Extraction of high-order post-Minkowskian results from self-force calculations

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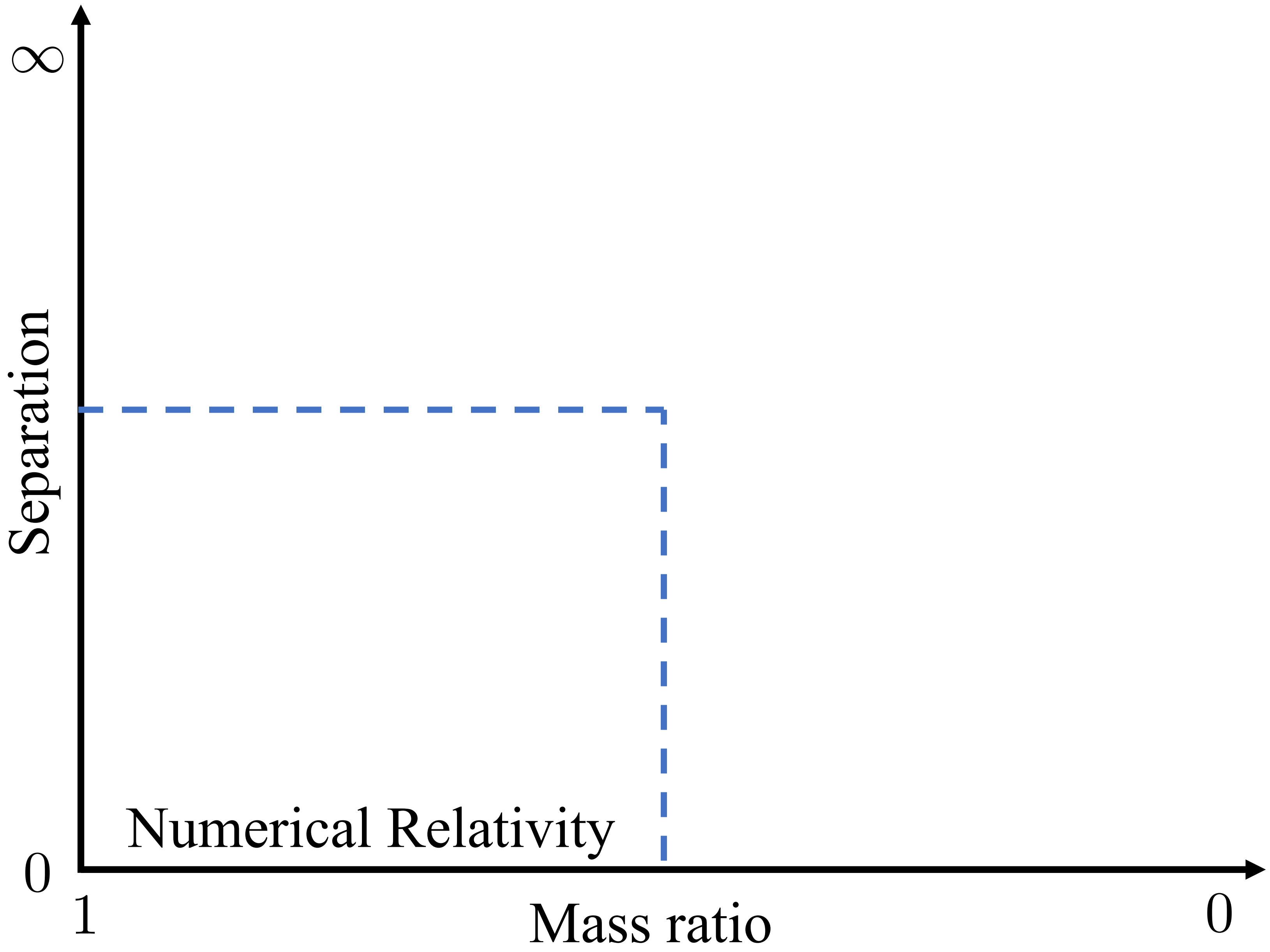
    - Probing higher-order PM terms.

Two-body parameter space for scattering



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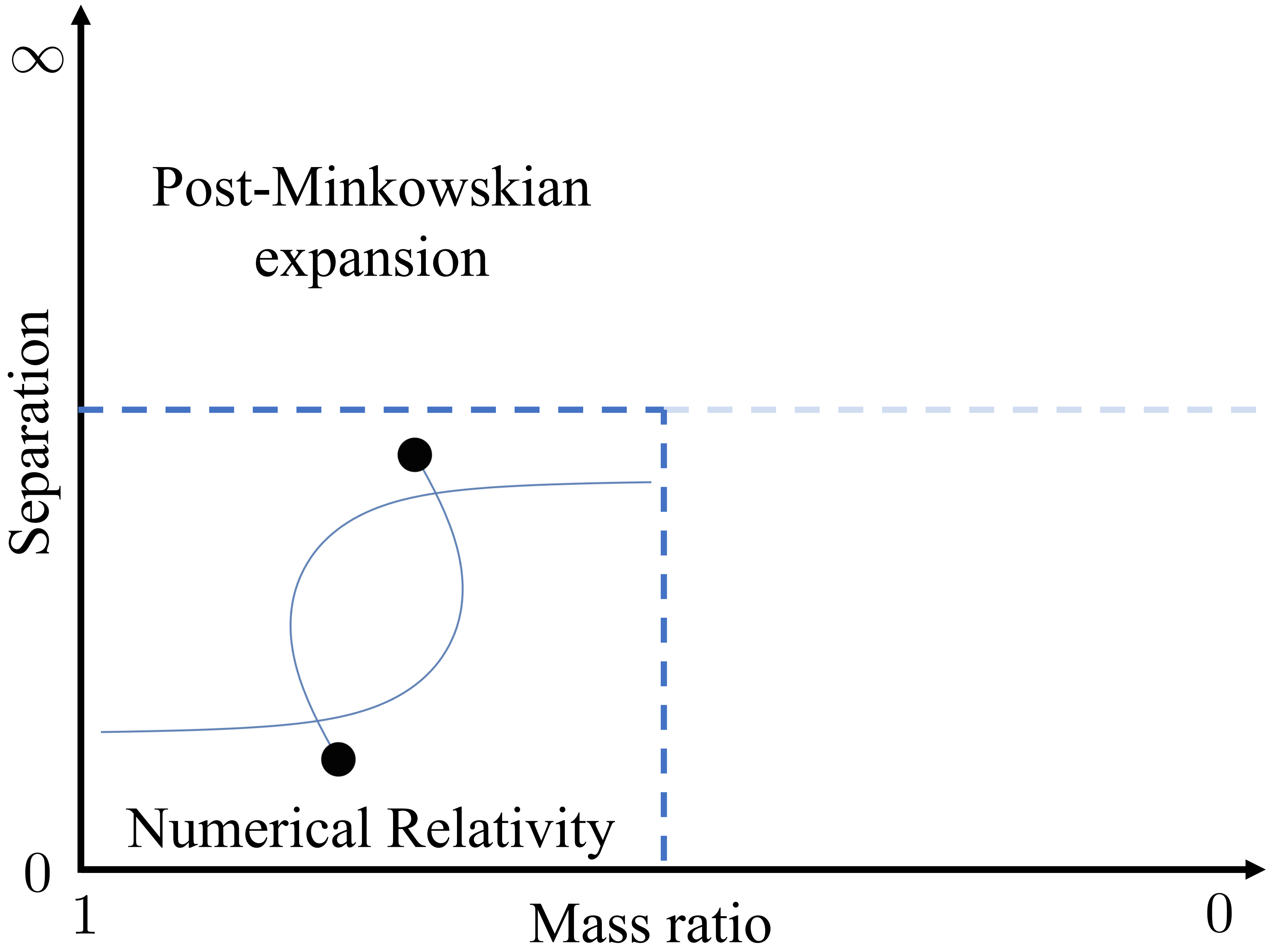

