Colour-kinematics duality, double copy, and homotopy algebras

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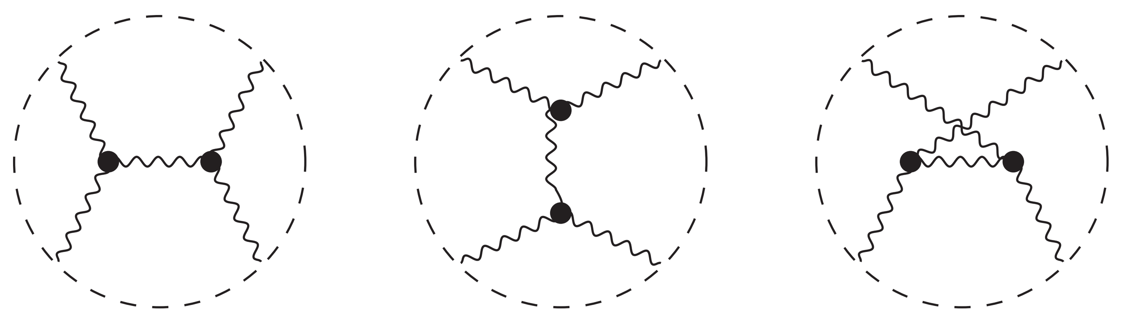
Colour–kinematic duality and $\mathrm{BV}^{\blacksquare}$-algebras

A colour–kinematic dual theory is such that

- Loop integrands: sum over cubic Feynman graphs γ

$$\sum_{\gamma} \frac{\mathsf{c}_\gamma \mathsf{n}_\gamma}{|\operatorname{Aut}(\gamma)| d_\gamma} \quad \text{with } \mathsf{c}_\gamma \text{ colour factor and } \mathsf{n}_\gamma \text{ kinematic factor}$$

- Kinematic Jacobi identity $\mathsf{c}_\alpha + \mathsf{c}_\beta + \mathsf{c}_\gamma = 0 \quad \Rightarrow \quad \mathsf{n}_\alpha + \mathsf{n}_\beta + \mathsf{n}_\gamma = 0$



What kind of theory gives rise to these amplitudes? Consider a cubic Batalin–Vilkovisky action

- Graded vector space $\mathfrak{L}$ as field space, $\mu_1 : \mathfrak{L} \to \mathfrak{L}$ encodes quadratic part, $\mu_2 : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ encodes interactions. $(\mathfrak{L}, \mu_1, \mu_2)$ is a differential graded Lie algebra

$$\mu_1 \frac{\mathsf{h}}{\blacksquare} + \frac{\mathsf{h}}{\blacksquare}\mu_1 + \Pi_{\text{on-shell}} = \mathsf{id} \quad \text{with propagator} \quad \frac{\mathsf{h}}{\blacksquare}$$

- Colour stripping: $(\mathfrak{L}, \mu_1, \mu_2)$ factorises into the tensor product of a colour Lie algebra and a differential graded commutative algebra $(\mathfrak{B}, \mathsf{d}, - \cdot -)$

$$\mathsf{dh} + \mathsf{hd} = \blacksquare$$

- Colour–kinematic duality: $(\mathfrak{B}, \mathsf{d}, - \cdot -)$ can be upgraded to a $\mathrm{BV}^{\blacksquare}$-algebra
- The kinematic Lie algebra is $(\mathfrak{B}[1], [-,-])$ with $[x, y] = (-1)^{|x|}(\mathsf{h}(x \cdot y) - (\mathsf{h}x) \cdot y - (-1)^{|x|} x \cdot (\mathsf{h}y))$

Examples

- Chern–Simons theory: off-shell CK duality (M. Ben-Shahar and H. Johansson, 2021)
- Colour stripped differential graded commutative algebra: de Rham complex $(\Omega^\bullet(\mathbb{M}^3), d, \wedge)$
- $\mathrm{BV}^{\blacksquare}$-algebra: $\blacksquare = \Box_{\mathbb{M}^3}$, $\mathsf{h} = -d^\dagger$,

$$\begin{aligned}[\alpha, \beta] &= (-1)^{|\alpha|} d^\dagger(\alpha \wedge \beta) - \\ &\quad -(-1)^{|\alpha|} d^\dagger \alpha \wedge \beta - \alpha \wedge d^\dagger \beta\end{aligned}$$

- Kinematic Lie algebra is isomorphic to the Schouten–Nijenhuis algebra $\mathbb{M}^3$. Restricting to physical fields, it is isomorphic to spacetime diffeomorphism algebra
- Analogous discussion for holomorphic CS theory: kinematic Lie algebra is holomorphic Schouten–Nijenhuis algebra
- On appropriate twistor spaces, CS and holomorphic CS organize and identify kinematic Lie algebras for self–dual and full Yang–Mills theory

A symmetry of the action

Consider a theory with tree-level CK duality

- With an appropriate choice of gauge and field redefinitions, it possible to extend tree-level CK duality to all BRST fields
- Manifesting off-shell CK duality may involve non-local field redefinitions, that generate unitarity-restoring counterterms
- Counterterms may violate CK duality

From our perspective, CK duality is a symmetry of the BRST action, generically anomalous at the loop level

Outlook

- Formulation of kinematic Lie algebra, CK duality and double copy in terms of homotopy algebras
- Lagrangian incarnation of CK duality and double copy
- Kinematic Lie algebras for CS and YM theories

