RATIONAL ALGORITHMS FOR THE DECOMPOSITION OF FEYNMAN INTEGRALS VIA INTERSECTION THEORY

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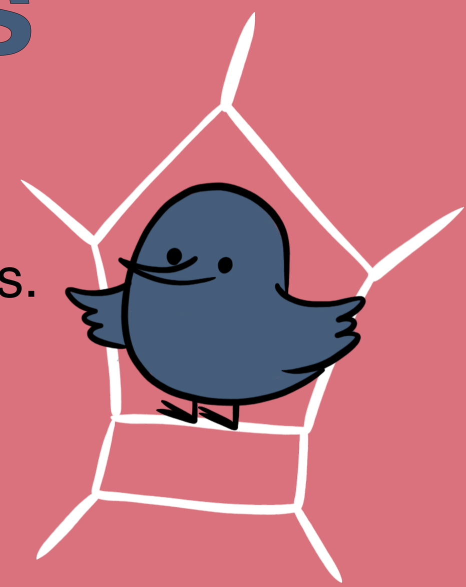
Feynman integrals as hypergeometric functions



Hypergeometric integrals are integrals of the form $\int u\varphi$ where u is a multivalued function and φ a differential $n-$form.

By making use of an appropriate integral representation, we can write multiloop Feynman Integrals as hypergeometric integrals.

We use the Baikov representation, which corresponds to the change of variables: $k_i \to z_i \equiv D_i$

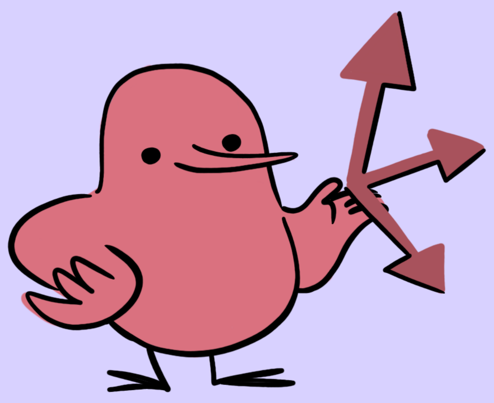
$$\int \prod_i \mathrm{d}^d k_i \prod_j \frac{1}{D_j^{a_j}} \longrightarrow K \int \mathrm{d}^n z\, B^\gamma \prod_{j=1}^{n} \frac{1}{z_j^{a_j}} = K \int u\varphi \qquad u = B^\gamma \quad \varphi = \frac{\mathrm{d}^n z}{z_1^{a_1}\dots z_n^{a_n}}$$

Decomposition = projections

Hypergeometric functions obey a vector space structure: **twisted cohomology group.**

Elements are equivalence classes of integrands evaluating to the same result

$$_\omega\langle\varphi| : \varphi \sim \varphi + \nabla_\omega \xi \qquad \nabla_\omega = \mathrm{d} + \omega\wedge \qquad \omega = \mathrm{d}\log u$$



Vector space structure characterized by

. dimension ν (number of **master integrals)**

. basis $\langle e_i|$ and dual basis $|h_j\rangle$

. scalar products $\langle\varphi_L|\varphi_R\rangle$

Therefore we can introduce a **metric**, $C_{ij} = \langle e_i|h_j\rangle$

$$\langle\varphi| = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle\,(C^{-1})_{ji}\,\langle e_i|$$

The problem of decomposition into master integrals is turned into the problem of **finding the projections on the basis vectors** using scalar products.

Calculating intersection numbers

Intersection numbers of differential 1-forms

$$\langle\varphi_L|\varphi_R\rangle = \sum_{p\in\mathcal{P}_\omega} \mathrm{Res}_{z=p}(\psi\varphi_R)$$

where ψ is the local solution of $\frac{\mathrm{d}}{\mathrm{d}z}\psi + \omega\psi = \varphi_L$

around each $p \in \mathcal{P}_\omega$

$$\mathcal{P}_\omega = \{z\,|\,z \text{ is a pole of } \omega\} \cup \{\infty\}$$

Ansatz

$$\psi = \sum_{j=\min}^{\max} \psi_p^{(j)}(z-p)^j + \mathcal{O}\left((z-p)^{\max+1}\right)$$

Multivariate intersection numbers

Recursive procedure: intersection numbers of $n-$forms rely on the calculation of intersection numbers of $(n-1)-$forms

$$z_1 \to \cdots \to z_n$$

.Generalization of univariate procedure

.Variables need to be ordered and one proceeds one fibration at time

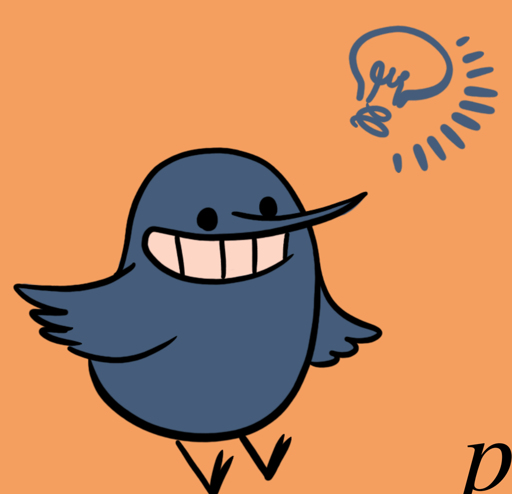
Irrational poles and rational algorithms

. **Starting point:** rational integrands

. Appearance of **non-rational poles** in intermediate steps of the calculation

. **Result**: intersection numbers are rational functions of the kinematic invariants and of the dimensional regulator

Based on:



⇨ Cancellations must happen in intermediate steps of the calculation

Non-rational poles $z^\star$ satisfy polynomial equations $p(z^\star) = 0$

$p(z)$ polynomial irreducible (**prime**) over $\mathbb{Q}$

$p(z)-$adic expansion

Consists in the series expansion of a rational function in powers of a polynomial "prime" over the field of rational numbers

$$\psi = \sum_i c_i(z)p(z)^i \qquad c_i(z) = \sum_{j=1}^{\deg p-1} c_{ij}z^j \qquad c_{ij} \in \mathbb{Q}$$

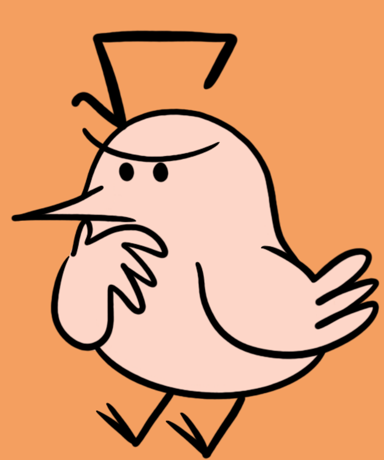
This allows to calculate the contributions to $\langle\varphi_L|\varphi_R\rangle_{p(z)}$ of **all the roots** of $p(z)$ **at once** using global residue theorem

$$\mathrm{Res}_{p(z)}(\psi\varphi_R) = \mathrm{Res}_{p(z)}\left(\dots + \frac{\sum_{j=0}^{\deg p-1}\tilde{c}_j z^j}{p(z)} + \dots\right) = \frac{\tilde{c}_{\deg p-1}}{l_c}$$

Motivations for purely rational algorithms

. Avoid algebraic bottlenecks

. Cutting-edge techniques require rational algorithms (e.g. finite fields, rational reconstruction)



Analogy with $p-$adic numbers

$$r = \sum_{i=k}^{\infty} a_i p^i \qquad r \in \mathbb{Q}$$

Formal series expansion of a rational number in powers of a prime p