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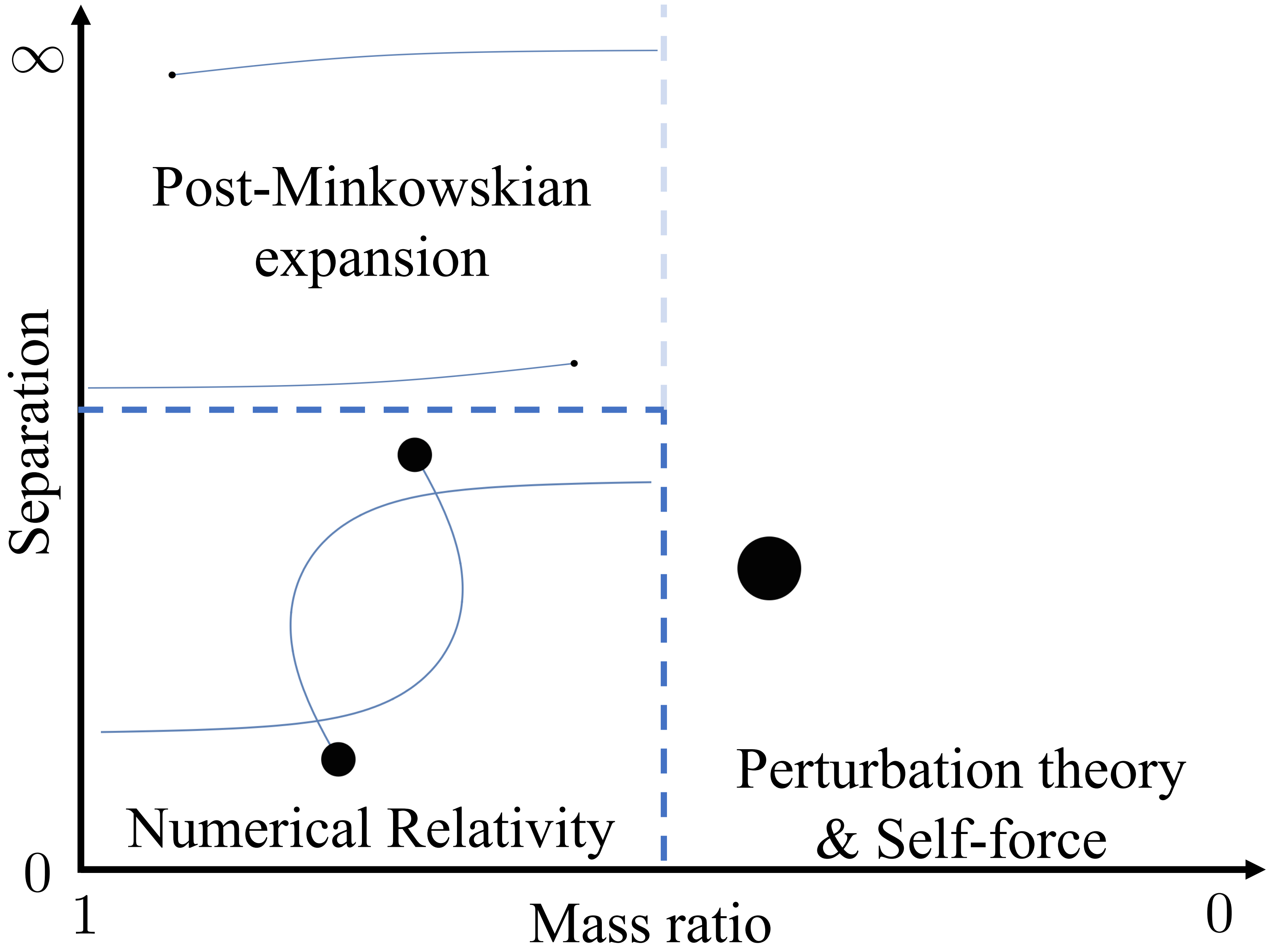

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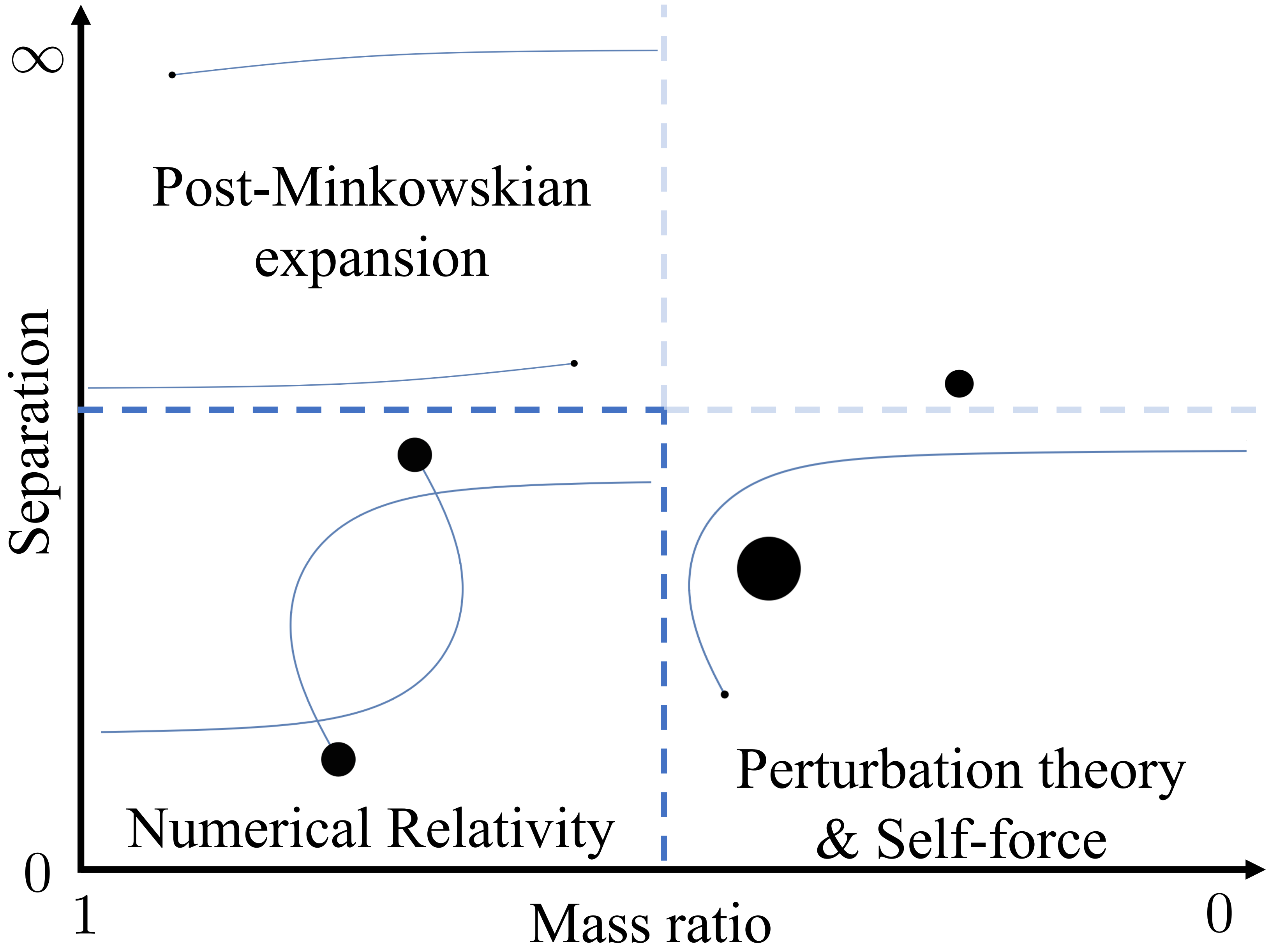

Motivation: Extreme-mass ratio scatter orbits

- Clean environment with well-defined asymptotic states.
- Exact post-Minkowskian calculations for bound orbits [Damour ’19; Bini, Damour & Geralico ’20].

