Root-$T\overline{T}$ Deformations

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Motivation: nice deformations.

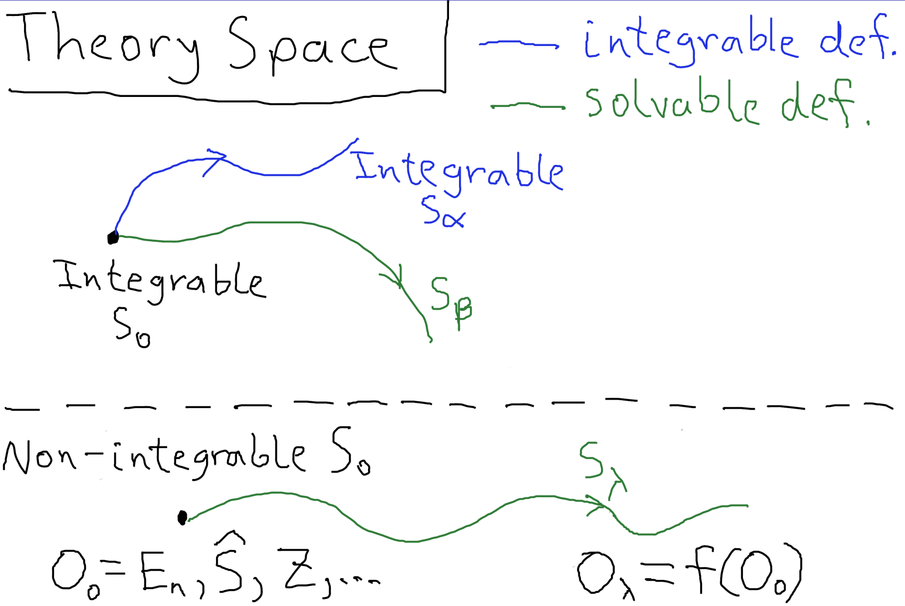
We like integrability because it allows us to compute things exactly.

Better yet, integrability-preserving deformations give us infinite families of integrable theories when we deform appropriate seed theories.

We will be interested in an *even nicer* subclass of deformations which are:

1. **universal**, in the sense that they can be applied to any member of a large class of theories;
2. **symmetry-preserving**, so that they leave desirable properties of the seed theory (like integrability or supersymmetry) intact; and
3. **solvable**, which means that we can compute quantities in the deformed theory in terms of those in the undeformed theory.

Integrable vs. solvable.



Stress tensor deformations.

We can achieve property (1), **universality**, by deforming a QFT using an integrated local operator constructed from the stress-energy tensor:

$$S_0 \longrightarrow S_0 + \lambda \int d^d x \, \mathcal{O}(x) \, , \qquad \mathcal{O} = f (T_{\mu\nu}) \, .$$

Every translation-invariant field theory admits a stress tensor $T_{\mu\nu}$.

Thus we can always consider a deformation of this form, for any scalar function $f(T_{\mu\nu})$ and any seed action S_0, at least classically.

Example. If $f(T_{\mu\nu}) = T^{\mu}{}_{\mu}$ is the trace of the stress tensor, then this deformation is a scale transformation, which is always well-defined.

Quantum deformations.

So far we have only discussed classical deformations of the action. Not all such deformations exist in the quantum theory.

A famous solvable deformation which *does exist* quantum mechanically is $T\overline{T}$. This was nicely introduced in Horatiu's talk.

In any translation-invariant $2d$ QFT there is an operator

$$\mathcal{O}_{T\overline{T}}(x) = \lim_{y \to x} \left(T^{\mu\nu}(x) T_{\mu\nu}(y) - T^{\mu}{}_{\mu}(x) T^{\nu}{}_{\nu}(y) \right) .$$

Despite involving a coincident-point limit of local operators, this point-splitting procedure gives a well-defined result [Zamolodchikov 2004].

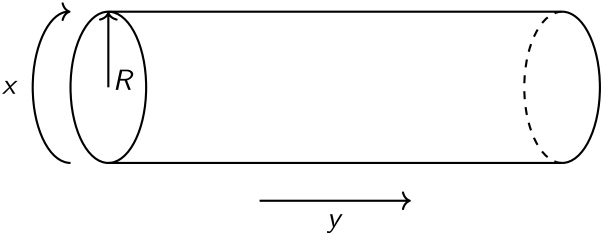
One can therefore deform any translation-invariant $2d$ QFT by this operator $\mathcal{O}_{T\overline{T}}$, even at the quantum level.

