Non-perturbative S-matrices from dispersive iterations in d=3 & 4

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In collaboration with A. Zhiboedov:
JHEP 2021, and 2023 (to appear)


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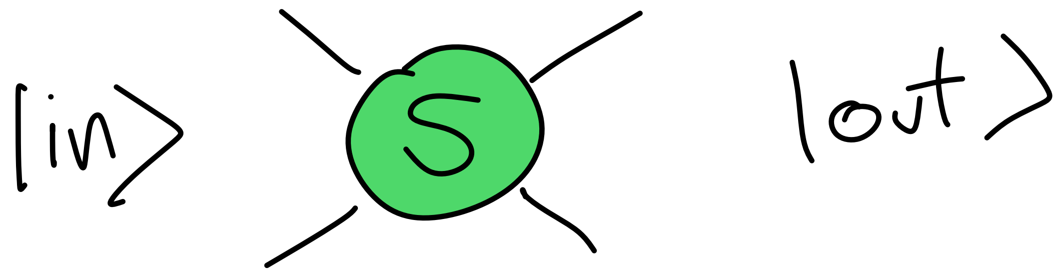
Outline

- Motivations and introduction
  - Unitarity
  - Scattering from production and Atkinson
- Results
  - numerical implementation
  - Aks physics (“scattering implies production”)
  - Regge physics

Introduction

Goal : compute S-matrix *functions* which satisfy analyticity, crossing, and non-perturbative unitarity



- Perturbative unitarity : amplitudes' methods

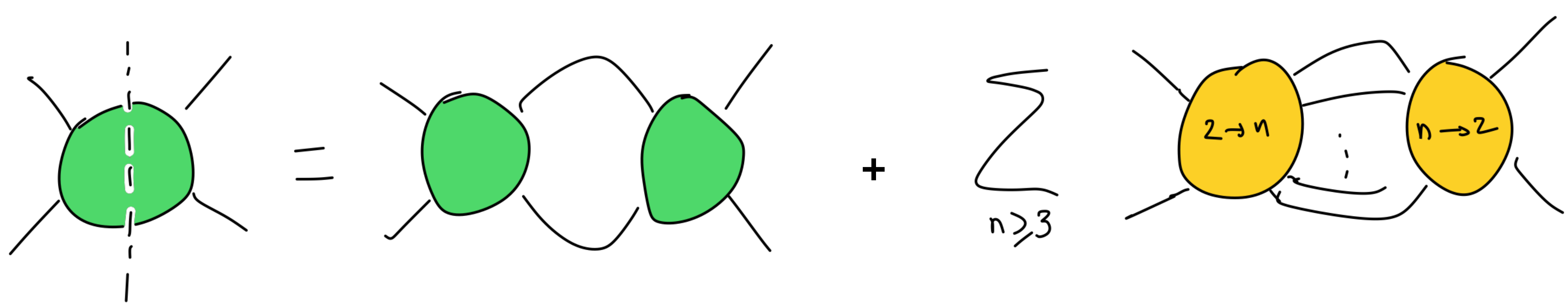
- Non-pert. unitarity : extremely hard to combine with crossing. As of today, no consistent amplitudes have been built in d≥3

- CFT numerical bootstrap has revived the hope that the S-matrix bootstrap of the 60's can be revisited today with modern computers' power.

Introduction

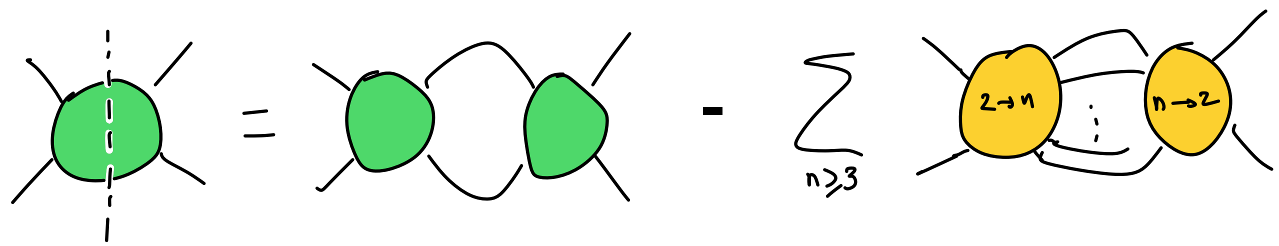
Restrict to $2 \to 2$ amplitudes

Still very hard, and they contain info about all $2 \to n$ processes, via optical theorem