$$1\text{SF} \rightarrow 4\text{PM}$$

$$2\text{SF} \rightarrow 6\text{PM}$$

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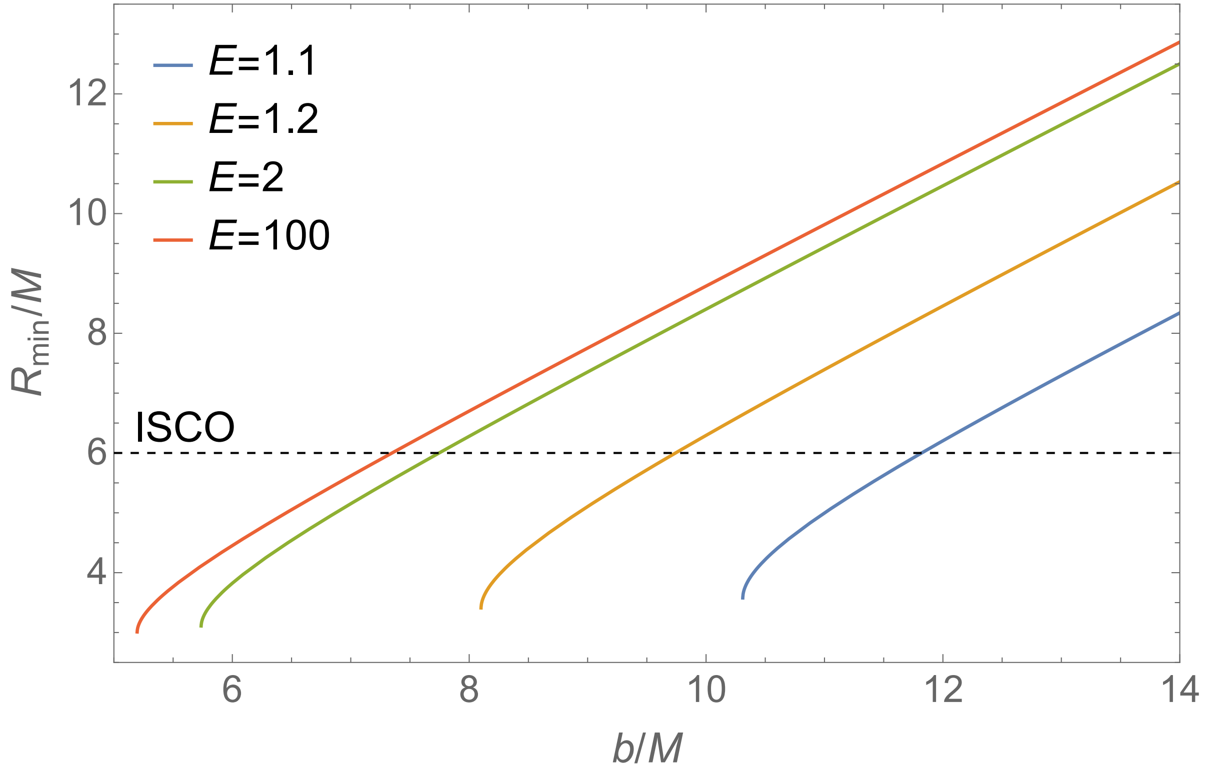
- Calibration of Effective-One-Body (EOB) in the ultra-strong field.



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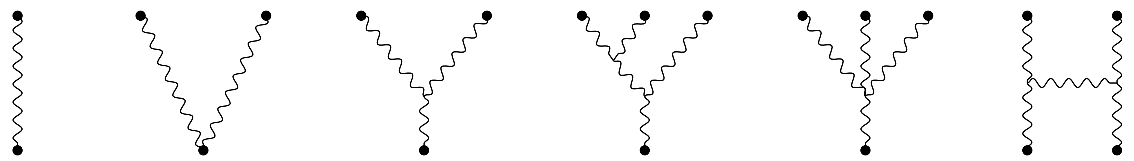
General Relativity

High-energy physics

- Calibration of Effective-One-Body (EOB) in the ultra-strong field.
- Comparisons with scatter amplitude calculations:
  - QFT and EFT [Bern et al. ‘21].



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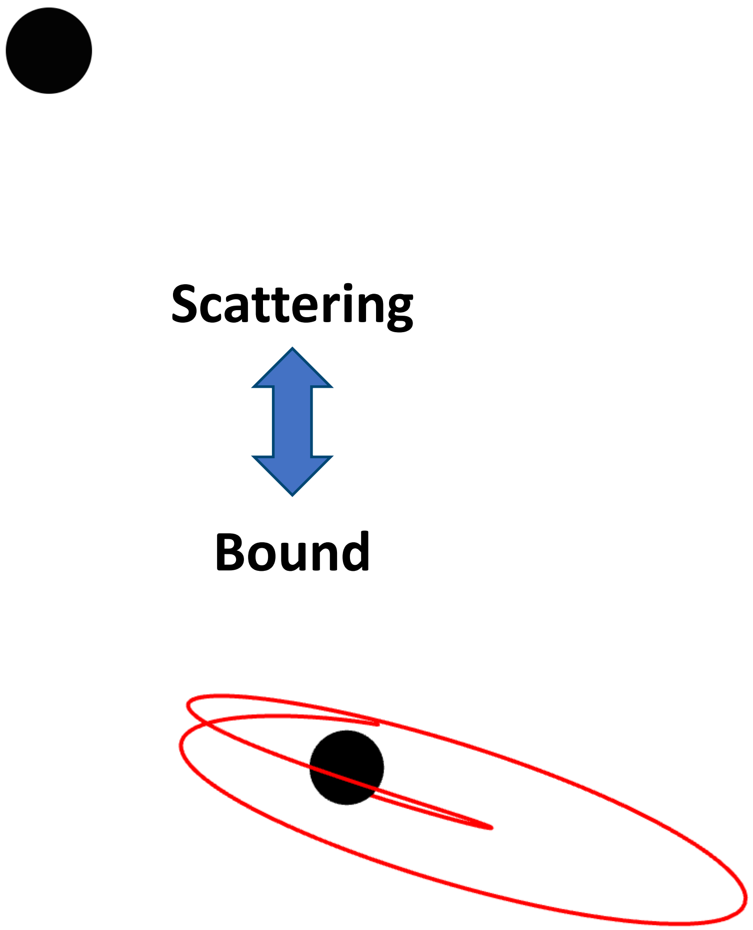


- Calibration of Effective-One-Body (EOB) in the ultra-strong field.
- Comparisons with scatter amplitude calculations:
  - QFT and EFT [Bern et al. ‘21].
- Dictionary between scatter and bound [Cho et al. ‘21].
  - Scattering angle ↔ periastron advance.

Scattering geodesics

Energy and angular momentum:

$$E > 1 \qquad\qquad L > L_{\text{crit}}(E)$$

Scattering geodesics

Energy and angular momentum:

$$E > 1 \qquad L > L_{\rm crit}(E)$$

Velocity at infinity and impact parameter:

$$v := \left.\frac{dr}{dt}\right|_{r\to\infty} \qquad b := \lim_{r\to\infty} r \sin|\varphi(r) - \varphi(\infty)|$$

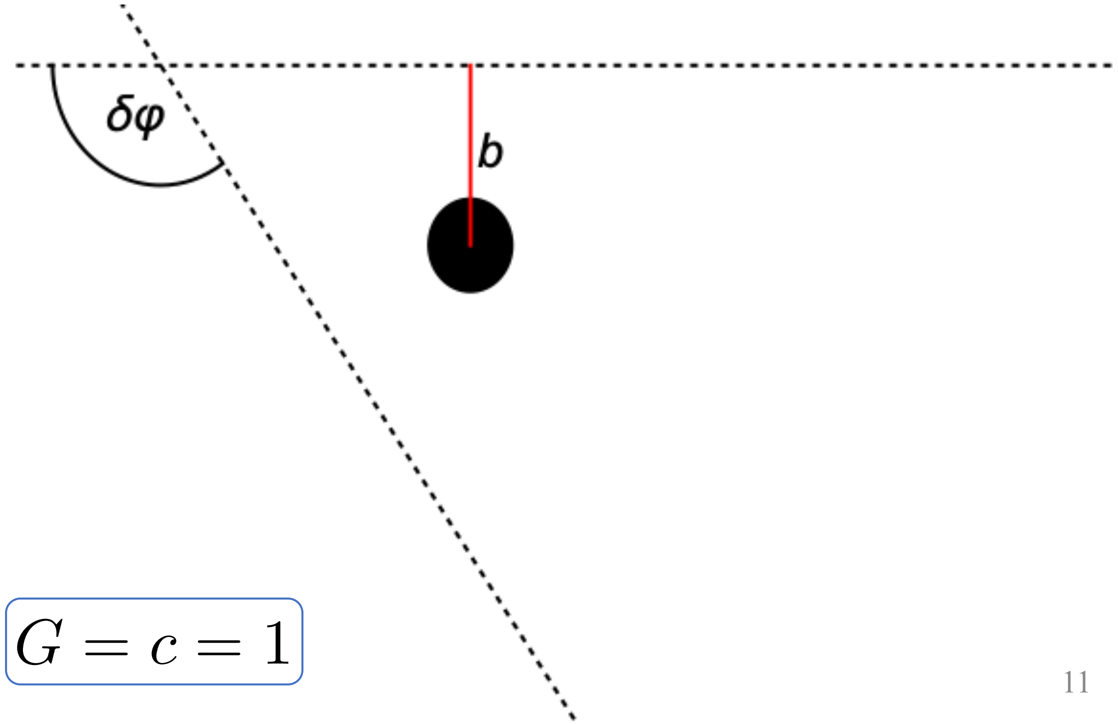
Geodesics:

$$\frac{dt}{d\tau} = \frac{\boxed{E}\, r}{r - 2M} \qquad \frac{d\varphi}{d\tau} = \frac{\boxed{L}}{r^2}$$

$$\left(\frac{dr}{d\tau}\right)^2 = \boxed{E^2} - V(\boxed{L}; r)$$

Scattering angle:

$$\delta\varphi := \int_{-\infty}^{\infty} \frac{d\varphi}{dt} dt - \pi$$



$$\boxed{G = c = 1}$$

Self-forced equations of motion

Expansion in the mass ratio:

$$\mathrm{g}_{\alpha\beta} = g_{\alpha\beta} + \eta h^{(1)}_{\alpha\beta} + \eta^2 h^{(2)}_{\alpha\beta} + \ldots \qquad \eta := \frac{\mu}{M}$$