Double copy as syngamy

- A cubic theory that manifest CK duality and its BRST operator

$$\begin{aligned} S &= \tfrac{1}{2}\mathsf{g}_{\alpha\beta}\, \bar{\mathsf{g}}_{\bar\alpha\bar\beta} \Phi^{\alpha\bar\alpha} \Box \Phi^{\beta\bar\beta} + \tfrac{1}{3!}\mathsf{f}_{\alpha\beta\gamma}\, \bar{\mathsf{f}}_{\bar\alpha\bar\beta\bar\gamma} \Phi^{\alpha\bar\alpha}\Phi^{\beta\bar\beta}\Phi^{\gamma\bar\gamma} \\ Q_{\mathrm{BRST}}\Phi^{\alpha\bar\alpha} &= \mathsf{q}^\alpha_\beta \delta^{\bar\alpha}_{\bar\beta} \Phi^{\beta\bar\beta} + \delta^\alpha_\beta \bar{\mathsf{q}}^{\bar\alpha}_{\bar\beta} \Phi^{\beta\bar\beta} + \tfrac{1}{2}\mathsf{f}_{\beta\gamma}{}^{\alpha}\, \bar{\mathsf{q}}^{\bar\alpha}_{\bar\beta\bar\gamma} \Phi^{\beta\bar\beta}\Phi^{\gamma\bar\gamma} + \tfrac{1}{2}\mathsf{q}^\alpha_{\beta\gamma}\, \bar{\mathsf{f}}_{\bar\beta\bar\gamma}{}^{\bar\alpha} \Phi^{\beta\bar\beta}\Phi^{\gamma\bar\gamma} + \cdots \end{aligned}$$

- A second CK dual theory

$$\begin{aligned} S &= \tfrac{1}{2}\mathsf{g}_{\mathsf{ab}}\, \bar{\mathsf{g}}_{\bar{\mathsf{a}}\bar{\mathsf{a}}} \Phi^{\mathsf{a}\bar{\mathsf{a}}} \Box \Phi^{\mathsf{b}\bar{\mathsf{b}}} + \tfrac{1}{3!}\mathsf{f}_{\mathsf{abc}}\, \bar{\mathsf{f}}_{\bar{\mathsf{a}}\bar{\mathsf{b}}\bar{\mathsf{c}}} \Phi^{\mathsf{a}\bar{\mathsf{a}}}\Phi^{\mathsf{b}\bar{\mathsf{b}}}\Phi^{\mathsf{c}\bar{\mathsf{c}}} \\ Q_{\mathrm{BRST}}\Phi^{\mathsf{a}\bar{\mathsf{a}}} &= \cdots \end{aligned}$$

- Syngamy: double copy both action and BRST operator
- Four possible combinations, e.g.

$$\begin{aligned} S &= \tfrac{1}{2}\mathsf{g}_{\alpha\beta}\, \mathsf{g}_{\mathsf{ab}} \Phi^{\alpha\mathsf{a}} \Box \Phi^{\beta\mathsf{b}} + \tfrac{1}{3!}\mathsf{f}_{\alpha\beta\gamma}\, \mathsf{f}_{\mathsf{abc}} \Phi^{\alpha\mathsf{a}}\Phi^{\beta\mathsf{b}}\Phi^{\gamma\mathsf{c}} \\ Q_{\mathrm{BRST}}\Phi^{\alpha\mathsf{a}} &= \mathsf{q}^\alpha_\beta \delta^{\mathsf{a}}_{\mathsf{b}} \Phi^{\beta\mathsf{b}} + \tfrac{1}{2}\mathsf{q}^\alpha_{\beta\gamma}\, \mathsf{f}_{\mathsf{bc}}{}^{\mathsf{a}} \Phi^{\beta\mathsf{b}}\Phi^{\gamma\mathsf{c}} + \delta^\alpha_\beta \mathsf{q}^{\mathsf{a}}_{\mathsf{b}} \Phi^{\beta\mathsf{b}} + \tfrac{1}{2}\mathsf{f}_{\beta\gamma}{}^{\alpha}\, \mathsf{q}^{\mathsf{a}}_{\mathsf{bc}} \Phi^{\beta\mathsf{b}}\Phi^{\gamma\mathsf{c}} + \cdots \end{aligned}$$

- Does the syngamy theories satisfy $Q^2_{\mathrm{BRST}} = 0$ and $Q_{\mathrm{BRST}} S = 0$? If $Q^2_{\mathrm{BRST}}\phi$ and $Q_{\mathrm{BRST}} S$ are proportional to generic algebraic relations satisfied by metric and structure constant of a CK dual theory, yes.

From two "parents" Yang–Mills theories, we obtain the following syngamy theories:

- copies of the parent theories
- a theory of biadjoint scalars
- a theory with the same field content as $\mathcal{N} = 0$ supergravity, and perturbatively quantum equivalent to this theory

References

[1] Borsten, Jurco, Kim, Macrelli, Saemann, and Wolf. Tree-Level Color–Kinematics Duality Implies Loop-Level Color–Kinematics Duality up to Counterterms, arXiv:2108.03030.

[2] Borsten, Jurco, Kim, Macrelli, Saemann, and Wolf. Kinematic Lie Algebras From Twistor Spaces, arXiv:2211.13261.