Conservative and Dissipative Two-Body Dynamics from the Eikonal Operator

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Motivations

The eikonal exponentiation resums contributions to the $2 \to 2$ amplitude due to many graviton exchanges

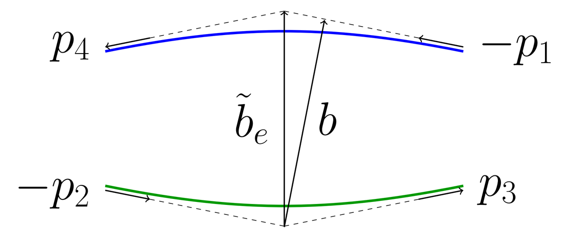
$$\widetilde{\mathcal{M}}(b) = \int \frac{d^D q}{(2\pi)^D} 2\pi\delta(2p_1\cdot q)2\pi\delta(2p_2\cdot q)\, e^{ib\cdot q}\mathcal{M}(q^2)\,,$$

$$1 + i\widetilde{\mathcal{M}}(b) = e^{2i\delta(b)}\,.$$

Combining $2i\delta$ with coherent graviton emissions encoded in the $2 \to 3$ amplitude,

$$\tilde{\mathcal{A}}^{\mu\nu}(b_1, b_2, k) \simeq \int \frac{d^D q_1}{(2\pi)^D} 2\pi\delta(2p_1\cdot q_1)2\pi\delta(2p_2\cdot q_2) \times e^{ib_1\cdot q_1 + ib_2\cdot q_2}\mathcal{A}^{\mu\nu}(q_1, q_2, k)\,,$$

the eikonal operator dictates the collision's final state. In this way, it provides a unified formalism to calculate classical observables associated to the collision's asymptotic states from scattering amplitudes [1, 2].



Conservative and Dissipative Effects in the 2-Body Problem

- $P^\alpha = \boldsymbol{P}^\alpha$: energy-momentum of the gravitational field after the collision,
- $Q^\alpha_i = \mathcal{Q}^\alpha_{(i)} + \boldsymbol{Q}^\alpha_{(i)} + Q^\alpha_{(i)c}$: momentum variation (*impulse*) of particle i (for $i = 1, 2$),
- $J^{\alpha\beta} = \mathcal{J}^{\alpha\beta} + \boldsymbol{J}^{\alpha\beta}$: angular momentum of the gravitational field after the collision,
- $\Delta L^{\alpha\beta}_{(i)} = \Delta\mathcal{L}^{\alpha\beta}_{(i)} + \Delta\boldsymbol{L}^{\alpha\beta}_{(i)} + \Delta L^{\alpha\beta}_{(i)c}$: angular momentum variation of particle i (for $i = 1, 2$).

In the following we calculate all such quantities to $\mathcal{O}(G^3)$ precision and check that the balance laws hold separately for each of the three types of quantities.

The Eikonal Operator

We start from a state with two massive particles with impact parameter $b = b_1 - b_2$ [3]

$$|\psi\rangle = \int_{-p_1}\int_{-p_2} \Phi_1(-p_1)\Phi_2(-p_2)e^{ip_1 b_1 + ip_2 b_2}|-p_1, -p_2\rangle\,.$$

The final state [4] is determined by the eikonal operator according to

$$S|\psi\rangle \simeq \int_{p_3,p_4} e^{-ib_1\cdot p_4 - ib_2\cdot p_3}|p_4, p_3\rangle \times \int\frac{d^DQ_1}{(2\pi)^D}\int\frac{d^DQ_2}{(2\pi)^D}\Phi_1(p_4 - Q_1)\,\Phi_2(p_3 - Q_2) \times \int d^Dx_1\int d^Dx_2\, e^{i(b_1-x_1)\cdot Q_1 + i(b_2 - x_2)\cdot Q_2}\, e^{2i\hat\delta(x_1,x_2)}|0\rangle\,.$$

Letting $\tilde b_e$ denote the projection of $x = x_1 - x_2$ orthogonal to $p_i + (-)^i\frac{1}{2}\tilde Q$ as in [4],

$$e^{2i\hat\delta(x_1,x_2)} = e^{2i\tilde\delta(\tilde b_e)}e^{\int_k \theta(\omega^* - k^0)\left[f_j a_j(k)^\dagger - f_j^*(k)a_k(k)\right]} \times e^{i\int_k\theta(k^0-\omega^*)\left[\tilde{\mathcal{A}}_j(x_1,x_2,k)a_j^\dagger(k) + \tilde{\mathcal{A}}_j^*(x_1,x_2,k)a_j(k)\right]}$$

where $f_j(k) = \varepsilon_j^{*\mu\nu}(k)F_{\mu\nu}(k)$ with

$$F^{\mu\nu}(k) = \sum\frac{\sqrt{8\pi G}\,p_n^\mu p_n^\nu}{p_n\cdot k - i0}\,.$$