Solvability of $T\overline{T}$.

The $T\overline{T}$ deformation is **solvable** in the sense described before.

Observables like the spectrum, torus partition function, and S-matrix of a $T\overline{T}$-deformed theory can be expressed in terms of undeformed quantities.

As an example, consider the spectrum of energies $E_n(R)$ for a $2d$ QFT on a cylinder of radius R:



Flow equation for energies.

Suppose that we deform the theory by

$$\frac{\partial S}{\partial \lambda} = \frac{1}{2} \int d^2 x \, \left(T^{(\lambda)}_{\mu\nu} T^{(\lambda)\mu\nu} - \left(T^{(\lambda)\mu}{}_{\mu} \right)^2 \right) \, .$$

Using the expressions

$$T_{yy} = -\frac{1}{R} E_n(R) \, , \qquad T_{xx} = -\frac{\partial E_n(R)}{\partial R} \, , \qquad T_{xy} = \frac{i}{R} P_n(R) \, ,$$

for stress tensor components, one finds that the spectrum flows according to the inviscid Burgers' equation,

$$\frac{\partial E_n}{\partial \lambda} = E_n \frac{\partial E_n}{\partial R} + \frac{P_n^2}{R} \, ,$$

as explained in [Cavaglià, Negro, Szécsényi, Tateo '16].

Connections to string theory.

One way to see that $T\overline{T}$ is related to string theory is to solve the flow equation for the Lagrangian beginning from a seed theory

$$\mathcal{L}_0 = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi \, ,$$

which gives

$$\mathcal{L}_\lambda = \frac{1}{2\lambda}\left(1 - \sqrt{1 - 2\lambda \partial^\mu \phi \partial_\mu \phi}\right) \, .$$

This is the Lagrangian for a static gauge Nambu-Goto string with a three-dimensional target space [Cavaglià, Negro, Szécsényi, Tateo '16].

The ability to find a closed-form expression for the deformed Lagrangian is another incarnation of solvability.

Are there other such deformations?

We want other **universal**, **symmetry-preserving**, **solvable** deformations.

We would especially like a multi-parameter family* extending $T\overline{T}$:

$$\partial_\lambda S(\lambda, \gamma) = \int d^2x \, \mathcal{O}_{T\overline{T}} \, , \qquad \partial_\gamma S(\lambda, \gamma) = \int d^2x \, \mathcal{O}_{\text{new}} \, .$$

If S is a smooth function of λ and γ, then $\partial_\lambda \partial_\gamma S = \partial_\gamma \partial_\lambda S$.

$$\begin{array}{ccc} S_0 & \xrightarrow{\mathcal{O}_{\text{new}}} & S(\gamma) \\ \Big\downarrow {\scriptstyle \mathcal{O}_{T\overline{T}}} & & \Big\downarrow {\scriptstyle \mathcal{O}_{T\overline{T}}} \\ S(\lambda) & \xrightarrow[\mathcal{O}_{\text{new}}]{} & S(\lambda, \gamma) \end{array}$$

*As Jules said in his talk on Monday: "if you can deform in two ways simultaneously, you learn even more."

Roadmap.

We will be led to propose and study a *marginal* stress tensor deformation,

$$\mathcal{R} \sim \sqrt{T^{\mu\nu} T_{\mu\nu} - \frac{1}{2} \left(T^{\mu}{}_{\mu} \right)^2} .$$

of 2d field theories. We call this the **root-$T\overline{T}$ deformation**.

The plan is as follows:

- ☑ Part 1: Introduction and context.
- ☐ Part 2: Stress tensor flows for PCM-like models.
- ☐ Part 3: A root-$T\overline{T}$ deformed spectrum from holography.
- ☐ Part 4: Summary and future directions.

This concludes the comprehensible portion of the talk. Please ask questions.

Part 2: Stress tensor flows for PCM-like models.

Sigma models.

Let us think about stress tensor deformations in the playground of integrable $2d$ sigma models like the principal chiral model (PCM).

