Momentum kernel and Numerators From Bilinear Forms

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Momentum Kernel from Shapovalov form

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Work in collaboration with Chih-hao Fu

In our recent proceeding paper, we present a natural way to construct momentum kernel from Shapovalov form.

Momentum Kernel

1. Notations

The momentum Kernel appears as the transition matrix of amplitudes and numerators, the inverse matrix of the Berends-Giele current, and the paring coefficients of the double copy relation. It is a function of the independent external momenta $\{k_i\}, i = 1, ..., n-1$

Or more specifically a function of the Mandelstam variables $s_{ij} = k_i \cdot k_j, \quad s_A = \sum_{i,j} s_{ij}, \quad A \subset \{1, ..., n\}$

2. Expression

The explicit expression for momentum kernel is a polynomial of Mandelstam variables depending on two permutations of the indices. Such dependence and the fact that it serves as a transition matrix between amplitudes and numerators implies the bilinear structure of momentum kernel

$$S[\sigma(2), \ldots, \sigma(n) | k_1, \tau(2), \ldots, \tau(n)]_{k_1} = \prod \left(s_{1\sigma(t)} + \sum \theta(\sigma(t), \sigma(q))\, s_{\sigma(t)\sigma(q)} \right), \quad \theta(\sigma(t), \sigma(q)) = \begin{cases} 1 & (\sigma(t) - \sigma(q))\left(\sigma\tau^{-1}(t) - \sigma\tau^{-1}(q)\right) < 0 \\ 0 & (\sigma(t) - \sigma(q))\left(\sigma\tau^{-1}(t) - \sigma\tau^{-1}(q)\right) > 0 \end{cases}.$$

3. Examples

Here we list the explicit expression for the first a few simple cases of momentum kernel for simple comparison with the Shapovalov forms

$$S[23, 23]_{k_1} = (s_{23} + s_{13})\, s_{12}$$
$$S[23, 32]_{k_1} = s_{13} s_{12}$$
$$S[234, 234]_{k_1} = -(s_{34} + s_{24} + s_{14})(s_{23} + s_{13})\, s_{12}$$

4. String KLT kernel

As the string-lift of momentum kernel, the string KLT kernel can be identified with the q-deformed version of Shapovalov form. The following example is a demonstration for this identification

$$\mathcal{S}_{\alpha'}[234|423]_{k_1} = \sin(\pi\alpha' s_{12}) \sin(\pi\alpha'(s_{13} + s_{23})) \sin(\pi\alpha'(s_{14} + s_{24} + s_{34}))$$

Shapovalov Form

1. Underlying Algebra

The Shapovalov form is a bilinear form defined on Lie algebra. To construct momentum kernel for n-pt amplitudes, we need the underlying algebra G to have a simple root system that reproduces the Mandelstam variables. We define algebra G to be the Lie algebra with generators implementing the following Lie-bracket

$$[E_i, F_j] = \delta_{ij} H_{k_i}, \ [H_{k_i}, E_j] = k_i \cdot k_j E_j, \ [H_{k_i}, F_j] = -k_i \cdot k_j F_j, \quad i = 1, ..., n-1$$

2. Definition:

The Shapovalov form is a bilinear form defined recursively on G by the following rules

$$\langle E_i, E_j \rangle = \delta_{ij}, \langle E_i, X \rangle = 0, \quad i, j \in \{1, ..., n\}, X \in G, X \notin \mathrm{Span}(\{E_1, ..., E_{n-1}\})$$
$$\langle [E_i, X], Y \rangle = \langle X, [F_i, Y] \rangle, \quad i, j \in \{1, ..., n\}, X, Y \in G$$

3. Example

The following expressions for the first a few nontrivial cases of Shapovalov form can be derived recursively by repetitive application of the defining property of Shapovalov form, the Lie brackets between generatros and the Jacobi identity.

