Determinants in self-dual SYM and twistor space

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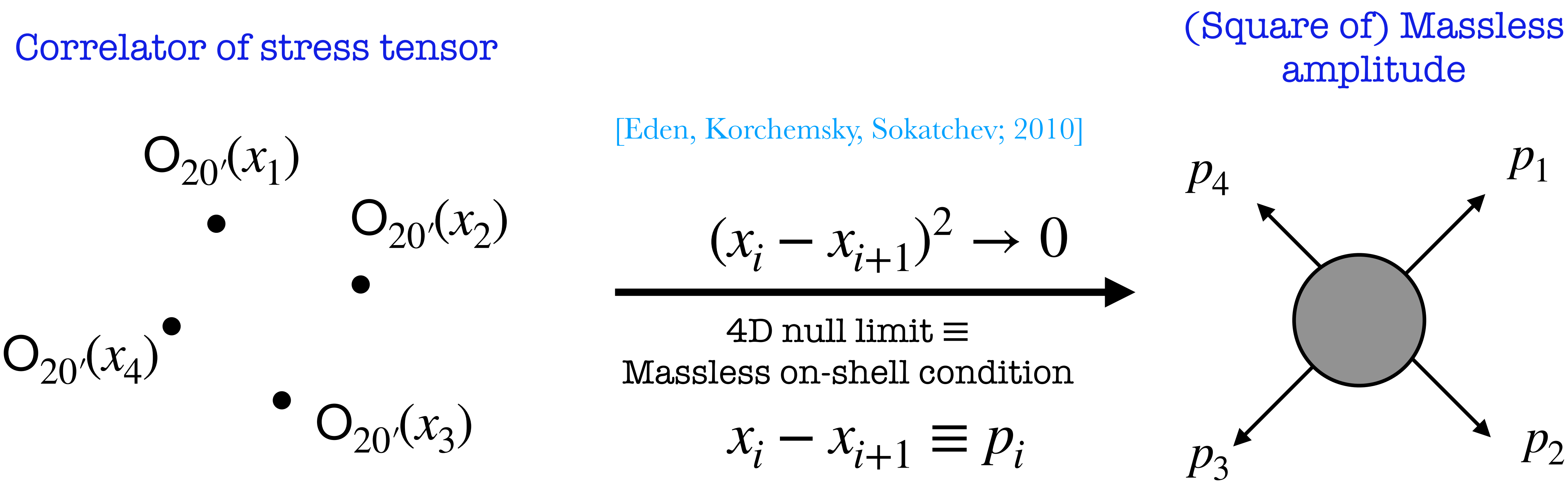
Motivation: hidden symmetries in planar N=4 SYM

- **Correlator-Amplitude duality** relating correlation functions of Stress tensor multiplet and scattering amplitudes of massless gluons. Hidden dual conformal symmetry of amplitudes.



- **Ten-dimensional symmetry** in SUGRA, a hidden 10D (conformal) symmetry emerges in the four-point correlator of the single-trace generating function of 1/2 BPS operators:

$$O(x,y) = \mathrm{Tr}(y\,.\,\Phi(x)) + \underbrace{\frac{1}{2}\mathrm{Tr}(y\,.\,\Phi(x))^2}_{O_{20'}} + \frac{1}{3}\mathrm{Tr}(y\,.\,\Phi(x))^3 + \cdots \quad \text{all KK modes}$$

The 10D symmetry combines spacetime and R-charge space kinematics. The four-point correlator is a functions of 10D distances $X_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$

[Caron-Huot, Trinh, 2018;
Aprile, Drummond, Heslop, Paul]

10D symmetry of loop-integrands

$$S_{\mathcal{N}=4} = S_{\text{self-dual}} + g^2 \int \frac{\mathrm{d}^4 x}{\pi^2} L_{\text{int}}(x)$$

- The loop-integrand is computed in the self-dual sector by including extra Lagrangian

$$\left\langle \prod_{i=1}^{n} \mathsf{O}(x_i, y_i) \right\rangle_{\text{SYM}} = \sum_{\ell=0}^{\infty} \frac{(-g^2)^\ell}{\ell!} \int \frac{\mathrm{d}^4 x_{n+1}}{\pi^2} \cdots \frac{\mathrm{d}^4 x_{n+\ell}}{\pi^2} \left\langle \prod_{i=1}^{n} \mathsf{O}(x_i, y_i) \prod_{k=1}^{\ell} L_{\text{int}}(x_{n+k}) \right\rangle_{\text{SDYM}}$$

[Eden, Petkou, Schubert, Sokatchev]

- At weak coupling we observe the same 10D symmetry on the loop-integrands of four-point correlators of the half-BPS generating function:

$$R_{1234} = \frac{(y_{13}^2\, y_{24}^2)^2}{x_{13}^2\, x_{24}^2} + \frac{y_{12}^2\, y_{23}^2\, y_{34}^2\, y_{41}^2}{x_{12}^2\, x_{23}^2\, x_{34}^2\, x_{41}^2} \left(x_{13}^2 x_{24}^2 - x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2\right) + (1 \leftrightarrow 2) + (1 \leftrightarrow 4)\,.$$

$$\langle \mathsf{O}(x_1, y_1)\mathsf{O}(x_2, y_2)\mathsf{O}(x_3, y_3)\mathsf{O}(x_4, y_4)\mathcal{L}(x_5)\cdots\mathcal{L}(x_l)\rangle_{SDYM}$$

$$= R_{1234}(2x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2) \times \mathcal{H}^{(l)}$$