Schwarzschild/Kerr

Perturbed equations of motion:

$$E = E_\infty - \eta \int_{-\infty}^{\tau} F_t \, d\tau \qquad L = L_\infty + \eta \int_{-\infty}^{\tau} F_\varphi \, d\tau$$

Geodesic

Can split self-force into conservative and dissipative pieces:

$$F^{\text{cons}}_\alpha(r, \dot{r}) = -F^{\text{cons}}_\alpha(r, -\dot{r}) \qquad F^{\text{diss}}_\alpha(r, \dot{r}) = F^{\text{diss}}_\alpha(r, -\dot{r}) \qquad \alpha = t, \varphi$$

Dissipative self-force removes energy and angular momentum from the system.

Self-force correction to the scattering angle

Scattering angle as a radial integral:

$$\delta\varphi = \sum_{\pm} \int_{r_p^\pm}^{\infty} \frac{\dot\varphi^\pm}{\dot r^\pm} dr - \pi = \sum_{\pm} \int_{r_p^\pm}^{\infty} \frac{H^\pm(r; E, L)}{\sqrt{r - r_p^\pm}} dr - \pi$$

Perturb equation [Barack & OL '22]:

$$\delta\varphi = \delta\varphi^{(0)} + \eta\delta\varphi^{(1)}$$

$$\delta\varphi^{(1)} = \sum_{\pm} \int_{R_{\min}}^{\infty} \left[\mathcal{G}_E^\pm(r; E_\infty, L_\infty) F_t^\pm - \mathcal{G}_L^\pm(r; E_\infty, L_\infty) F_\varphi^\pm\right] dr$$

Can split into conservative and dissipative pieces on outgoing leg:

$$\delta\varphi^{(1)}_{\rm cons} = \int_{R_{\min}}^{\infty} \left[\mathcal{G}_E^{\rm cons} F_t^{\rm cons} - \mathcal{G}_L^{\rm cons} F_\varphi^{\rm cons}\right] dr \qquad \delta\varphi^{(1)}_{\rm diss} = \int_{R_{\min}}^{\infty} \left[\mathcal{G}_E^{\rm diss} F_t^{\rm diss} - \mathcal{G}_L^{\rm diss} F_\varphi^{\rm diss}\right] dr$$

Scalar self-force model

Endow particle with a spin-0 scalar charge q.

New small parameter:

$$\eta := \frac{q}{\mu M}$$

Scalar field obeys the Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}}\partial_\alpha \left(\sqrt{-g}\,\partial^\alpha \Phi\right) = -4\pi q \int_{-\infty}^{\infty} \frac{1}{\sqrt{-g}}\,\delta^4\left(x^\mu - x_p^\mu(\tau)\right)\, d\tau$$

Scalar self-force:

$$F_\mu := q\nabla_\mu \Phi\Big|_{x_p}$$

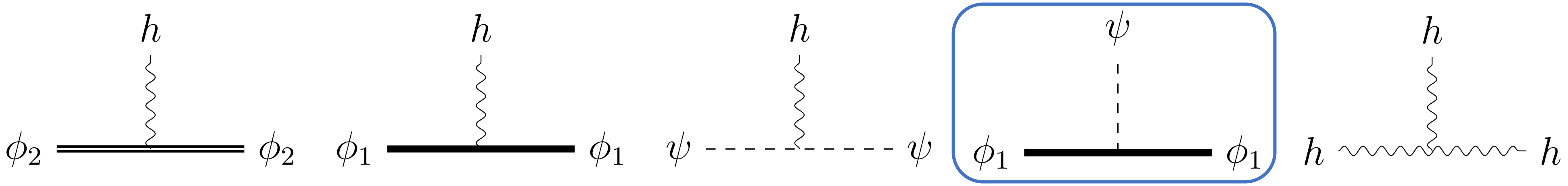
Scalar self-force in terms of amplitudes

Lagrangian:

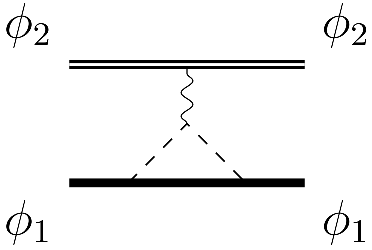
$$S = \int d^D x \sqrt{-g} \left[-\frac{2}{\kappa^2} R + \frac{1}{2}\phi_1(\Box + m_1^2)\phi_1 + \frac{1}{2}\phi_2(\Box + m_2^2)\phi_2 + \frac{1}{2}\psi \Box \psi + \boxed{\frac{1}{2} q \psi \phi_1^2} \right]$$

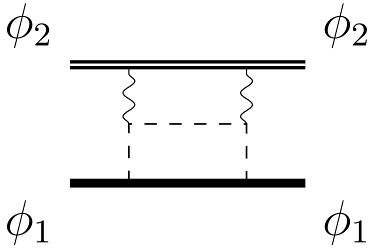
$\phi_{1,2}$: black holes $\qquad\qquad$ ψ : scalar field

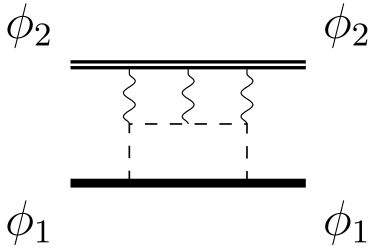
Three-point interaction vertices:



Only keep terms which are linear in mass-ratio and proportional to q^2 :

2PM :


3PM :


4PM :


[Cheung, Rothstein, Solon] [Bern, Cheung, Roiban, Shen, Solon, Zeng] [Bern, Cheung, Para-Martinez, Roiban, Ruf, Shen, Solon, Zeng]

Scattering angle correction: PM expansion

2PM [Gralla & Lobo '22]:

$$\delta\varphi^{2\mathrm{PM}}_{\mathrm{cons}} = -\frac{\pi}{4}\left(\frac{M}{b}\right)^2 \qquad \delta\varphi^{2\mathrm{PM}}_{\mathrm{diss}} = 0$$

3PM:

$$\delta\varphi^{3\mathrm{PM}}_{\mathrm{cons}} = -\frac{4\left(3-v^2\right)}{3v^2\sqrt{1-v^2}}\left(\frac{M}{b}\right)^3 \qquad \delta\varphi^{3\mathrm{PM}}_{\mathrm{diss}} = \frac{2\left(v^2+1\right)^2}{3v^3\sqrt{1-v^2}}\left(\frac{M}{b}\right)^3$$

4PM dissipative:

$$\delta\varphi^{4\mathrm{PM}}_{\mathrm{diss}} = \left(r_1 + r_2\,\mathrm{arcsech}\left(\sqrt{1-v^2}\right) + r_3\log\left[\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right]\right)\left(\frac{M}{b}\right)^4$$