$$\langle [E_3, [E_2, E_1]], [E_3, [E_2, E_1]] \rangle = (s_{23} + s_{13})\, s_{12}$$
$$\langle [E_3, [E_2, E_1]], [E_2, [E_3, E_1]] \rangle = s_{13} s_{12}$$
$$\langle [E_4, [E_3, [E_2, E_1]]], [E_4, [E_3, [E_2, E_1]]] \rangle = -(s_{34} + s_{24} + s_{14})(s_{23} + s_{13})\, s_{12}$$

4. q-deformed Shapovalov form

The Shapovalov form naturally generalizes to the modules over q-deformation of G. The explicit expression can be calculated following similar approach. The result turns out to be the q-deformation of the classical one in the sense that each term in the multiplication is replaced by its q-defromation for example:

$$\langle e_4 e_3 e_2 v_1, e_3 e_2 e_4 v_1 \rangle_q = -[s_{14}]_q [s_{13} + s_{23}]_q [s_{12}]_q \qquad [z]_q = q^z - q^{-z}$$

This formula can be identified with the KLT kernel when $q \to e^{i\pi\alpha'}$

As the inversion of momentum kernel, the Berends-Giele current can be naturally identified as the Shapovalov form between two dual elements of the half-ladder basis or coefficients in the expression of Shapovalov dual of half ladder basis when spanning by half ladder basis, as in the following example.

$$\begin{aligned}([E_4, [E_3, [E_2, E_1]]])^* = & \frac{1}{s_{1234} s_{123} s_{12}} [E_4, [E_3, [E_2, E_1]]] + \frac{1}{s_{1234} s_{12} s_{34}} [[[E_4, E_3], E_2], E_1] \\ & + \frac{1}{s_{1234} s_{234} s_{34}} [[[E_4, E_3], E_2], E_1] + \frac{1}{s_{1234} s_{234} s_{23}} [[E_4, [E_3, E_2]], E_1] \\ & + \frac{1}{s_{1234} s_{123} s_{23}} [[E_4, [E_3, E_2]], E_1]\end{aligned}$$

This identification combined with the recursive property of Shapovalov dual elements leads to an alternative proof for the inversion relation between momentum kernel and Berends-Giele currents. This proof also has a q-deformed version for the KLT kernel and twisted intersection numbers.

Numerators as Shapovalov Forms

Nonlinear Sigma Model

work in progress

Jacobi-manifest representation of NLSM numerators

The numerators of non-linear sigma model takes the form of a momentum kernel on half-ladder basis.

$$N(\ell(\sigma)) = (-1)^{n/2} \pi^2 S[\sigma|\sigma]_{k_1}, \quad n \in 2\mathbb{Z}, \sigma \in S_{n-2}$$

In fact with the values on half-ladder basis, any homomorphism of the cubic graphs, such as the numerators can be written in the following way in terms of Shapovalov form, which manifestly satisfies the Jacobi identity.

$$N(X) = \langle X, U_n \rangle, \quad X \in G \qquad U_n = (-1)^{n/2} \pi^2 \sum_{\tau \in S_{n-2}} \langle \ell(\tau), \ell(\tau) \rangle (\ell(\tau))^*, \quad n \in 2\mathbb{Z}$$

For 4pt the universal vector on the left simplifies to

$$\frac{s_{12}+s_{23}}{s_{123}} [[E_1, E_2], E_3] + \frac{s_{13}+s_{23}}{s_{123}} [[E_1, E_3], E_2]$$

Color-kinematic duality of the Shapovalov form expression of NLSM numerators

The Shapovalov form expression is naturally dual to the color factor in the sense that the color factor, as a successive product of structure constant is in fact a Shapovalov form or a Kiling form itself. The trace definition of Killing form reads:

$$B(X, Y) = \mathrm{tr}(T_{adj}[X] T_{adj}[Y]), \quad X, Y \in G$$

and it is straightforward to check it is equivalent to the recursive definition, which is equivalent to Shapovalov form upto a interchange of E and F

$$B(E_\alpha, F_\beta) = \delta_{\alpha\beta}, \quad B([X, Y], Z) = B(X, [Y, Z])$$