[Eden, Heslop, Korchemsky, Sokatchev, 2011]
[Chicherin, Drummond, Heslop, Sokatchev, 2015]
[Chicherin, Georgoudis, Goncalves, Pereira, 2018]

- The first three-loop integrands (we consider $y_i = 0$ for $i \geq 5$):

$$\mathcal{H}^{(1)} = \frac{1}{\prod_{1 \leq i < j \leq 5} X_{ij}^2}\,,$$

[Caron-Huot, FC; 2021]

$$\mathcal{H}^{(2)} = \frac{1}{48} \frac{X_{12}^2 X_{34}^2 X_{56}^2 + S_6 \text{ permutations}}{\prod_{1 \leq i < j \leq 6} X_{ij}^2}\,,$$

$$\mathcal{H}^{(3)} = \frac{1}{20} \frac{(X_{12}^2)^2 \left(X_{34}^2 X_{45}^2 X_{56}^2 X_{67}^2 X_{73}^2\right) + S_7 \text{ permutations}}{\prod_{1 \leq i < j \leq 7} X_{ij}^2}\,.$$

Generalization of correlator/massive amplitude duality

- The 10D null limit of the "master" correlator is equal to a scattering amplitude of massive W-bosons in the Coulomb branch(CB).

Generating function of all four-point correlators

$O(x_1, y_1)$

$O(x_2, y_2)$

$O(x_4, y_4)$

$O(x_3, y_3)$

10D vector $X_i = (x_i, y_i)$

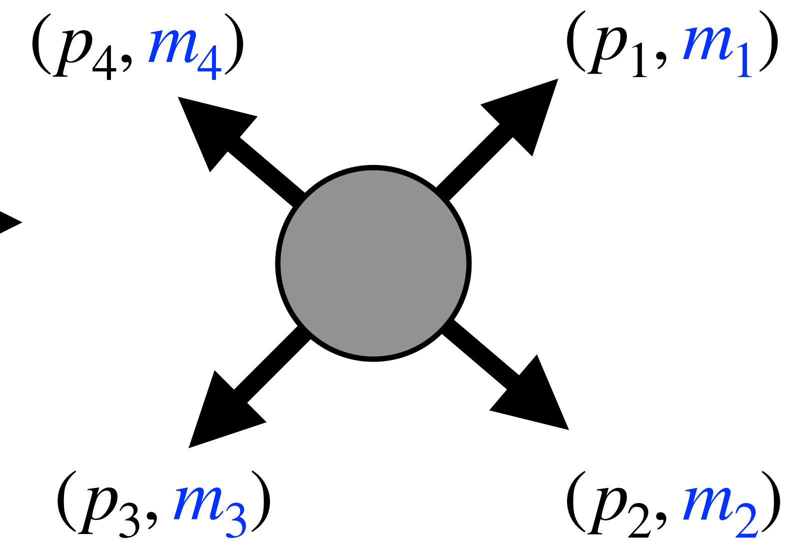
[Caron-Huot, FC; 2021]

$$(X_i - X_{i+1})^2 \to 0$$

10D null limit ≡ massive on-shell condition

$$x_i - x_{i+1} \equiv p_i$$
$$y_i - y_{i+1} \equiv m_i$$

(Square of) four-point massive amplitude



- The vev of the scalar in CB and the masses of the W-bosons are set by y. The 10D null limit $x_{12}^2 + y_{12}^2 = 0$ is equivalent to the mass on-shell condition $p_1^2 + m_1^2 = 0$.
- Thanks to the masses the amplitude is finite and the duality carries over to the integrated form of the correlator and amplitude.
- A special four-point amplitude is known from this duality: the octagon.
- Checked only for four-points. There are not many results for higher-point integrands.

[Bargheer, Fleury, Gonçalves; 2022]

Outline

- Half-BPS determinants as generating functions
- Determinant operators from twistor space
- Correlation functions of determinants
- Graph and Matrix duality
- Some results in planar limit with 10D distances
- Outlook

Half-BPS operators and chiral superspace

- Scalar operators: $\text{Tr}\left(y^{AB}\phi_{AB}(x)\right)^K$ ϕ_{AB} is vector of six scalars

- 6D null polarization vector: $y^{AB}y_{AB} = 0 \longrightarrow y^{AB} = \epsilon^{a'b'} Y^A_{a'} Y^B_{b'}$

- The vector $Y^A_{a'}$ determines which chiral half is absent in the BPS operator.

$$\theta^A_\alpha = W^A_a \left(\theta^+\right)^a_\alpha + Y^A_{a'} (\theta^-)^{a'}_\alpha \ ,$$

$$Y^A_{a'} \frac{\partial\, \mathbb{O}(x, y, \theta)}{\partial \theta^A_\alpha} = \frac{\partial\, \mathbb{O}(x, y, \theta)}{\partial\, (\theta^-)^{a'}_\alpha} = 0 \qquad \text{with} \quad \alpha = 1{,}2,\ a' = 1{,}2$$

- Example: the chiral part of the stress tensor multiplet

$$\mathcal{T}(x, y, \theta^+) = \text{Tr}\left(y^{AB}\,\Phi_{AB}(x)\right)^2 + \theta^{+\,\alpha}_a\, O^{+++,\alpha}_a(x) + \cdots + \left(\theta^+\right)^4 L_{int}$$

- Extensively studied correlator: [Chicherin, Doobary, Eden, Heslop, Korchmesky Mason, Sokatchev, 2015]

$$\langle \mathcal{T}_1 \mathcal{T}_2 \cdots \mathcal{T}_{n+l} \rangle_{SDYM} \rightarrow \langle \mathcal{T}_1 \mathcal{T}_2 \cdots L_{int} L_{int} \rangle_{SDYM} \rightarrow \text{Massless amplitude}$$

- In this talk we turn on SUSY for all higher BPS operators in a single generating function.