$$r_i = \text{rational coefficients}$$

Scalar field evolution scheme

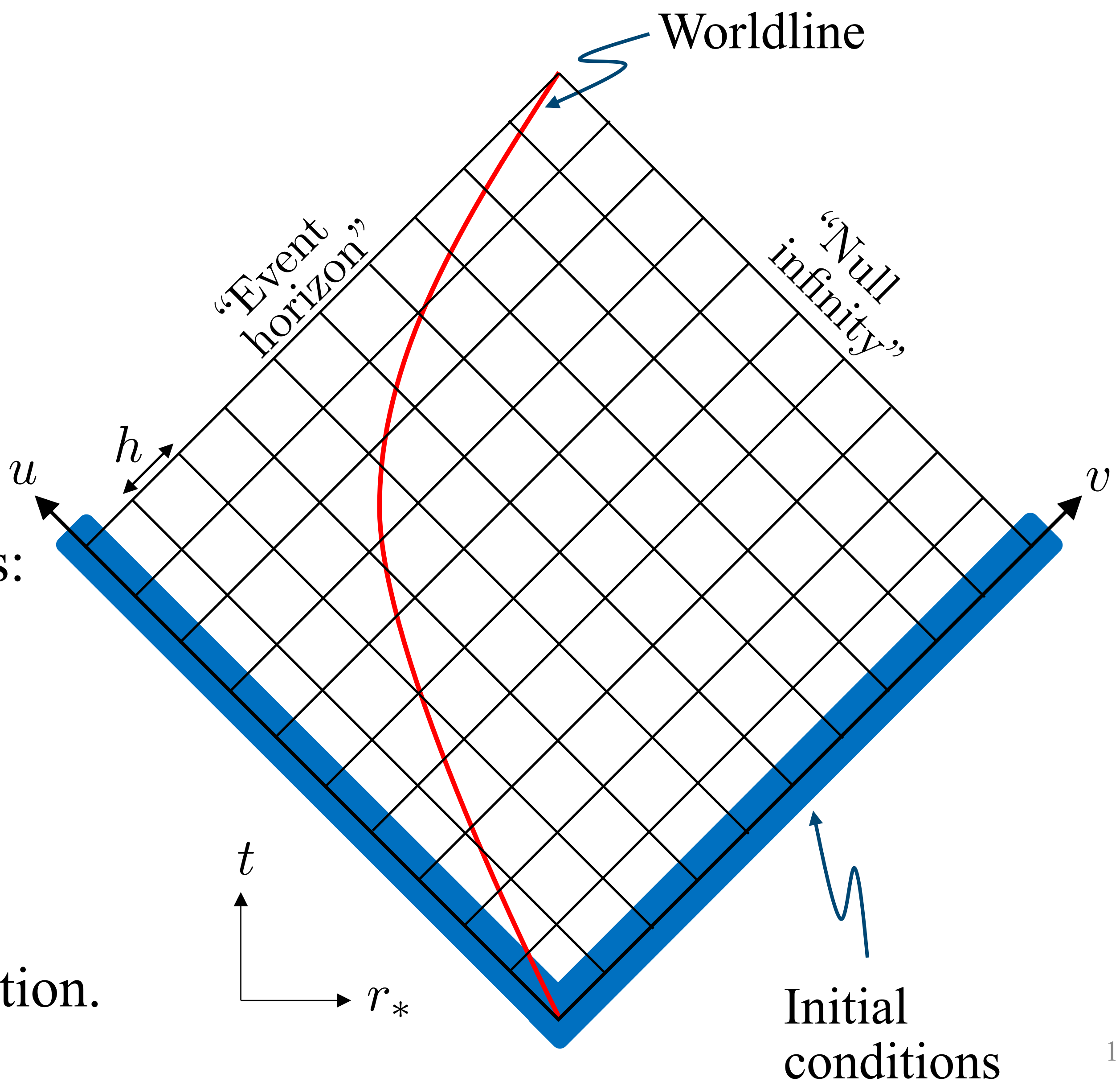
Decompose scalar field in the time-domain:

$$\Phi = \frac{2\pi q}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \psi_{\ell m}(t,r) Y_{\ell m}(\theta,\varphi)$$

1+1D scalar wave equation in null-coordinates:

$$\psi_{,uv} + V(\ell;r)\psi = S_\psi(\ell;x_p^\mu)\,\delta\,(r-R)$$

Evolve finite-difference version of 1+1D equation.



Post-processing: Truncation at finite radius

Can only numerically determine the self-force up to a finite radius $R = R_{\text{final}}$:

$$\delta\varphi^{(1)} = \int_{R_{\text{min}}}^{R_{\text{final}}} [\mathcal{G}_E F_t - \mathcal{G}_L F_\varphi]\, dr + \int_{R_{\text{final}}}^{\infty} [\mathcal{G}_E F_t - \mathcal{G}_L F_\varphi]\, dr$$

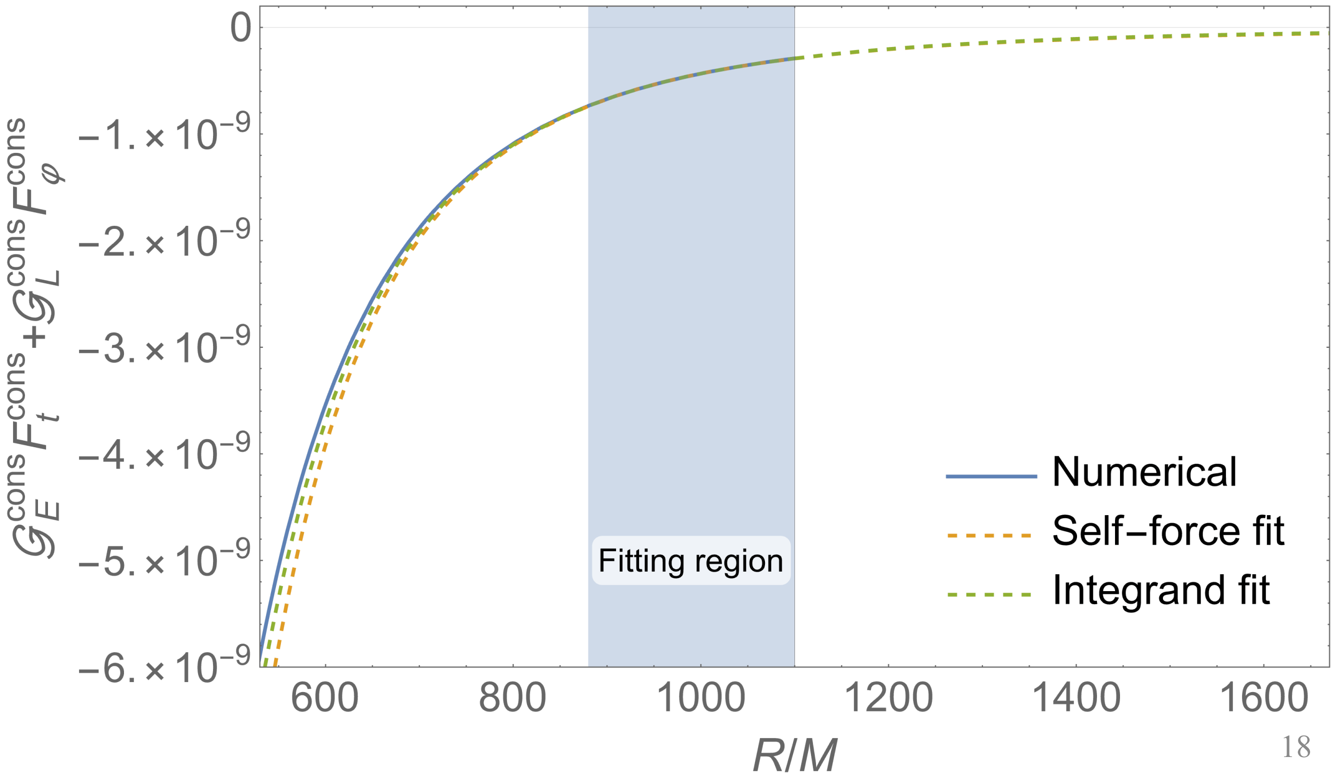
Numerical

Error (~1%)

Form an analytic tail by fitting to the data:

- Fit the self-force data.
- Fit the integrand directly.

Tail contributes an error ~ 0.01%.



Post-processing: Richardson extrapolation

Next dominant error due to finite resolution ~ 0.1%.

Can increase the convergence from quadratic to cubic using Richardson extrapolation.