For example, the Shaopvalov form expression for the following color factor reads

$$f^{ab}{}_{e_1} f^{e_1 c}{}_{e_2} \delta_{e_2 d} = \langle [[E^a, E^b], E^c], E^d \rangle$$

The other way to manifest the color-kinematic duality is to find a orthonormal basis for Shapovalov form for the kinematic algebra, as in the

$$N([[E_1, E_2], E_3]) = c_{(6)} \sum_{i=1}^{8} f^{(i)}_{(1)(3)} f^{(6)}_{(i)(2)} + c_{(7)} \sum_{i=1}^{8} f^{(i)}_{(1)(3)} f^{(7)}_{(i)(2)}$$

where

$$c_{(6)} = \sqrt{s_{12}(s_{13} + s_{23})}, \quad c_{(7)} = -\frac{s_{13}\sqrt{s_{13}(s_{13}+s_{23})}}{\sqrt{s_{13}s_{23}(s_{12}+s_{13}+s_{23})}}$$

$$f_{(2)(5)}{}^{(6)} = -\sqrt{\frac{s_{12}s_{13}}{s_{13}+s_{23}}} \quad f_{(2)(5)}{}^{(7)} = \sqrt{\frac{s_{23}}{(s_{13}+s_{23})s_{12}}} \quad f_{(1)(2)}{}^{(4)} = \sqrt{s_{12}}, \quad f_{(1)(3)}{}^{(5)} = \sqrt{s_{13}} \quad f_{(3)(4)}{}^{(6)} = -\sqrt{(s_{13} + s_{23})}$$

Yang-Mills Theory

work in progress with Chih-hao Fu

Root system for algebraic construction of Yang-Mills numerators

The root system used was proposed in P. Goddard and D. I. Olive, DAMTP-83/22. For a n-point amplitude the root system is generated by 2n-1 simple roots and the underlying Lie algebra is defined by the following Lie brackets:

$$[E_{k_i - \epsilon_i}, E_{\epsilon_i}] = E^{\mathrm{gluon}}_{\{\epsilon_i, k_i\}}, \quad \left[H_\mu, E^{\mathrm{gluon}}_{\{\epsilon_i, k_i\}}\right] = k_\mu E^{\mathrm{gluon}}_{\{\epsilon_i, k_i\}}$$

where

$$\epsilon_i^2 = (k_i - \epsilon_i)^2 = -2, \quad \epsilon_i \cdot (k_i - \epsilon_i) = 2$$

The onshell condition $k_i^2 = 0$ and the gauge invariance $\epsilon_i \cdot k_i = 0$ follows naturally from the normalization of the simple roots.

Jacobi-manifest reperesentation for Yang-mills numerators

The n-point Yang-Mills numerators can be written as the Shapovalov form on a universeral vector and a nested Lie bracket of $E^{\mathrm{gluon}}_{\{\epsilon_i, k_i\}}$

$$N(12...n) = \langle U_n, r([E_{\epsilon_1}, E_{k_1 - \epsilon_1}], ..., [E_{\epsilon_{n-1}}, E_{k_{n-1} - \epsilon_{n-1}}], E_{\epsilon_n}) \rangle$$

For example, the 4-point numerator takes the following form

$$N(1234) = \langle U_4, [[E_{\epsilon_1}, E_{k_1 - \epsilon_1}], [[E_{\epsilon_2}, E_{k_2 - \epsilon_2}], [[E_{\epsilon_3}, E_{k_3 - \epsilon_3}], E_{\epsilon_4}]]] \rangle$$

The universal vector is not unique as the linear space spanned by numerators is (n-2)! dimensional while the linear space spanned by the algebra element of suitable weight is (2n-4)! dimensional.