Determinant as generating function

- The bosonic generating function is the logarithm of a determinant operator:

$$\mathrm{O}(x, y) = -\log \mathrm{Det}(1 - y\,.\,\Phi(x)) = -\,\mathrm{Tr}\log(1 - y\,.\,\Phi(x))$$

$$= \mathrm{Tr}(y\,.\,\Phi(x)) + \frac{1}{2}\mathrm{Tr}(y\,.\,\Phi(x))^2 + \frac{1}{3}\mathrm{Tr}(y\,.\,\Phi(x))^3 + \cdots \quad \text{all KK modes}$$

- We consider the supersymmetric extensions:

$$\mathbb{O}(x, y, \theta) = \mathrm{O}(x, y) + \text{all susy descendants} \qquad \mathbb{D}(x, y, \theta) = \exp\left[-\mathbb{O}(x, y, \theta)\right]$$

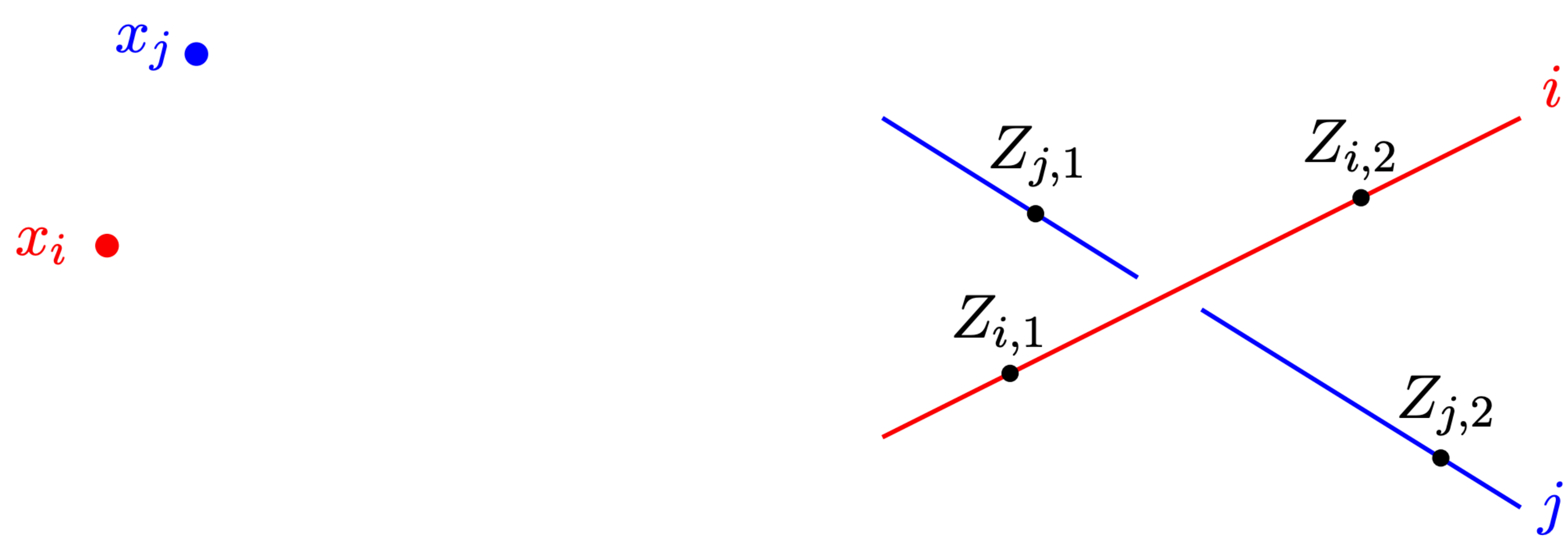
- We turn on all θ and y dependence and get all operators in the same footing:

$$\left\langle \prod_{i=1}^{n} \mathbb{O}(x_i, y_i, \theta_i) \prod_{i=n+1}^{n+\ell} L_{\mathrm{int}}(x_i) \right\rangle_{\mathrm{SDYM}} \xrightarrow{\mathrm{SUSY}} \left\langle \prod_{i=1}^{n+\ell} \mathbb{O}(x_i, y_i, \theta_i) \right\rangle_{\mathrm{SDYM}}$$

- We have a novel construction of the determinant operator $\mathbb{D}(x, y, \theta)$ in twistor space.

- We use matrix duality to make the 10D structure more manifest.