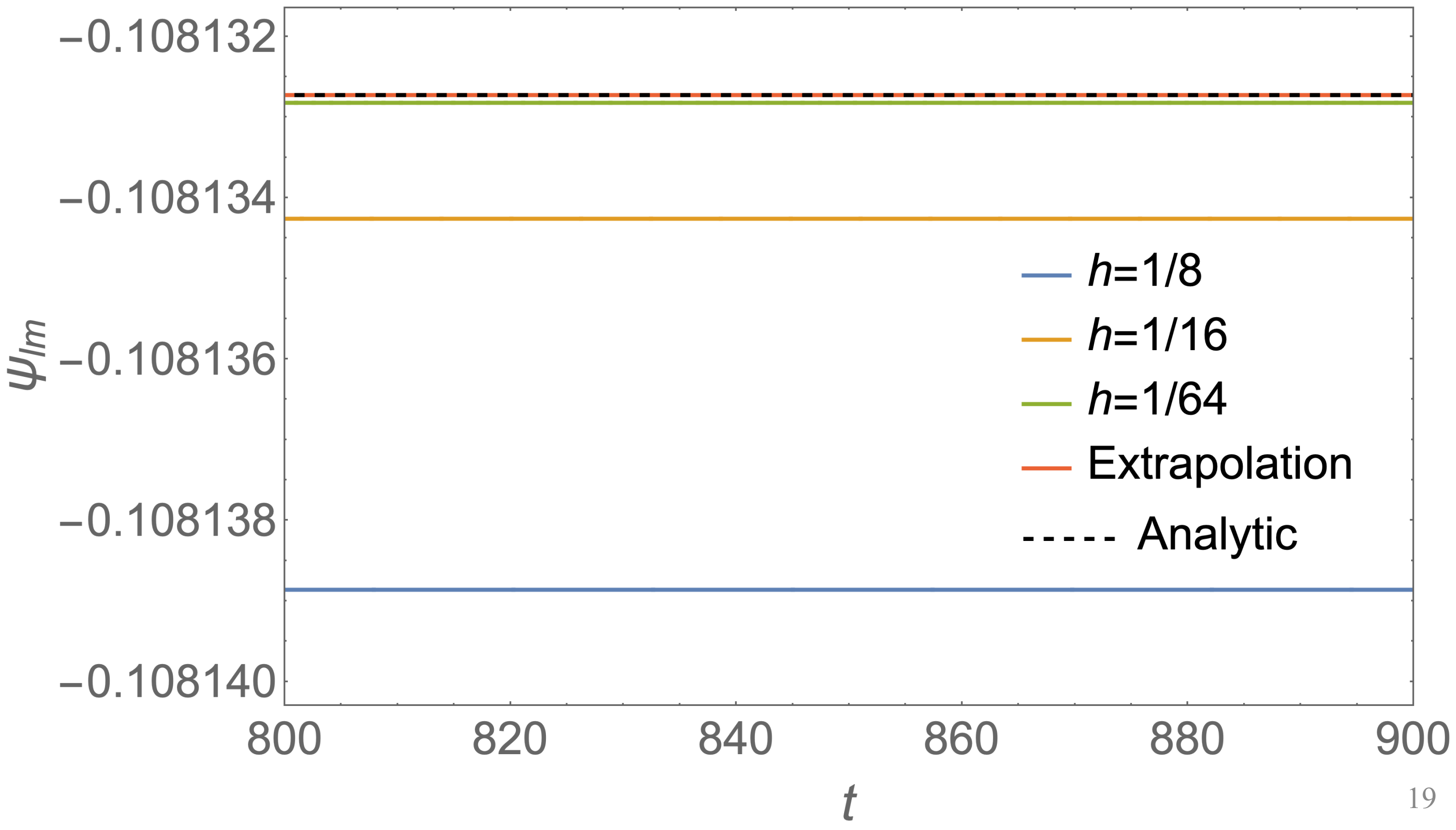
Model:

$$A(h) = A_{\text{exact}} + Ch^n + O\left(h^{n+1}\right)$$

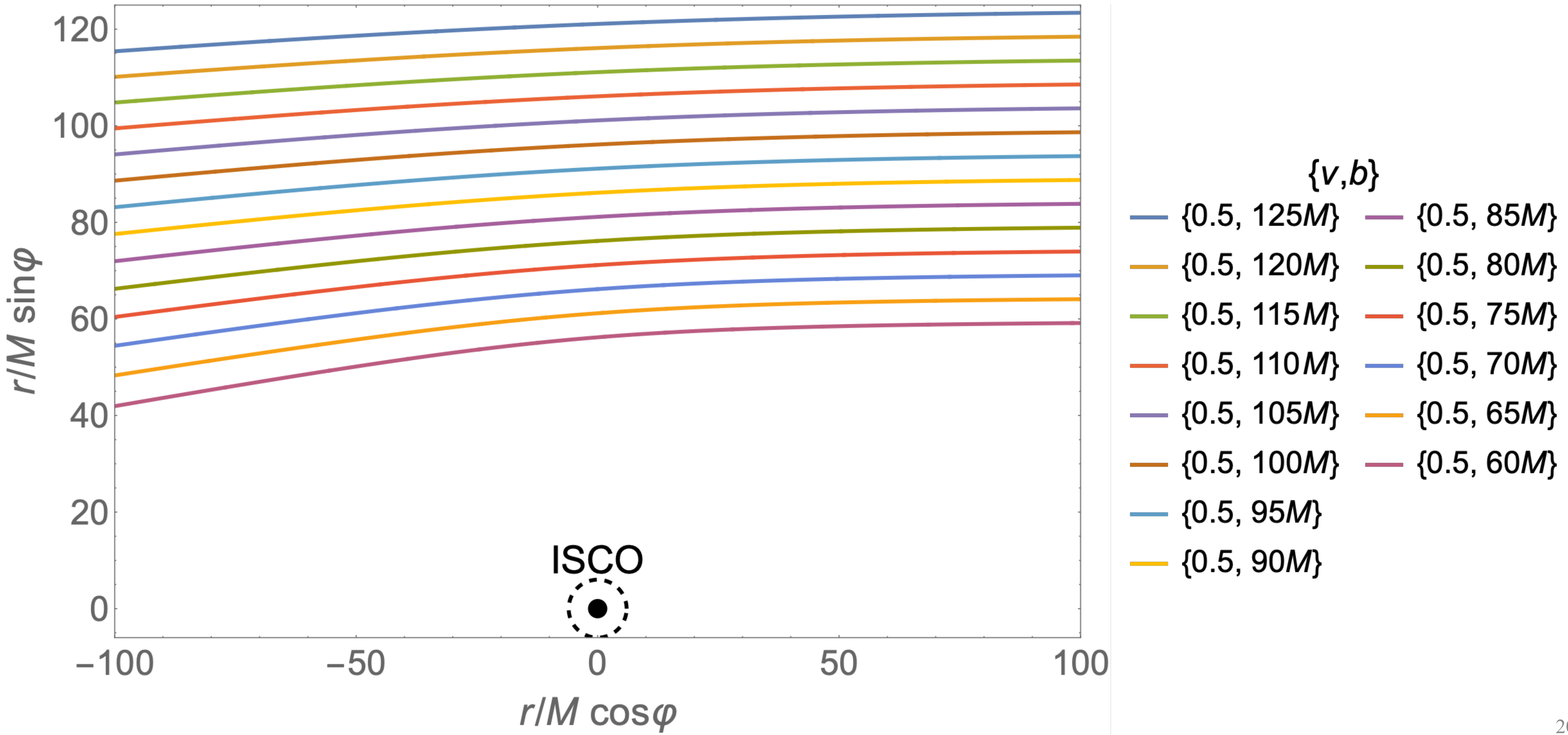
Extrapolation:

$$A_{\text{Extr}} = \frac{t^n A(\frac{h}{t}) - A(h)}{t^n - 1} = A_{\text{exact}} + O(h^{n+1})$$

Error in extrapolation < 0.001%.