Let $g(x,t) \in G$ be a group-valued field and define the left-invariant current

$$j_\mu = g^{-1}\partial_\mu g \in \mathfrak{g} \, .$$

We will sometimes use light-cone coordinates

$$x^{\pm} = \frac{1}{2}\left(t \pm x\right) \, ,$$

as in Riccardo's talk, and write the components of j_μ as $j_\pm$.

The usual PCM has

$$\mathcal{L} \sim \operatorname{tr}\left[j_+ j_-\right] \, , \qquad T_{\pm\pm} \sim \operatorname{tr}\left[j_\pm j_\pm\right] \, .$$

PCM-like theories.

We focus on stress tensor deformations of the PCM and related models; these have Lagrangians that can be written in terms of quantities like

$$\operatorname{tr}\left[j_\mu j_\nu \right] ,$$

but *not* traces with more fields, such as $\operatorname{tr}\left[j_\mu j_\nu j_\rho \right]$.

Any Lorentz invariant constructed from traces of products of two j_μ can be written as a function of the basis elements

$$x_1 = -\operatorname{tr}[j_+ j_-] \, , \qquad x_2 = \frac{1}{2} \left(\operatorname{tr}[j_+ j_+] \operatorname{tr}[j_- j_-] + \left(\operatorname{tr}[j_+ j_-] \right)^2 \right) \, .$$

Let us refer to any Lagrangian

$$\mathcal{L}(x_1, x_2)$$

as a **PCM-like model**. If $\mathcal{L} \sim x_1$, this is the usual principal chiral model.

Modified equations of motion.

For the PCM, the current j_μ is flat (by the Maurer-Cartan identity) and conserved (by the equation of motion). It is easy to write down a Lax

$$\mathfrak{L}_\pm = \frac{j_\pm}{1 \mp z} .$$

Flatness of this Lax for any z is equivalent to the equations of motion.

For a general PCM-like model $\mathcal{L}(x_1, x_2)$, the equation of motion is

$$\partial^\mu \mathfrak{J}_\mu = 0 , \qquad \mathfrak{J}_\mu = 2 \frac{\partial \mathcal{L}}{\partial x_1} j_\mu + 4 \frac{\partial \mathcal{L}}{\partial x_2} \operatorname{tr} \left[j_\mu j^\rho \right] j_\rho .$$

Thus j_μ is flat but not conserved, and $\mathfrak{J}_\mu$ is conserved but not flat.

Classical integrability.

Claim 1. Given any PCM-like model with Lagrangian $\mathcal{L}(x_1, x_2)$ which, up to overall scaling, satisfies the differential equation

$$\left(\frac{\partial \mathcal{L}}{\partial x_1} + x_1 \frac{\partial \mathcal{L}}{\partial x_2} \right)^2 - \left(\frac{\partial \mathcal{L}}{\partial x_2} \right)^2 \left(2 x_2 - x_1^2 \right) = 1 \, , \qquad (\star)$$

the equations of motion are equivalent to flatness of the Lax connection

$$\mathfrak{L}_{\pm} = \frac{j_{\pm} \pm z \, \mathfrak{J}_{\pm}}{1 - z^2} \, ,$$

for any value of the spectral parameter z, where $\mathfrak{J}_{\mu}$ is the current whose conservation expresses the equation of motion.

Commutator magic.

Why is claim 1 true? Equation ($\star$) implies nice relations for commutators:

$$[\mathfrak{J}_+, \mathfrak{J}_-] \sim \left(\left(\frac{\partial \mathcal{L}}{\partial x_1} + x_1 \frac{\partial \mathcal{L}}{\partial x_2} \right)^2 - \left(\frac{\partial \mathcal{L}}{\partial x_2} \right)^2 (2x_2 - x_1^2) \right) [j_+, j_-] \\ \sim [j_+, j_-] \, ,$$

and

$$[\mathfrak{J}_+, j_-] = [j_+, \mathfrak{J}_-] \, .$$

Using these relations, it is straightforward to check that the Lax works.

Deforming PCM-like models.

Now suppose that we deform a PCM-like model by a function of the stress tensor. That is, consider a one-parameter family of theories $\mathcal{L}_\lambda$ obeying

$$\frac{\partial \mathcal{L}_\lambda}{\partial \lambda} = f\left(T_{\mu\nu}^{(\lambda)} \right) ,$$

where f is any scalar constructed from the stress tensor $T_{\mu\nu}^{(\lambda)}$ of $\mathcal{L}_\lambda$.