Introduction

Our approach : Scattering from production



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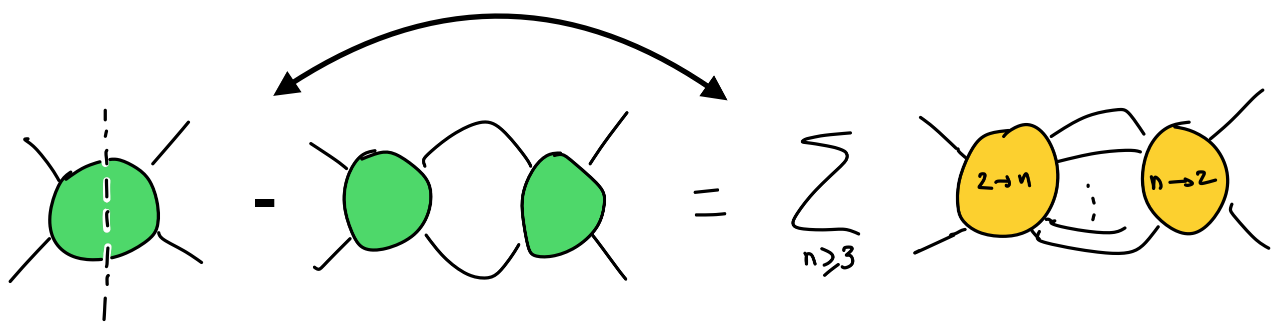

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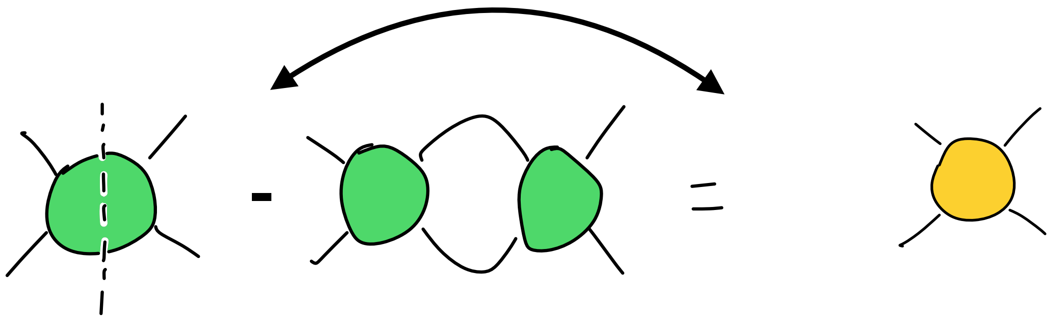

= input = f(s,t)

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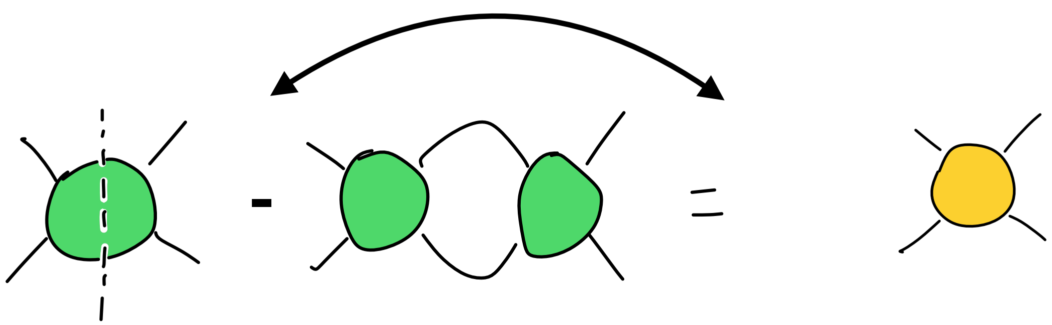
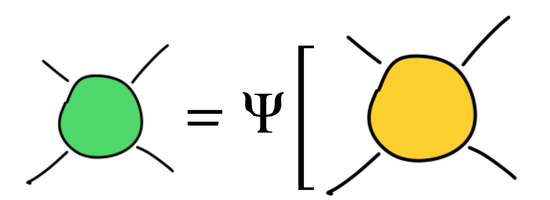


= input = f(s,t)

Gives scattering as a function (formally) of production



⇨ alleviates complicated multipoint physics

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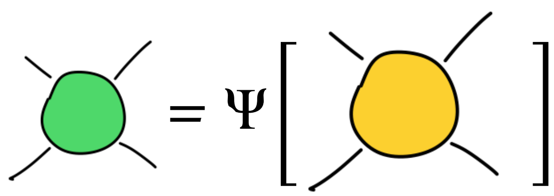