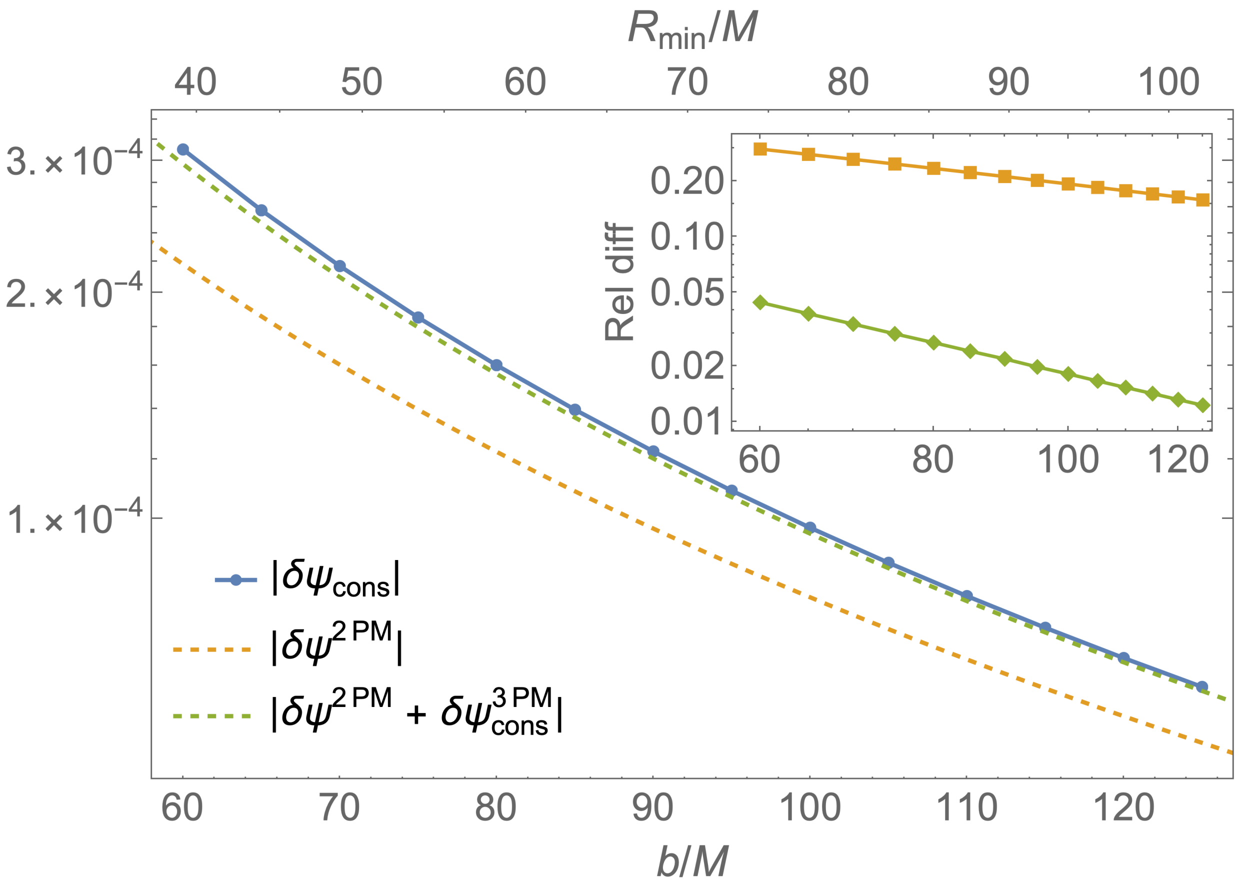


Sample orbits

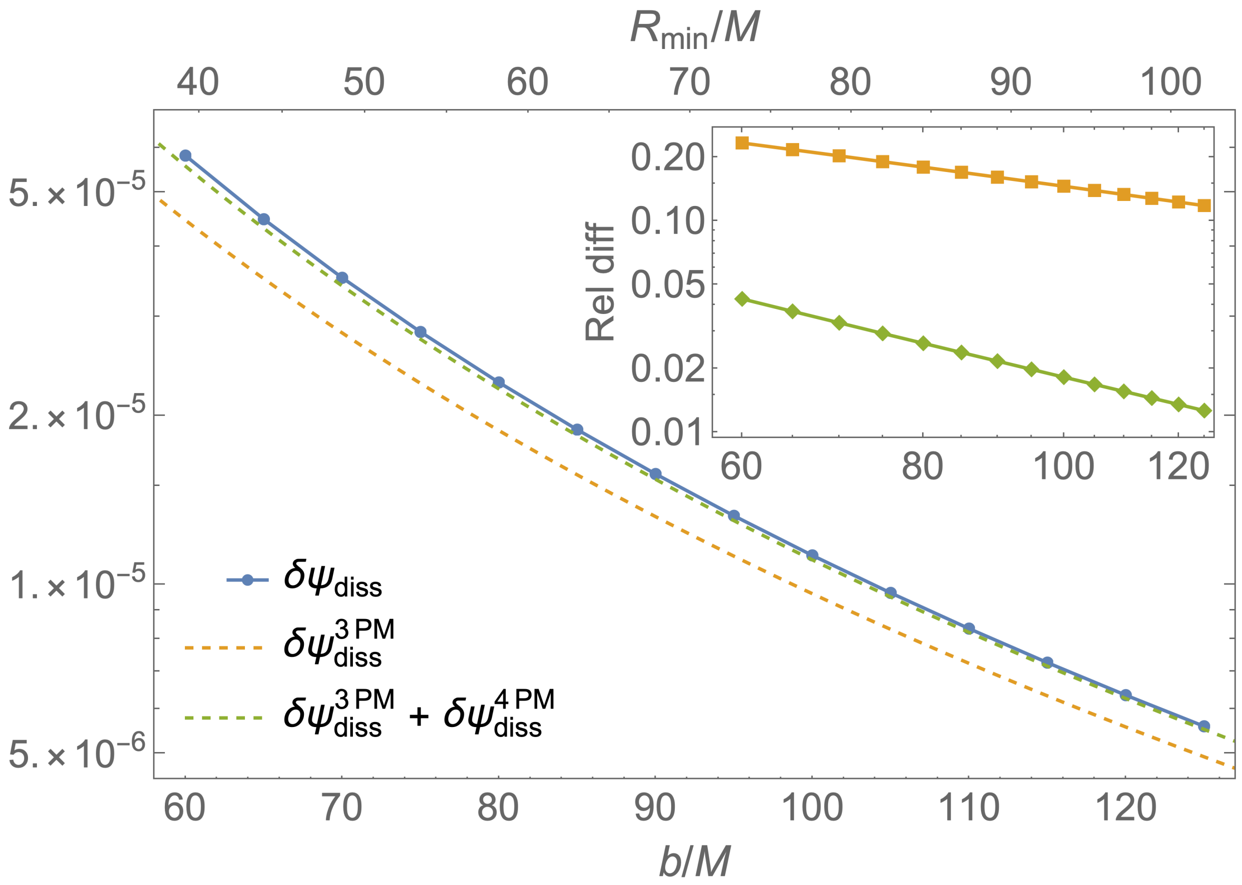


PM comparison: $v = 0.5$

Conservative



Dissipative



Scattering angle correction: 4PM conservative

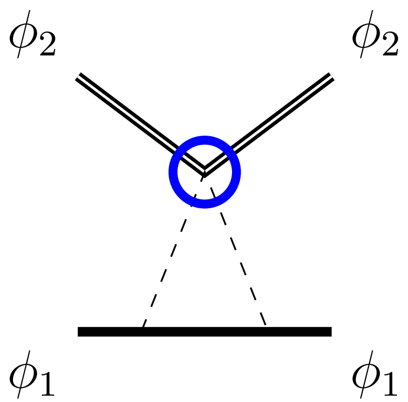
$$
\begin{aligned}
\delta\varphi^{\rm 4PM}_{\rm cons} = \Bigg(& r_1 + r_2 \,\mathrm{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right) + r_3 \,\mathrm{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right)^2 + r_4 \mathsf{E}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)^2 \\
& + r_5 \mathsf{K}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right) \mathsf{E}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right) + r_6 \mathsf{K}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)^2 \\
& + r_7 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) + r_8 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) \,\mathrm{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right) \\
& + r_9 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) \log\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) + r_{10} \log\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) \\
& + r_{11} \log^2\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) + r_{12}\,\boxed{\alpha} + r_{13}\frac{\boxed{\beta}}{v^2} + r_4 \log(b)\Bigg)\left(\frac{M}{b}\right)^4
\end{aligned}
$$