Claim 2. If the initial theory $\mathcal{L}_0$ at $\lambda = 0$ satisfies the partial differential equation $(\star)$ then so does $\mathcal{L}_\lambda$ for any λ and function f.

Therefore *any* stress tensor deformation preserves classical integrability for these models, and we can write down the Lax explicitly in terms of $\mathcal{L}$.

Is any deformation special?

This result makes it sound like all stress tensor deformations are "equally good" for the purposes of classical integrability of PCM-like models.

However, let us return to the motivation of multi-parameter families:

$$
\begin{array}{ccc}
S_0 & \xrightarrow{\mathcal{O}_{\text{new}}} & S(\gamma) \\
\Big\downarrow{\scriptstyle \mathcal{O}_{T\overline{T}}} & & \Big\downarrow{\scriptstyle \mathcal{O}_{T\overline{T}}} \\
S(\lambda) & \xrightarrow[\mathcal{O}_{\text{new}}]{} & S(\lambda, \gamma)
\end{array}
$$

We know that $T\overline{T}$ is special because it exists quantum-mechanically.

Question. Is there a marginal $\mathcal{O}_{\text{new}} = f\left(T_{\mu\nu}\right)$ which commutes with $T\overline{T}$ for PCM-like models, giving a two-parameter family of Lax connections?

Root-$T\overline{T}$ appears.

Answer. There is a unique marginal $f(T_{\mu\nu})$ which commutes with $T\overline{T}$,

$$\mathcal{R} \sim \sqrt{T^{\mu\nu} T_{\mu\nu} - \frac{1}{2} (T^{\mu}{}_{\mu})^2} .$$

This is the root-$T\overline{T}$ operator studied in [CF, Sfondrini, Smith, Tartaglino - Mazzucchelli '22]. It preserves classical conformal invariance when applied to a CFT seed, unlike $T\overline{T}$.

An example Lagrangian flow.

To build intuition, let us solve the flow equation

$$\frac{\partial S}{\partial \gamma} = \frac{1}{\sqrt{2}} \int d^2x \, \mathcal{R} = \int d^2x \sqrt{\frac{1}{2} T^{(\gamma)\mu\nu} T^{(\gamma)}_{\mu\nu} - \frac{1}{4} \left(T^{(\gamma)\mu}{}_{\mu} \right)^2}$$

with initial condition

$$\mathcal{L}_0 = - \operatorname{tr} \left[j_+ j_- \right] .$$

The result is

$$S_\gamma = \int d^2x \, \left(- \cosh(\gamma) \operatorname{tr} \left[j_+ j_- \right] + \sinh(\gamma) \sqrt{\operatorname{tr} \left[j_+ j_+ \right] \operatorname{tr} \left[j_- j_- \right]} \right) .$$

The argument of the square root is, in complex coordinates, exactly $T\overline{T}$.

Beyond PCM.

This talk focused on the simplest case of the PCM.

In [Borsato, CF, Sfondrini '22] we show that integrable two-parameter $T\overline{T}$ and root-$T\overline{T}$ flows exist for

1. PCM with WZ term;
2. symmetric space sigma model (with WZ term);
3. semi-symmetric space sigma model (with WZ term).

This result singles out root-$T\overline{T}$ as the unique marginal deformation which gives rise to integrable two-parameter families along with $T\overline{T}$ for this (fairly large) class of examples.

Part 3: A root-$T\overline{T}$ deformed spectrum from holography.

Energy flow for root-$T\overline{T}$?

We have seen that root-$T\overline{T}$ has some nice properties, such as universality, preserving integrability, and commuting with $T\overline{T}$.

What about **solvability**?

For instance, is there any analogue of the energy flow equation

$$\frac{\partial E_n}{\partial \lambda} = E_n \frac{\partial E_n}{\partial R} + \frac{P_n^2}{R} \, ,$$

associated with the $T\overline{T}$ deformation, for the root-$T\overline{T}$ flow?

A naïve approach.

To set expectations, let us look for a candidate flow equation for a root-$T\overline{T}$ deformed field theory on a cylinder of radius R as follows.

Question. Does there exist any differential equation of the form

$$\frac{\partial E_n(R)}{\partial \gamma} = f\left(E_n, \partial_R E_n, P_n\right) ,$$

with the following properties?