The phase $2\tilde\delta$ does does not contain the radiation-reaction terms [5, 6, 7, 8, 9],

$$\mathrm{Re}\, 2\delta_2(\tilde b_e) = 2\tilde\delta_2(\tilde b_e) + \tfrac{1}{4}G\,Q^2_{\text{1PM}}\,\mathcal{I}(\sigma)\,.$$

The saddle-point conditions impose $Q_1 = p_1 + p_4$ and $Q_2 = p_2 + p_3$ with

$$x_{i\mu} - b_{i\mu} = \frac{\partial\,\mathrm{Re}\,2\delta(s,\tilde b_e)}{\partial Q_i^\mu} - i\int_k\tilde{\mathcal{A}}^*(x_1,x_2,k)\frac{\overleftrightarrow{\partial}}{\partial Q_i^\mu}\tilde{\mathcal{A}}(x_1,x_2,k)\,,$$

$$Q_{i\mu} = \frac{\partial\,\mathrm{Re}\,2\delta(s,\tilde b_e)}{\partial x_i^\mu} - i\int_k\tilde{\mathcal{A}}^*(x_1,x_2,k)\frac{\overleftrightarrow{\partial}}{\partial x_i^\mu}\tilde{\mathcal{A}}(x_1,x_2,k)\,.$$

Static Modes

Angular momentum of the static gravitational field [10, 11],

$$\mathcal{J}_{\alpha\beta} = -i\int_k\left(k_{[\alpha}F^*\frac{\overleftrightarrow{\partial}}{\partial k^{\beta]}}F + 2F^*_{\mu[\alpha}F^\mu_{\beta]}\right) = -\sum_{n=1,2}\sum_{m=3,4}c_{nm}\,p_n^{[\alpha}p_m^{\beta]}$$

so that in the center-of-mass frame [6, 10, 11]

$$\frac{\mathcal{J}^{yz}}{bp} = G(Q_{\text{1PM}} + Q_{\text{2PM}})\,\mathcal{I}(\sigma) + \mathcal{O}(G^4)\,.$$

Static part of the radiation-reaction impulse,

$$\mathcal{Q}^\alpha_{(1)} = \mathrm{Im}\int_k F^*\frac{\partial F}{\partial b_1^\alpha} = \frac{1}{2}\frac{\partial\tilde Q^2}{\partial b_\alpha}\mathcal{G} = -\frac{b^\alpha}{4b^2}G\,Q_{\text{1PM}}(2Q_{\text{1PM}} + 3Q_{\text{2PM}})\,\mathcal{I}(\sigma) + \mathcal{O}(G^5)\,.$$

Static contribution to the mechanical angular momentum or particle 1,

$$\Delta\mathcal{L}^{\alpha\beta}_{(1)} = \mathrm{Im}\int_k F^*\,p_{4[\alpha}\frac{\partial F}{\partial p_4^{\beta]}} + b_1^{[\alpha}\mathcal{Q}^{\beta]}_{(1)}\,.$$

Defining

$$2\eta_m J^{\alpha\beta}_{(m)} = \sum_{\eta_n = -\eta_m}c_{nm}\,p_n^{[\alpha}p_m^{\beta]} - \sum_{\substack{\eta_n = \eta_m\\ n\neq m}}d_{nm}\,p_n^{[\alpha}p_m^{\beta]}\,,$$

we find the following result,

$$\Delta\mathcal{L}^{\alpha\beta}_{(1)} = J^{\alpha\beta}_{(1)} + J^{\alpha\beta}_{(4)} + b_1^{[\alpha}\mathcal{Q}^{\beta]}_{(1)}\,.$$

Balance laws:

$$\mathcal{J}^{\alpha\beta} + \Delta\mathcal{L}^{\alpha\beta}_{(1)} + \Delta\mathcal{L}^{\alpha\beta}_{(2)} = 0\,,\qquad \mathcal{Q}_{(1)} + \mathcal{Q}_{(2)} = 0\,.$$

$$\sigma_{nm} = -\eta_n\eta_m\frac{p_n\cdot p_m}{m_n m_m}\,,\qquad \Delta_{nm} = \frac{\operatorname{arccosh}\sigma_{nm}}{\sqrt{\sigma_{nm}^2 - 1}}\,,$$

$$c_{nm} = 2G\left[\left(\sigma^2_{nm} - \tfrac{1}{2}\right)\frac{\sigma_{nm}\Delta_{nm} - 1}{\sigma^2_{nm} - 1} - 2\sigma_{nm}\Delta_{nm}\right],$$

$$d_{nm} = 2G\,\frac{\sigma^2_{nm} - \frac{1}{2}}{\sigma^2_{nm} - 1}\,,$$

$$2\mathcal{G} = c_{14} + c_{23} - 2c_{13}\,,\qquad 2a_0 = c_{13} + d_{12}\,,$$

$$\tfrac{1}{2}\mathcal{I}(\sigma) = \frac{8 - 5\sigma^2}{3(\sigma^2 - 1)} + \frac{\sigma(2\sigma^2 - 3)\operatorname{arccosh}\sigma}{(\sigma^2 - 1)^{3/2}}\,.$$