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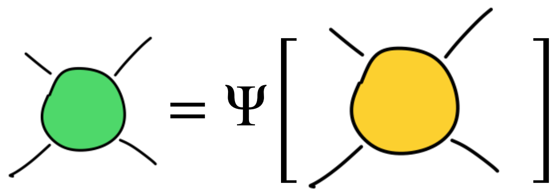

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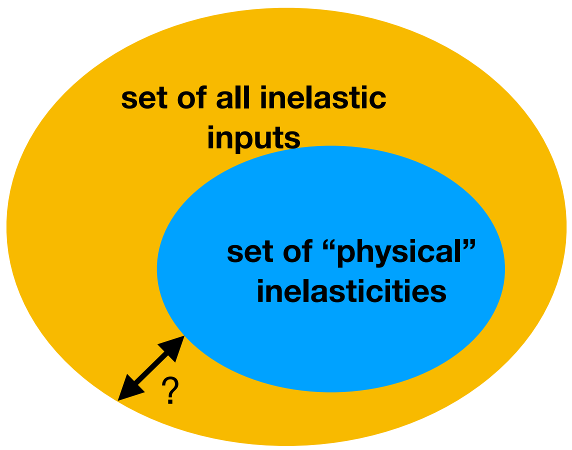
Conjecture: scans all physical theories



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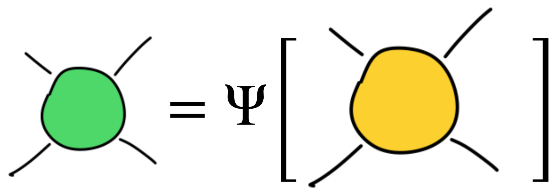

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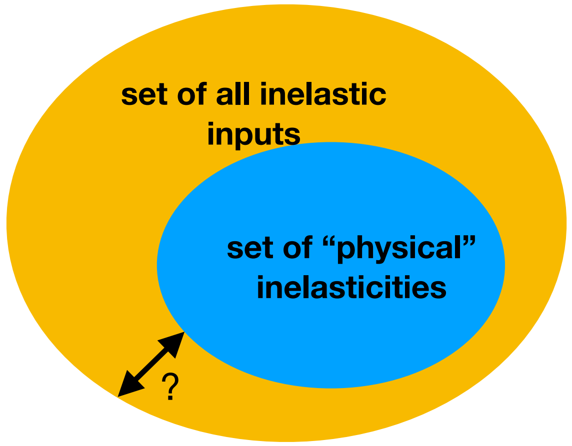
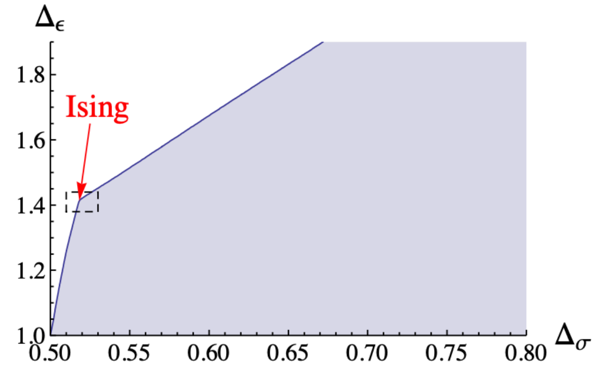


- Change of paradigm compared to 60s: explore space of theories, rather than solve one theory

- Then, maybe, find and solve *extremal* theories



[arXiv:1203.6064] Phys.Rev. **D86** (2012) 025022
Solving the 3D Ising Model with the Conformal Bootstrap
S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, A. Vichi

Other approaches

- A lot of activity on S-matrix bootstrap, starting with PPTvRV '16

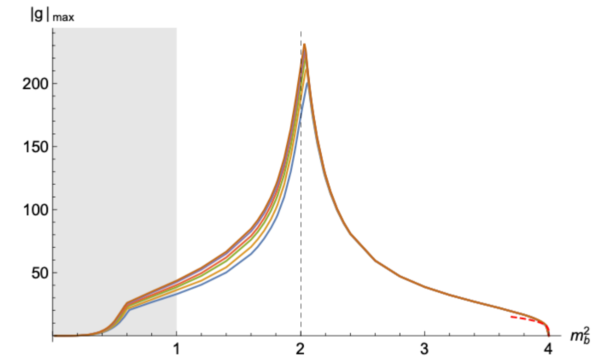
- One difficulty: numerics does not allow to control inelasticity and seem in tension with Aks theorem.