1. The flow is generated by a *marginal* stress tensor deformation, so γ is dimensionless and f is a Lorentz scalar constructed from $T_{\mu\nu}$;
2. the momentum P_n is undeformed, so $P_n(\gamma) = P_n(0)$; and
3. the flow gives a two-parameter family of commuting deformations with the inviscid Burgers' equation of $T\overline{T}$.

Candidate root-$T\overline{T}$ energy flow.

Answer. There exists a unique differential equation with these properties,

$$\frac{\partial E_n}{\partial \gamma} = \sqrt{\frac{1}{4}\left(E_n - R\frac{\partial E_n}{\partial R}\right)^2 - P_n^2}\,.$$

The right side is exactly the root-$T\overline{T}$ operator $\mathcal{R}$ when components of $T_{\mu\nu}$ are expressed in terms of energies and momenta:

$$T_{yy} = -\frac{1}{R}E_n(R)\,, \qquad T_{xx} = -\frac{\partial E_n(R)}{\partial R}\,, \qquad T_{xy} = \frac{i}{R}P_n(R)\,.$$

Deformed spectrum for CFT seed.

Suppose that we root-$T\overline{T}$ deform the spectrum of a CFT on a cylinder of radius R. All energies and momenta scale like

$$E_n = \frac{a_n}{R} \,, \qquad P_n = \frac{b_n}{R} \,,$$

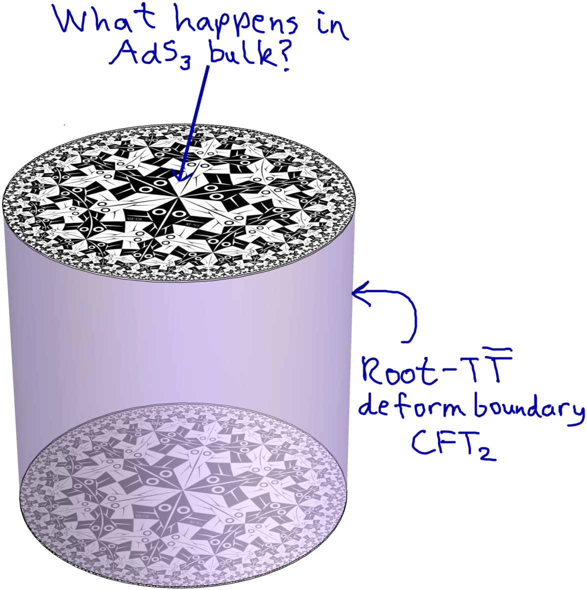
for dimensionless constants a_n, b_n. One can solve the flow and find

$$E_n(\gamma) = \cosh(\gamma) E_n + \sinh(\gamma) \sqrt{E_n^2 - P_n^2} \,.$$

However, just because we can write down a differential equation for the spectrum does not mean that a quantum deformation exists.

Can we find evidence for this flow?

Studying the spectrum using holography.



AdS_3 boundary conditions.

Conventional deformations of CFT_d are often equivalent to mixed boundary conditions in the AdS_{d+1} bulk dual [Witten 2001].

A general asymptotically AdS_3 metric admits the expansion

$$ds^2 = g_{\alpha\beta}(\rho, x^\alpha)\, dx^\alpha\, dx^\beta + \ell^2 \frac{d\rho^2}{4\rho^2}\,,$$

$$g_{\alpha\beta}(\rho, x^\alpha) = \frac{g^{(0)}_{\alpha\beta}(x^\alpha)}{\rho} + g^{(2)}_{\alpha\beta}(x^\alpha) + \rho g^{(4)}_{\alpha\beta}(x^\alpha)\,,$$

in terms of a Fefferman-Graham coordinate ρ with the boundary at $\rho = 0$.

The expansion coefficient $g^{(0)}_{\alpha\beta}$ is identified with $h_{\alpha\beta}$, the boundary metric, and the subleading term $g^{(2)}_{\alpha\beta}$ is related to the boundary stress tensor $T_{\alpha\beta}$.

Variational principle.