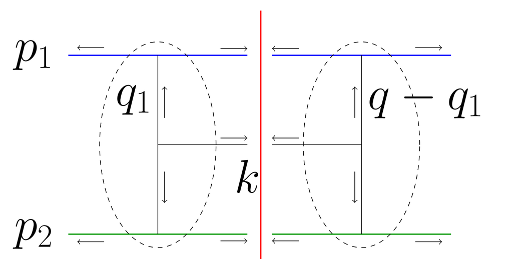
Reverse Unitarity

For the observables

$$\boldsymbol{P}^\alpha = \int_k\tilde{\mathcal{A}}\,k^\alpha\tilde{\mathcal{A}}^*\,,\qquad \boldsymbol{Q}_{(i)\alpha} = \mathrm{Im}\int_k\frac{\partial\tilde{\mathcal{A}}}{\partial b_i^\alpha}\tilde{\mathcal{A}}^*\,,$$

we can apply reverse unitarity [12, 13] via

$$\boldsymbol{O} = \mathrm{FT}\int d(\text{LIPS})f_{\boldsymbol{O}}$$



where $f^\alpha_{\boldsymbol{P}} = k^\alpha$, $f^\alpha_{\boldsymbol{Q}_{(1)}} = q_1^\alpha - \frac{1}{2}q^\alpha$.

For the observables $\boldsymbol{J}_{\alpha\beta} = \boldsymbol{J}^{(o)}_{\alpha\beta} + \boldsymbol{J}^{(s)}_{\alpha\beta}$ [10]

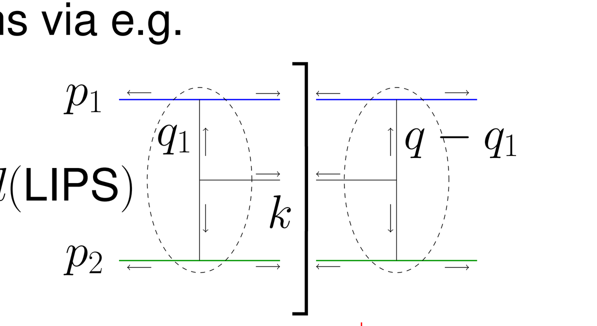
$$i\boldsymbol{J}^{(o)}_{\alpha\beta} = \int_k k_{[\alpha}\frac{\partial\tilde{\mathcal{A}}}{\partial k^{\beta]}}\tilde{\mathcal{A}}^*\,,\qquad \boldsymbol{J}^{(s)}_{\alpha\beta} = i\int_k 2\tilde{\mathcal{A}}^\mu_{[\alpha}\tilde{\mathcal{A}}^*_{\beta]\mu}$$

and $\Delta\boldsymbol{L}^{\alpha\beta}_{(i)} = \mathrm{Im}\,\boldsymbol{J}^{\alpha\beta}_{(i)} + b_i^{[\alpha}\boldsymbol{Q}^{\beta]}_{(i)}$,

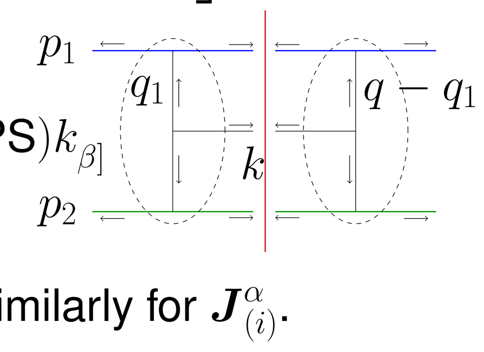
$$\boldsymbol{J}^{\alpha\beta}_{(i)} = \int_k p_{i[\alpha}\frac{\partial\tilde{\mathcal{A}}}{\partial p_i^{\beta]}}\tilde{\mathcal{A}}^*\,,$$

we need to take into account the action of the derivatives on the δ functions via e.g.

$$i\boldsymbol{J}^{(o)}_{\alpha\beta} = \mathrm{FT}\int k_{[\alpha}\frac{\partial}{\partial k^{\beta]}}\left[d(\text{LIPS})\ \ldots\right]$$



$$-u_{2[\alpha}\mathrm{FT}\frac{\partial}{\partial q_{\|2}}\int d(\text{LIPS})k_{\beta]}$$



where $q_{\|2} = -u_2\cdot q$, and similarly for $\boldsymbol{J}^\alpha_{(i)}$.