Elliptic integrals

Free coefficients

Log term

r_i = rational coefficients



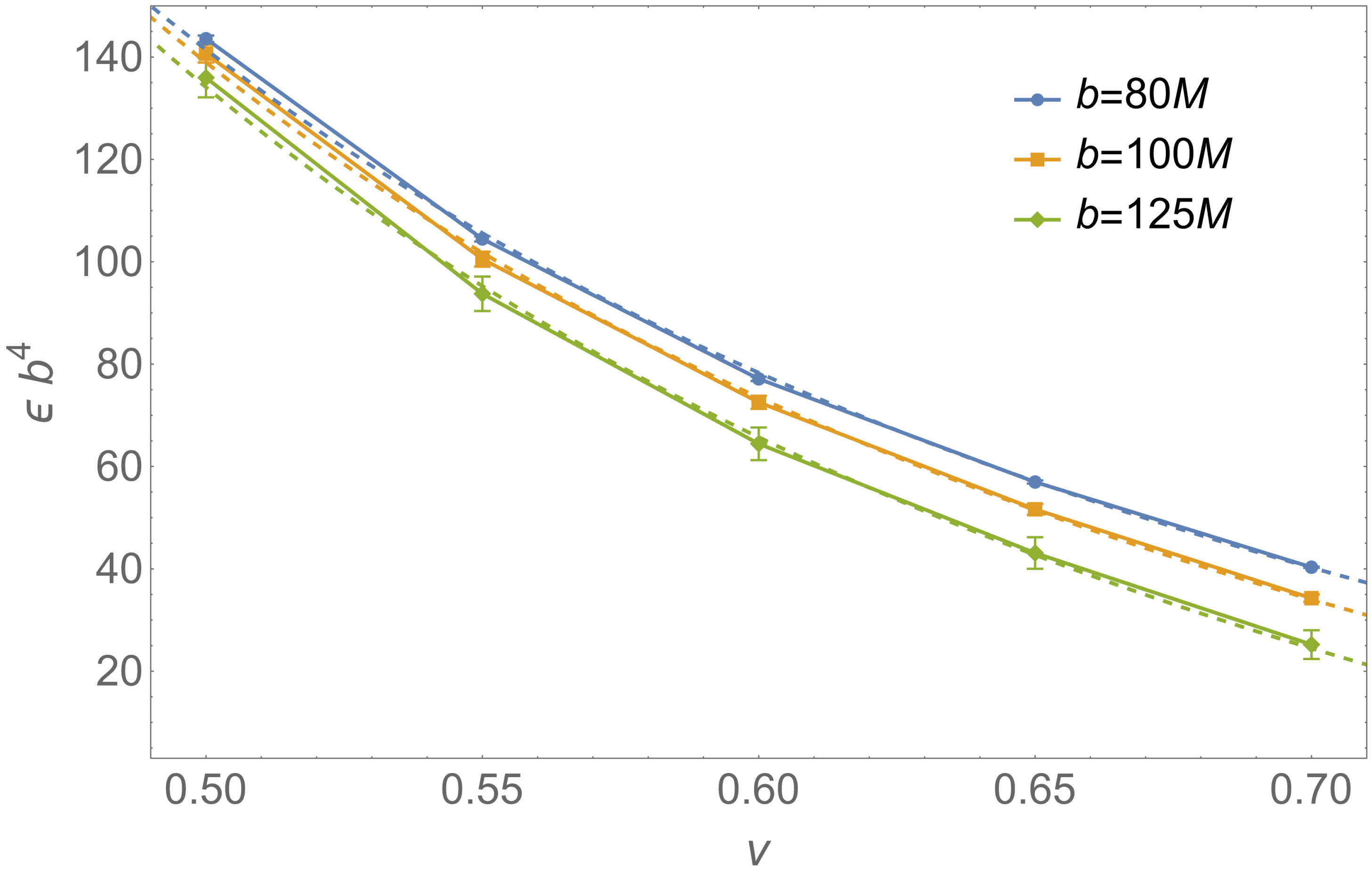
Preliminary

Extraction of high-order conservative PM results

Subtract known analytic parts:

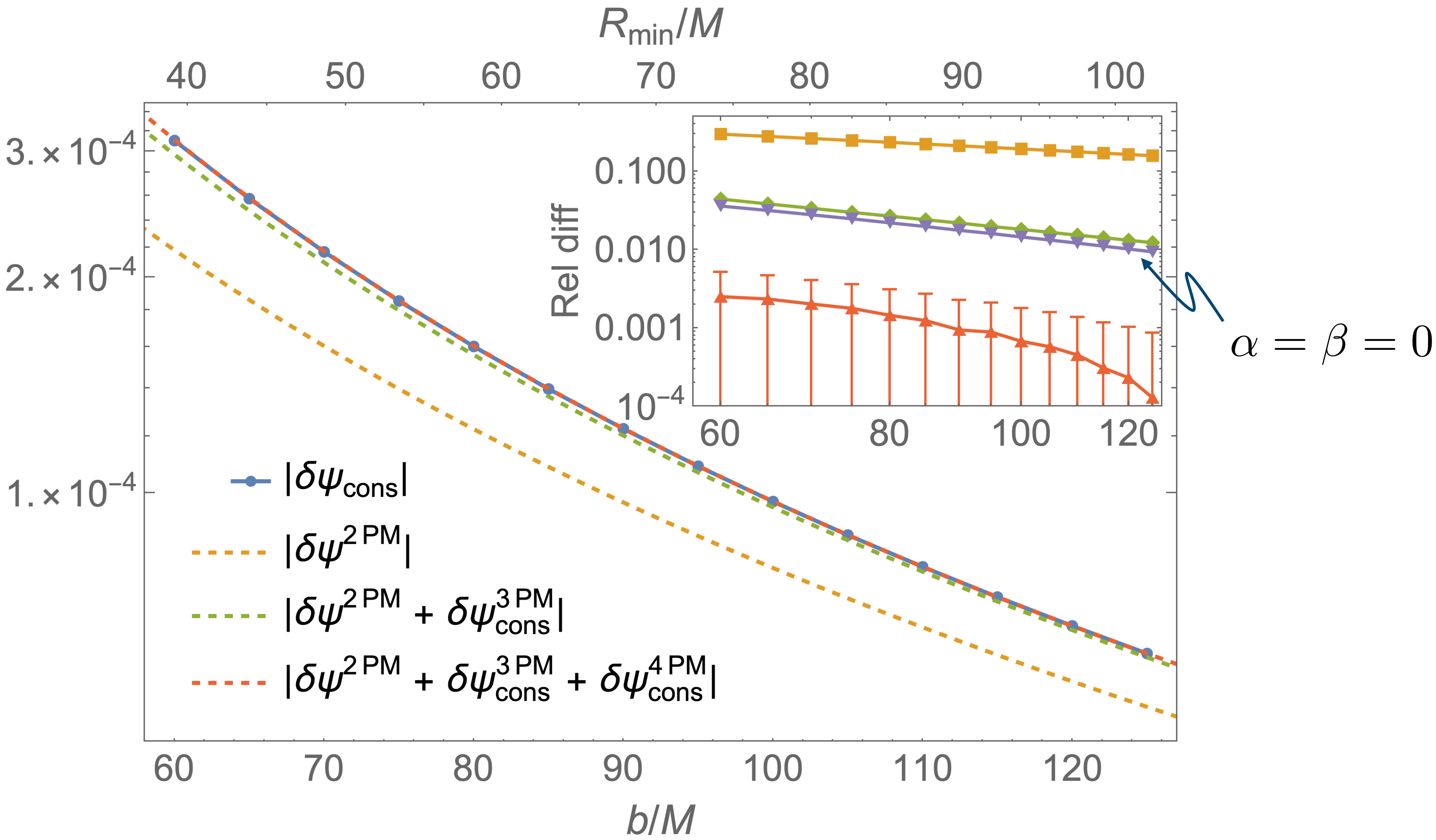
$$\epsilon := \delta\phi_{\text{cons}} - \delta\phi^{2\text{PM}}_{\text{cons}} - \delta\phi^{3\text{PM}}_{\text{cons}} - \frac{A(v) + B\log(b)}{b^4} = \frac{\pi}{32}\left(\alpha + \frac{\beta}{v^2}\right)\frac{1}{b^4} + O\left(\frac{1}{b^5}\right)$$

b/M	α	β
80	-202.6 ± 1.7	20.07 ± 0.09
100	-96.8 ± 2.7	8.52 ± 0.13
125	-114 ± 8	8.91 ± 0.35



Preliminary

PM comparison: Conservative $v = 0.5$



Preliminary

Extraction of high-order dissipative PM results

PM expansion with free parameters:

$$\delta\varphi^{\rm PM}_{\rm diss} = \frac{a_3}{b^3} + \frac{a_4}{b^4} + \frac{a_5}{b^5} + \frac{a_6}{b^6} + \ldots$$

Up to 4PM can fit value or use **analytic value**.

a_3	a_4	a_5	a_6
11.2 ± 0.1			
9.439 ± 0.01	184 ± 1		
9.632 ± 0.002	143.8 ± 0.4	1847 ± 17	
9.593 ± 0.002	152.7 ± 0.3	990 ± 31	25900 ± 900
9.6225	166.7 ± 1.3		
9.6225	146.82 ± 0.09	1707 ± 7	
9.6225	148.0 ± 0.02	1476 ± 34	-62100 ± 8700
9.6225	**143.344**	1987 ± 17	
9.6225	**143.344**	2372 ± 29	-25500 ± 2100
$< 1\%$	$\sim 1\%$	$\sim 1600(?)$	???

Preliminary

Summary and future work

Compared numerical scalar self-force results with analytic PM results.

Good agreement for both conservative and dissipative:

- ~10% for LO.
- ~ 1% for NLO.

Have investigated extractions of higher-order components from numerics:

- Fixed free parameters in 4PM conservative: agreement with numerics to < 0.5%.
- Extraction of 5PM dissipative coefficient.

Future work:

- Calculate the gravitational self-force correction to the scattering angle.
- Compare to PM(/NR/EOB) in the gravitational case.

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