If we vary the bulk Einstein-Hilbert action, with appropriate boundary term, it reduces to an on-shell boundary integral:

$$\delta S\Big|_{\text{on-shell}} = \frac{1}{2}\int_{\partial\mathcal{M}} d^2x\,\sqrt{h}\,T_{\alpha\beta}\,\delta h^{\alpha\beta}\,.$$

To have a good variational principle, we demand $\delta h^{\alpha\beta} = 0$. The boundary metric is held fixed.

In a deformed theory, we expect that there will be some *other* variational principle where a different object $h_{\alpha\beta}(\gamma)$ is held fixed.

Is there such a modified variational principle which corresponds to a boundary root-$T\overline{T}$ deformation?

Holding fixed a new metric.

Let

$$\widetilde{T}_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} h_{\alpha\beta} T^{\rho}{}_{\rho} \, .$$

be the traceless part of the stress tensor. Then define

$$\begin{aligned} h_{\alpha\beta}(\gamma) &= \cosh(\gamma) h_{\alpha\beta}(0) + \frac{\sinh(\gamma)}{\mathcal{R}(0)} \widetilde{T}_{\alpha\beta}(0) \, , \\ \widetilde{T}_{\alpha\beta}(\gamma) &= \cosh(\gamma) \widetilde{T}_{\alpha\beta}(0) + \sinh(\gamma) \mathcal{R}(0) h_{\alpha\beta}(0) \, , \\ \mathcal{R}(0) &= \sqrt{\frac{1}{2} T_{\alpha\beta}(0) T^{\alpha\beta}(0) - \frac{1}{4} \left(T^{\alpha}{}_{\alpha}(0) \right)^2} \, . \end{aligned}$$

We find that the boundary root-$T\overline{T}$ deformation corresponds to a new variational principle in which the metric $h_{\alpha\beta}(\gamma)$ is held fixed and acts as a source for the new stress tensor $\widetilde{T}_{\alpha\beta}(\gamma)$.

Computing masses.

Using standard gravity techniques, we can compute the mass of a bulk AdS_3 spacetime subject to these deformed boundary conditions.

For instance, we can begin with an undeformed spacetime

BTZ black hole with mass M, spin J

$$\Updownarrow$$

CFT state with energy $E \sim M$ and momentum $P \sim J$

How does the spacetime mass change when we turn on root-$T\overline{T}$ deformed boundary conditions?

Root-$T\overline{T}$ energies from gravity.

One finds that the spacetime mass satisfies

$$M(\gamma) = \cosh(\gamma) M(0) + \sinh(\gamma) \sqrt{M^2 - J^2} \, ,$$

which matches the solution to our root-$T\overline{T}$ flow equation with CFT seed,

$$E_n(\gamma) = \cosh(\gamma) E_n + \sinh(\gamma) \sqrt{E_n^2 - P_n^2} \, .$$

This provides evidence that, at least for large-c holographic CFTs, the candidate flow equation for the root-$T\overline{T}$ deformed spectrum is correct.

Part 4: Summary and future directions.

Summary.

We proposed and studied a new stress tensor deformation of $2D$ QFTs:

$$\frac{\partial S_\gamma}{\partial \gamma} \sim \int d^2x \sqrt{T^{\mu\nu} T_{\mu\nu} - \frac{1}{2} \left(T^{\mu}{}_{\mu} \right)^2} \, .$$

This root-$T\overline{T}$ operator shares some of the nice properties of $T\overline{T}$:

1. it is **universal** because it is constructed from the stress tensor;
2. it **preserves symmetries** like integrability in many examples; and
3. it may be **solvable**, as evidenced by a candidate flow equation for the cylinder spectrum.

Root-$T\overline{T}$ is singled out as the unique such marginal deformation which forms a two-parameter commuting family with $T\overline{T}$.

Future directions.

There is much more to do. Here are a few questions:

1. Can the root-$T\overline{T}$ deformation be defined directly at the quantum level, and if so, what are its properties?
   - As a toy example, one can dimensionally reduce to $(0+1)$-dimensions. See [García, Sánchez-Isidro '22] and upcoming work 2306.XXXXX.
2. There are interesting analogies between stress tensor flows for $2d$ PCM-like or scalar theories and $4d$ Abelian gauge theories. Can these be pushed further? What about non-Abelian gauge theories?
3. What is the interplay between root-$T\overline{T}$ and supersymmetry? Can root-$T\overline{T}$ be formulated in superspace like the usual $T\overline{T}$?

Thank you for your a$T\overline{T}$ention!