Twistor space



- Homogenous coordinates $\mathcal{Z} = (Z^I, \eta^A)$ with $I, A = 1, 2, 3, 4$.
- Non-local relation: a spacetime point maps to a CP^1 line in supertwistor space $CP^{3|4}$
- Incidence relation $Z^I = \left(\lambda_\alpha, \mu^{\dot\alpha}\right).$ $\mu^{\dot\alpha} = x^{\dot\alpha\beta}\lambda_\beta$ and $\eta^A = \theta^{A\alpha}\lambda_\alpha$
- Recovering spacetime point: $$Z_1^I Z_2^J - Z_2^I Z_1^J \propto \begin{pmatrix} \epsilon_{\alpha\beta} & -ix_\alpha^{\dot\beta} \\ ix_\beta^{\dot\alpha} & -\frac{1}{2}x^2\epsilon^{\dot\alpha\dot\beta} \end{pmatrix}$$

$\mathcal{N} = 4$ SYM in twistor space

- Gauge field in twistor space: $\mathsf{A} = a(Z) + \eta\,\Lambda(Z) + \eta^2\,\phi(Z) + \eta^3\,\Psi(Z) + \eta^4\,g(Z)$

  with $Z = (\lambda_\alpha, i\,x^{\dot\alpha\beta}\lambda_\beta)$ $\quad \eta^A = \theta^{A\alpha}\lambda_\alpha$

- In super-twistor space $CP^{3|4}$: local CS term plus a non-local part defined over a CP1 line

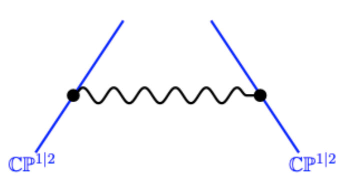
$$S^{twistor}_{\mathcal{N}=4} = \int d\Omega^{3|4}\,\mathsf{A}\bar\partial\mathsf{A} + \mathsf{A}^3 \;+\; g^2\int d^4x d^8\theta \log\det(\bar\partial + \mathsf{A})\Big|_{CP^1}$$

[Witten, 2004; Boels, Mason, Skinner 2006]

$$\overset{\textbf{gauge}}{=} \quad S_{self-dual} \quad + \quad g^2\int d^4x\, L_{int}(x)$$

- In spacetime gauge: $\mathcal{N} = 4$ SYM as a perturbation around its self-dual sector.

- Other gauge (CSW) simplifies the propagator at the cost of introducing a reference twistor:



[Mason, Skinner, 2010]

$$\langle \mathsf{A}(\mathcal{Z}_1)\,\mathsf{A}(\mathcal{Z}_2)\rangle \sim \delta^{2|4}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_*) \equiv \frac{1}{2\pi i}\int_{\mathbb{C}^2}\frac{ds}{s}\frac{dt}{t}\,\delta^{4|4}\left(s\,\mathcal{Z}_1 + t\,\mathcal{Z}_2 + Z_*\right)$$

- Construction of higher BPS operators in twistor space [Staudacher et al. , Sokatchev et al. 2016]

Half-BPS Determinant operator

- We add susy $\psi^b = \lambda_\alpha (\theta^-)^{\alpha b}$ to the twistor line $CP^1 \to CP^{1|2}$ and a mass term:

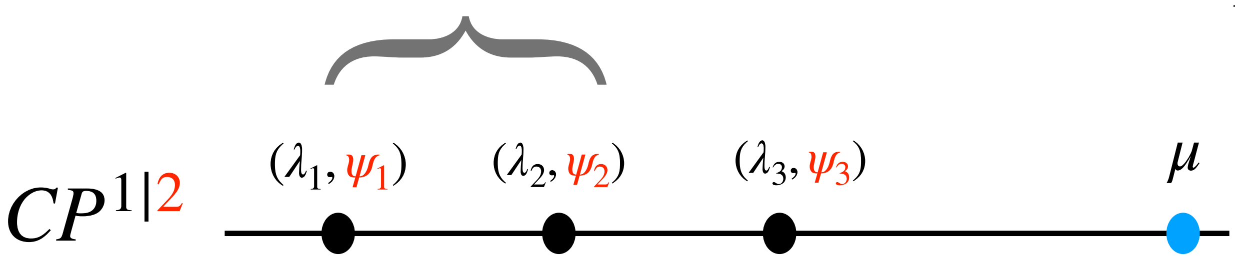
$$\mathbb{D}(x, y, \theta^+) = \int D\alpha\, D\beta\, e^{\int_{CP^{1|2}} d^2\psi\, \alpha(\lambda,\psi)\, (\bar{\partial} + \mathsf{A} + \delta^{1|2}_{\mu,\lambda})\, \beta(\lambda,\psi)}$$

$$= \det(1 - y\,.\,\Phi(x)) + \text{ all susy descendants}$$

- We can integrate-out α, β using their $CP^{1|2}$ propagator:

$$\langle \beta_1 \alpha_2 \rangle \equiv \Delta_{12\mu} = 1 + R_{12\mu}$$

$$R_{123} = \frac{\delta^{0|2}(\psi_1 \langle \lambda_2 \lambda_3 \rangle + \psi_2 \langle \lambda_1 \lambda_2 \rangle + \psi_3 \langle \lambda_1 \lambda_2 \rangle)}{\langle \lambda_1 \lambda_2 \rangle \langle \lambda_1 \lambda_3 \rangle \langle \lambda_2 \lambda_1 \rangle}$$