- Aks theorem ('64):

  "Scattering implies production" (in d>2)

- Also: some part of the analytic structure (Landau curves) not built in and convergence to them seems hard to achieve.

M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees, P. Vieira, 2016, 2017

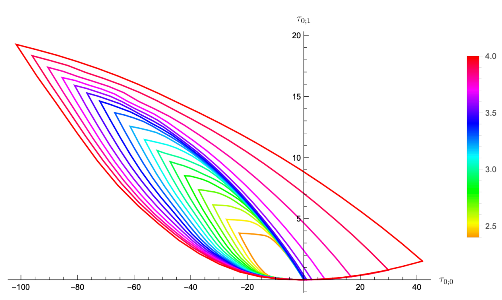


gold standard results on coupling maximization

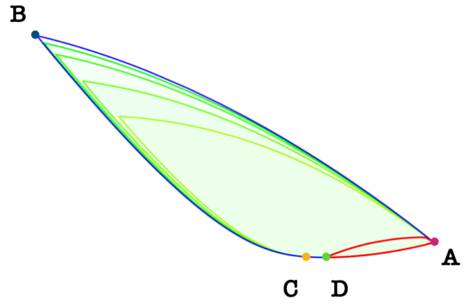
Recent works

(1) [arXiv:2207.12448]
Nonperturbative Bounds on Scattering of Massive Scalar Particles in d ≥ 2
H. Chen, A. L. Fitzpatrick, D. Karateev

(2) [arXiv:2210.01502]
Bridging Positivity and S-matrix Bootstrap Bounds
J. Elias Miro, A. Guerrieri, M. A. Gumus



(1)

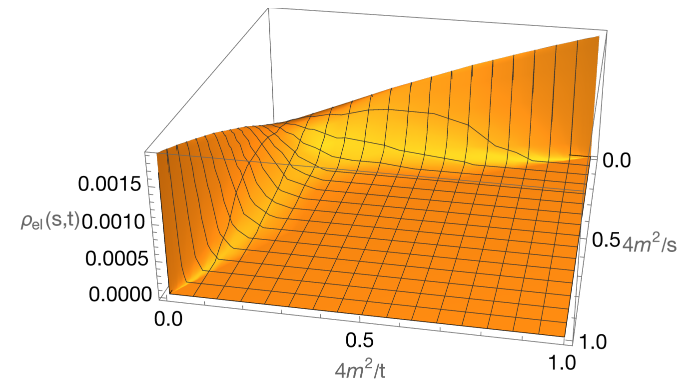


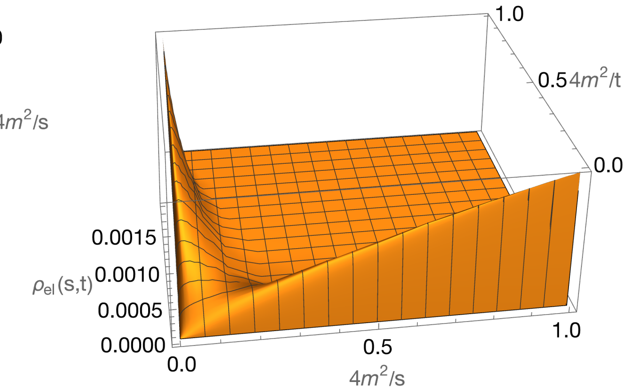
(2)

