{p}_3 \sim \frac{2\pi}{x}(-E - S_1 - S_2)$$

$$\hat{p}_4 \sim \frac{2\pi}{x}(-E + S_1 + S_2)$$

S = ANGULAR MOMENTUM IN AdS (J_{12})

E = ENERGY, TIME TRASL. BY $P_0 - K_0 - P_3 - K_3$ $\neq$ BMN $P_0 - K_0$

PERIODIC CASE:

$$\Omega(x) = 1 + \frac{1}{x}\hat{Q} + \mathcal{O}(1/x^2)$$

NOETHER CHARGE OF $psu(2,2|4)$

DIAGONALISED AFTER CHOICE OF CARTAN SUBALGEBRA

NON-COMPACT ALGEBRAS CAN HAVE INEQUIVALENT CHOICES OF CARTAN SUBALG.

EXAMPLE: $sl(2,\mathbb{R})$ CARTAN $= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ OR $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

DIFFERENT ADJOINT ORBITS

TWISTED CASE

$$\Omega(x) = W^{-1}\left(1 + \frac{1}{x}\hat{Q} + \mathcal{O}(1/x^2)\right)$$

<u>NOT</u> NOETHER

DIAGONALISATION OF W FORCES CHOICE OF CARTAN

SEMICLASSICAL CORRECTION TO CSC

SEMICLASSICAL QUANTIZATION À LA [GROMOV, VIEIRA '07]

$$p \to p + \delta p$$

NEW POLES FOR "MICRO CUTS"

δp FIXED BY CONSISTENCY CONDITIONS

(GLUING AT CUTS AND ASYMPTOTICS)

AROUND BMN-LIKE SOLUTION:

$$\delta E \neq 0, \quad \delta S = 0, \quad \delta Q = 0$$

CURRENT WORK AND FUTURE DIRECTIONS

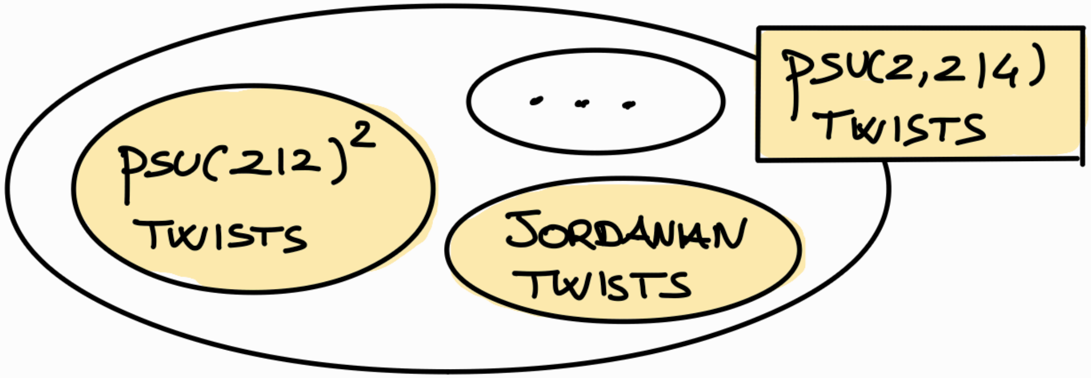
$\eta \to 0$ LIMIT: NOT THE USUAL INTEGRABILITY DESCRIPTION OF $AdS_5 \times S^5$

SUBTLE $Q \to 0$ LIMIT OF CSC

$\neq$ CHOICE OF CARTAN SUBALGEBRA

FINITE-GAP EQUATIONS ~ THERMODYNAMICAL LIMIT OF BETHE EQ'S

DRINFELD-TWISTED WORLDSHEET S-MATRIX ?



$\longrightarrow$ UNDERSTAND THE UNDEFORMED LIMIT

TREE-LEVEL WORLDSHEET S-MATRIX

NO BMN ISOMETRIES ($\neq$ DIAGONAL TsT)

DEFORMATION OF BMN SOLUTION $\Rightarrow$ DIFFERENT LIGHT-CONE GAUGE FIXING (ALSO WHEN $\eta \to 0$) $\Rightarrow$ SCATTERING PROBLEM

RELATION BETWEEN THIS S-MATRIX AND USUAL ONE ?

RELATION BETWEEN S-MATRICES OF DEFORMED AND TWISTED MODELS ?

UNIMODULAR VS NON-UNIMODULAR ?

TWISTED SPIN-CHAIN $\rightsquigarrow$ DEFORMATION OF $\mathcal{N}=4$ SYM ?

$$ds^2 = -\frac{(z^2+\rho^2)(4z^4+\eta^2)}{4z^6}\,dT^2$$

$$+\frac{dz^2+d\rho^2+\rho^2 d\Theta^2-2\,dT\,dV}{z^2} + ds^2_{S^5}$$

$$B=\frac{\eta}{2}\left(\frac{\rho\, d\rho\wedge dT}{z^4}+\frac{dz\wedge dT}{z^3}\right)$$

+ DILATON & RR FLUXES

BMN-LIKE SOLUTION

$$T = \frac{2\tau}{\sqrt{4-\eta^2}}\,, \qquad V = -\frac{\eta^2}{2\sqrt{4-\eta^2}}\,\tau\,, \qquad Z = 1\,, \qquad P = 0$$

$$\left(\text{AND } \phi = \tau \text{ ON } S^5\right)$$

SOLUTION ON THE TWISTED MODEL

$$\tilde{T} = \frac{2\tau}{\sqrt{4-\eta^2}}\,, \qquad \tilde{V} = 0\,, \qquad \tilde{Z} = \exp\left(-\frac{\eta\sigma}{\sqrt{4-\eta^2}}\right)\,, \qquad \tilde{P} = 0$$

$$\left(\text{AND } \phi = \tau \text{ ON } S^5\right)$$