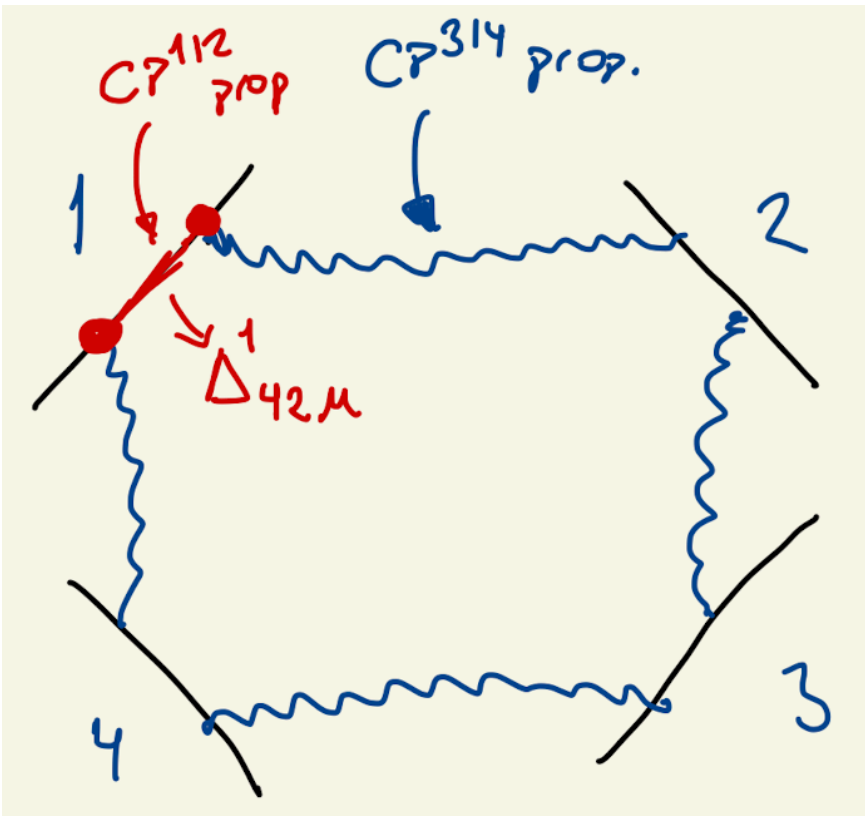


- A series of infinite single-trace vertices

$$\mathbb{O}(x, y, \theta^+) = \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \langle \lambda_i d\lambda_i \rangle\, d^2\psi_i\; \Delta_{12\mu} \Delta_{23\mu} \cdots \Delta_{n1\mu} \times \mathbf{Tr}(\mathsf{A}_1 \mathsf{A}_2\, \cdots\, \mathsf{A}_n) \Big|_{CP^{1|2}}$$

Correlation functions of determinants

- The operators with support on $CP^{1|2}$ superlines interact through the bulk propagator of the gauge field on $CP^{3|4}$.



- Integrals over insertions on the $CP^{1|2}$ are easily carried out thanks to the delta function form of the $CP^{3|4}$ propagators

- At large N_c, the combinatorics of graphs grows with the number of operators and genus.

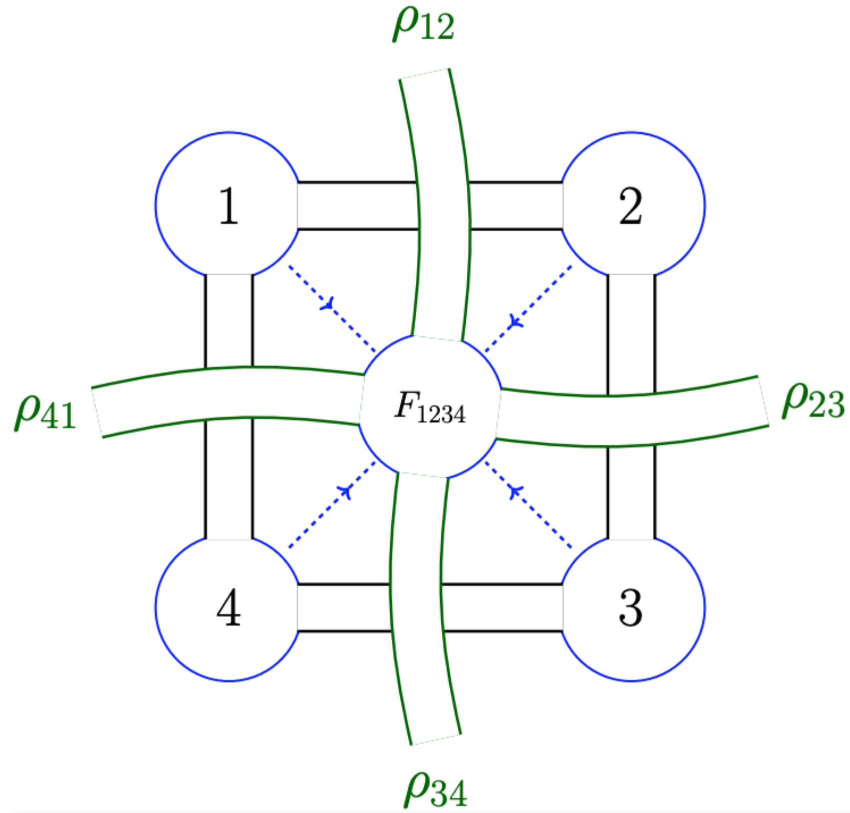
- Spurious poles depending on reference twistor only disappeared and after summing all graphs

- In the genus expansion, for each graph topology and for each pair of operators we obtain a geometric series of 4D scalar propagators

$$d_{ij} = \frac{y_{ij}^2}{x_{ij}^2} \qquad\qquad d_{ij} + d_{ij}^2 + d_{ij}^3 + \cdots = \frac{d_{ij}}{1 - d_{ij}} = \frac{y_{ij}^2}{x_{ij}^2 + y_{ij}^2} \equiv D_{ij}$$