Radiative Modes

In this way, we recover [13, 14]

$$\boldsymbol{P}^\alpha = \frac{G^3m_1^2m_2^2}{b^3}(\check u_1^\mu + \check u_2^\mu)\,\mathcal{E}\,,\qquad \boldsymbol{Q}^\alpha_{(1)} = -\frac{G^3m_1^2m_2^2}{b^3}\check u_2^\alpha\,\mathcal{E}\,.$$

Denoting $\mathcal{C} = \frac{-\mathcal{E}_+ + \sigma\mathcal{E}_-}{\sqrt{\sigma^2-1}}$, $\mathcal{F} = \mathcal{E}_+ - \frac{1}{2}\mathcal{E} = -\mathcal{E}_- + \frac{1}{2}\mathcal{E}$, in a frame where $b_1 + b_2 = 0$, we also recover [10]

$$\boldsymbol{J}^{\alpha\beta} = \frac{G^3m_1^2m_2^2}{b^3}\mathcal{F}\left(b^{[\alpha}\check u_1^{\beta]} - b^{[\alpha}\check u_2^{\beta]}\right),$$

and we obtain

$$\Delta\boldsymbol{L}^{\alpha\beta}_{(1)} = \frac{G^3m_1^2m_2^2}{b^3}\left[\frac{\mathcal{E}_+ b^{[\alpha}u_1^{\beta]}}{\sigma - 1} - \frac{1}{2}\mathcal{E}\,b^{[\alpha}\check u_2^{\beta]}\right].$$

Balance laws:

$$\boldsymbol{P}^\alpha + \boldsymbol{Q}^\alpha_{(1)} + \boldsymbol{Q}^\alpha_{(2)} = 0\,,\qquad \boldsymbol{J}^{\alpha\beta} + \Delta\boldsymbol{L}^{\alpha\beta}_{(1)} + \Delta\boldsymbol{L}^{\alpha\beta}_{(2)} = 0\,.$$

$$\frac{\mathcal{E}}{\pi} = f_1 + f_2\log\frac{\sigma+1}{2} + f_3\frac{\sigma\ \operatorname{arccosh}\sigma}{2\sqrt{\sigma^2-1}}$$

$$\frac{\mathcal{C}}{\pi} = g_1 + g_2\log\frac{\sigma+1}{2} + g_3\frac{\sigma\ \operatorname{arccosh}\sigma}{2\sqrt{\sigma^2-1}}$$

Tidal Effects

We include tidal effects by means of the $2 \to 3$ amplitude in [17]

$$\boldsymbol{P}^\alpha_{\text{tid}} = R_f\sum_X\frac{c_{X_1^2}}{m_1}\left(\mathcal{E}^X\check u_1^\alpha + \mathcal{F}^X\check u_2^\alpha\right)$$

$$\boldsymbol{J}^{\alpha\beta}_{\text{tid}} = R_f\sum_X\frac{c_{X_1^2}}{m_1}\left(\mathcal{C}^X b^{[\alpha}u_1^{\beta]} + \mathcal{D}^X u_2^{[\alpha}b^{\beta]}\right)$$

where

- $R_f = 15\pi G^3m_1^2m_2^2/(64\,b^7)$
- X can be either E (electric/mass-type) or B (magnetic/current-type)
- $\mathcal{E}^X$ stands for
$$\mathcal{E}^X = f_1^X + f_2^X\log\frac{\sigma+1}{2} + f_3^X\frac{\sigma\operatorname{arccosh}\sigma}{2\sqrt{\sigma^2-1}}$$
with $f_3^X = -\left(\sigma^2 - \frac{3}{2}\right)f_2^X/(\sigma^2 - 1)$ (and so on).

Nonrelativistic limit, $\sigma = \sqrt{1 + p_\infty^2}$ and $p_\infty \to 0$,

$$\mathcal{C}^E = \tfrac{1056}{5}p_\infty - \tfrac{349}{35}p_\infty^3 + \mathcal{O}(p_\infty^5)$$
$$\mathcal{D}^E = \tfrac{1056}{5}p_\infty - \tfrac{324}{7}p_\infty^3 + \mathcal{O}(p_\infty^5)$$
$$\mathcal{C}^B = 40p_\infty^3 + \tfrac{3833}{35}p_\infty^5 + \mathcal{O}(p_\infty^7)$$
$$\mathcal{D}^B = -\tfrac{168}{5}p_\infty^3 + \tfrac{1471}{10}p_\infty^5 + \mathcal{O}(p_\infty^7)\,.$$

This offers a cross-check of the result when compared with the energy or angular momentum obtained integrating the small-p_∞ expansion of the $2 \to 3$ amplitude.

Conclusions

We propose an eikonal operator and apply it to obtain conservative and radiative observables to $\mathcal{O}(G^3)$ precision [1], including tidal effects [2].

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