The resummation gives an effective propagator with ten-dimensional denominator

Graph duality and matrix duality

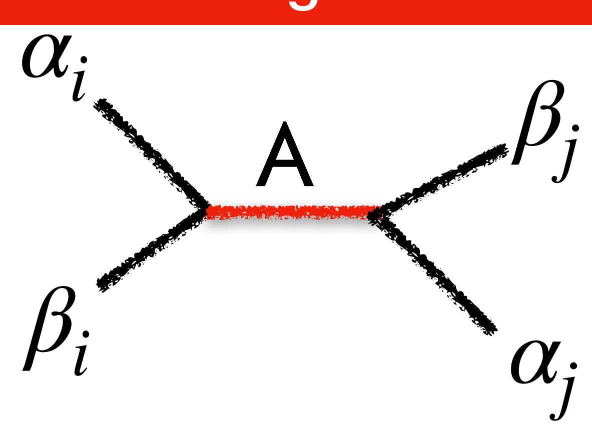


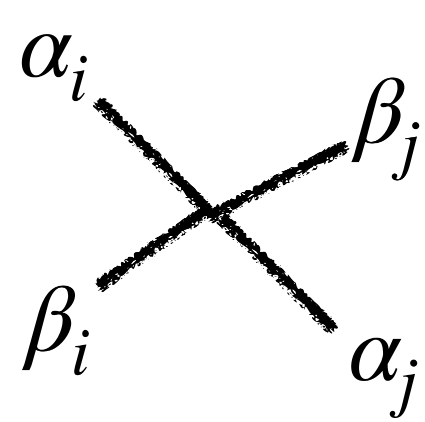
- Model with multi-local vertices representing the faces of the original SDYM graphs.

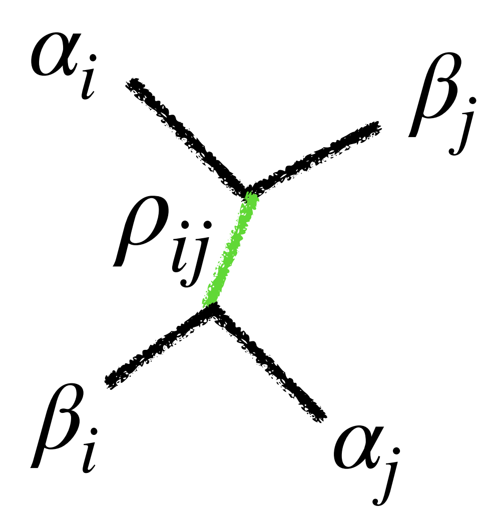
$$F_{1234} = \Delta^{1}_{42\mu_1}\,\Delta^{2}_{13\mu_2}\,\Delta^{3}_{24\mu_3}\,\Delta^{4}_{31\mu_4}\,\rho_{12}\,\rho_{23}\,\rho_{34}\,\rho_{41}$$

- For matrix duality we integrate in and out auxiliary fields

Here we integrate-out A first!







$$\langle \mathbb{D}(x_1, y_1, \theta_1^+) \;\cdots\; \mathbb{D}(x_n, y_n, \theta_n^+) \rangle \;=\; \int D\rho\, e^{-N_c \sum_{i<j}^{n} \frac{\rho_{ij}\,\rho_{ji}}{2\,d_{ij}}}\; \det(1-\Delta\rho)^{N_c}$$

- This duality exchanges the roles of the numbers of colors N_c and determinants n.

[Gopakumar 2010,
Jiang, Komatsu, Vescovi, 2019]

Advantages of new matrix model

- At large N_c, the gaussian term is modified and the new effective propagator has a ten-dimensional denominator:

$$\langle \mathbb{D}(x_1,y_1,\theta_1^+)\ \cdots\ \mathbb{D}(x_K,y_K,\theta_K^+)\rangle = \int D\rho\, e^{-N_c \sum_{i<j}^{K} \frac{\rho_{ij}\rho_{ji}}{2\,d_{ij}}} \det(1-\Delta\rho)^{N_c}$$

$$D_{ij} = \frac{y_{ij}^2}{x_{ij}^2 + y_{ij}^2} \qquad = \int D\rho\, e^{-N_c \sum_{i<j}^{K} \frac{\rho_{ij}\rho_{ji}}{2\,D_{ij}} + N_c\left(\frac{1}{3}Tr\left(\Delta\rho\right)^3 + \frac{1}{4}Tr\left(\Delta\rho\right)^4 + \cdots\right)}$$

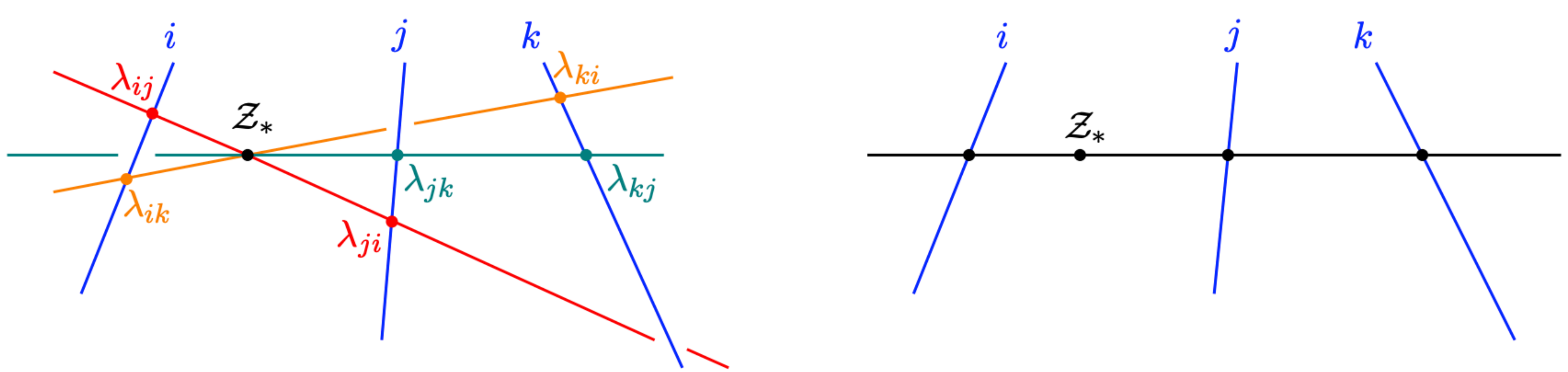
- SUSY can be pull out from the determinant and added as corrections to the Gaussian:

$$\langle\, \mathbb{D}_1 \ldots \mathbb{D}_n \,\rangle_{\mathrm{SDYM}} = \frac{1}{\mathcal{M}} \int [\mathcal{D}\rho] e^{N_c\left(\sum_{i<j} \frac{\rho_{ij}\rho_{ji}}{d_{ij}} + S_{\mathrm{eff}}^{(1)}(\rho,R) + S_{\mathrm{eff}}^{(2)}(\rho,R) + \ldots\right)} \det(\mathbb{I}_n - \rho)^{N_c}$$

$$S_{\mathrm{eff}}^{(1)} = \sum_{i<j<k} (f_{ijk} - f_{ikj}) \left(\frac{R^i_{jk\mu}}{d_{jk}} + \frac{R^j_{ki\mu}}{d_{ki}} + \frac{R^k_{ij\mu}}{d_{ij}} \right),$$

Free from spurious poles

$$S_{\mathrm{eff}}^{(2)} = \sum_{i\neq j\neq k} (f_{ijk} + f_{ij} f_{ik}) \frac{R^i_{jk\mu} R^j_{ik\mu}}{d_{ik}} + \sum_{i\neq j\neq k\neq l} f_{ijkl} \left[\frac{R^i_{jl\mu} R^j_{kl\mu} - R^i_{jkl} R^j_{ik\mu}}{d_{kl}} - \frac{R^i_{jl\mu} R^k_{jl\mu}}{2d_{jl}} \right]$$



Some new results:

$$\langle \mathbb{D}(x_1, y_1, \theta_1^+) \;\cdots\; \mathbb{D}(x_K, y_K, \theta_K^+)\rangle = (\textbf{poly in } D_{ij}) * (\textbf{superconformal invariants})$$

- Six-point NMHV correlator of determinants: with $D_{ij} = \dfrac{y_{ij}^2}{x_{ij}^2 + y_{ij}^2}$

$$\frac{\langle \mathbb{O}_1 \cdots \mathbb{O}_6 \rangle^{\rm NMHV}_{\rm SDYM}}{\prod_{i<j}^6 (1 + D_{ij})} = \prod_{i<j}^{6} D_{ij} \times \left(C^{(6a)} \mathcal{I}^{(6a)} + C^{(6b)} \mathcal{I}^{(6b)} \right) + \left[\prod_{i<j}^{5} D_{ij} \times C^{(5)} \tilde{\mathcal{I}}^{(5)} + 5 \text{ perm} \right]$$

$$N_c^4 C^{(6a)} = 2 + \mathcal{O}(N_c^{-1}) = -N_c^4 C^{(6b)} \ ,$$

$$N_c^4 C^{(5)} = 4 D_{16} D_{26} D_{36} D_{46} D_{56} + 2 \sum_{5} D_{16} D_{26} D_{36} D_{46} - 2 \sum_{10} D_{16} D_{26} (1 + D_{12}) + \mathcal{O}(N_c^{-1})$$

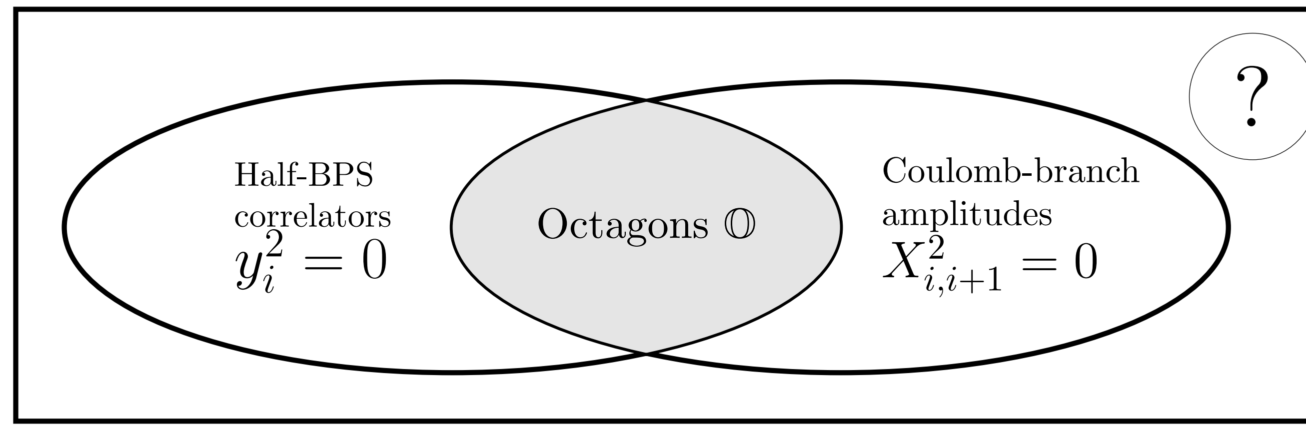
$$\begin{aligned}
\mathcal{I}^{(6b)}_{123456} = & \sum_{90} \frac{R^1_{234} R^4_{561}}{d_{23} d_{56}} \frac{\det [d_{ij}]^{i=1,2,3}_{j=4,5,6}}{\prod_{\substack{i=1,2,3 \\ j=4,5,6}} d_{ij}} \\
& + \sum_{360} \frac{R^1_{234} R^2_{135}}{d_{34} d_{35} d_{36} d_{12} d_{24} d_{45} d_{51}} \left[\frac{d_{12}}{d_{16} d_{26}} - \frac{d_{15}}{d_{16} d_{56}} - \frac{d_{24}}{d_{26} d_{46}} + \frac{d_{45}}{d_{46} d_{56}} \right] \\
& + \sum_{90} \frac{R^1_{234} R^2_{134}}{d_{12} d_{34} d_{35} d_{36} d_{45} d_{46}} \left[\frac{1}{d_{15} d_{26}} + \frac{1}{d_{16} d_{25}} \right] - \sum_{180} \frac{R^1_{234} R^1_{235}}{d_{16} d_{23} d_{24} d_{25} d_{34} d_{35} d_{46} d_{56}}
\end{aligned}$$

- In the limit $y_{ij}^2 \to 0$ the invariant reduces to the stress tensor case.

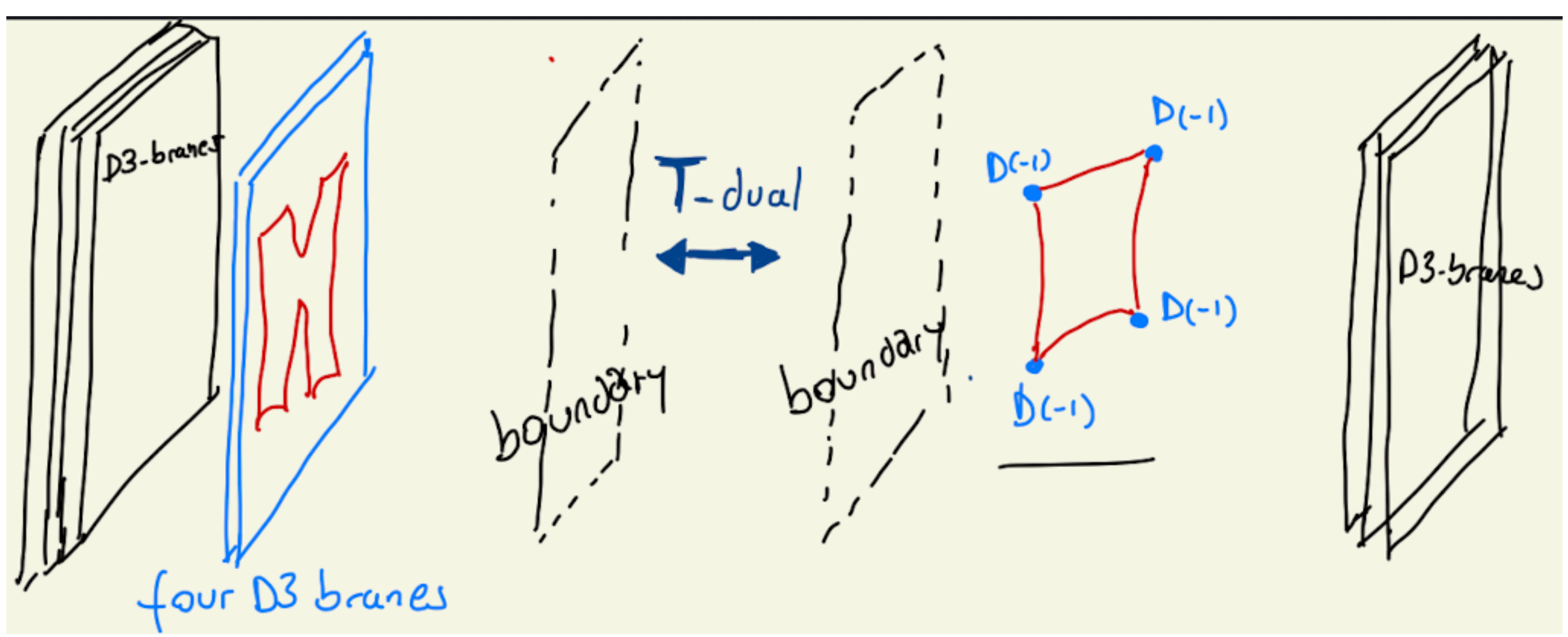
[Chicherin, Doobary, Eden, Heslop ,Korchemsky , Sokatcheva, 2015]

Outlook

- Explore 10D null limit and connection to massive amplitudes
- Relax BPS condition $y_i^2 = 0$



- String theory perspective on matrix duality
- Relationship between Giant Graviton (our Determinant) and D-instanton that appears in the T-duality explanation of correlator- Wilson loop- amplitude triality.



[Alday, Maldacena 2007]
[Berkovits, Maldacena 2008]