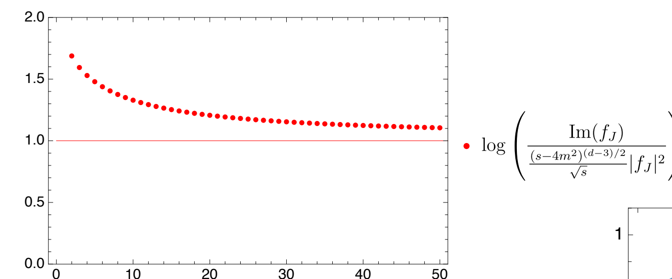
Main results

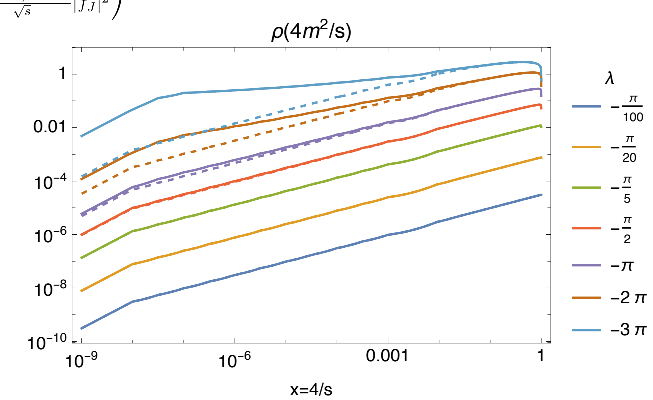
PT, Zhiboedov, 202(2/3), to appear

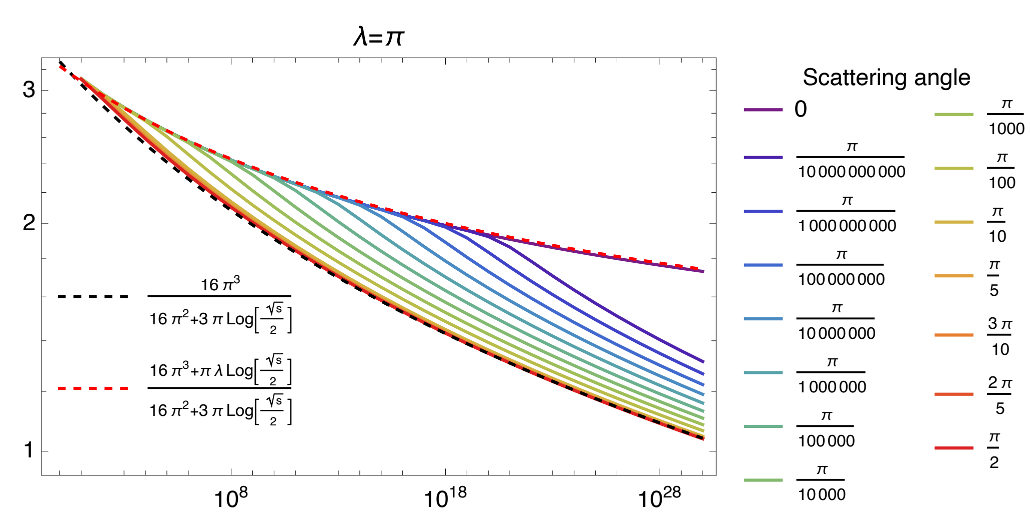
- Numerical implementation of dispersive iterations which produces S-matrices that satisfy all known constraints in d≥3 (mostly d=3 and d=4)
- Inelastic physics is correctly produced, and matches perturbation theory (3 loops) at low energies.
- Probe non-trivial Regge (d=4) threshold (d=3) physics and log-resummation.
- Conjecture that *quasi-elastic* amplitudes possibly saturate some bootstrap bounds.









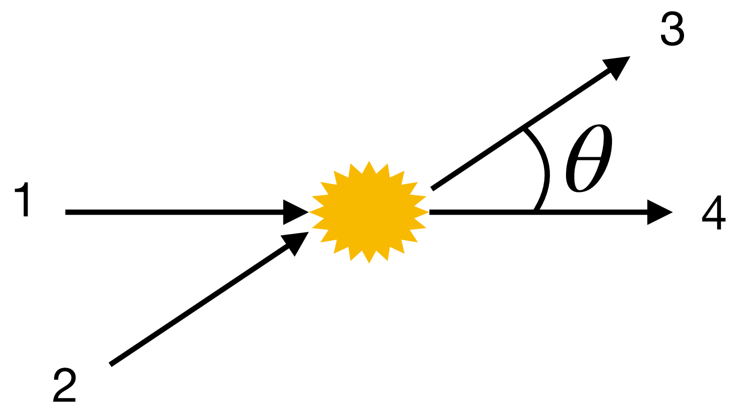


Unitarity, analyticity

Unitarity

Setup

Scattering of identical, scalar, lightest particles of a gapped QFT in *d>2* dimensions of mass m^2



Enforce some $\mathbb{Z}_2$ symmetry to remove 3-pt vertex

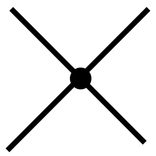
Result: massive, ϕ^4-like theories

$$z = \cos(\theta) = 1 + \frac{2t}{s - 4m^2}$$

$$s = (p1 + p2)^2$$

$$t = (p1 - p3)^2$$

$$u = (p1 - p4)^2$$



$\sim \lambda$

Unitarity

- $S^\dagger S = 1$
- $S = 1 + iT \implies 2i\,\mathrm{Im}\,T = T^\dagger T$

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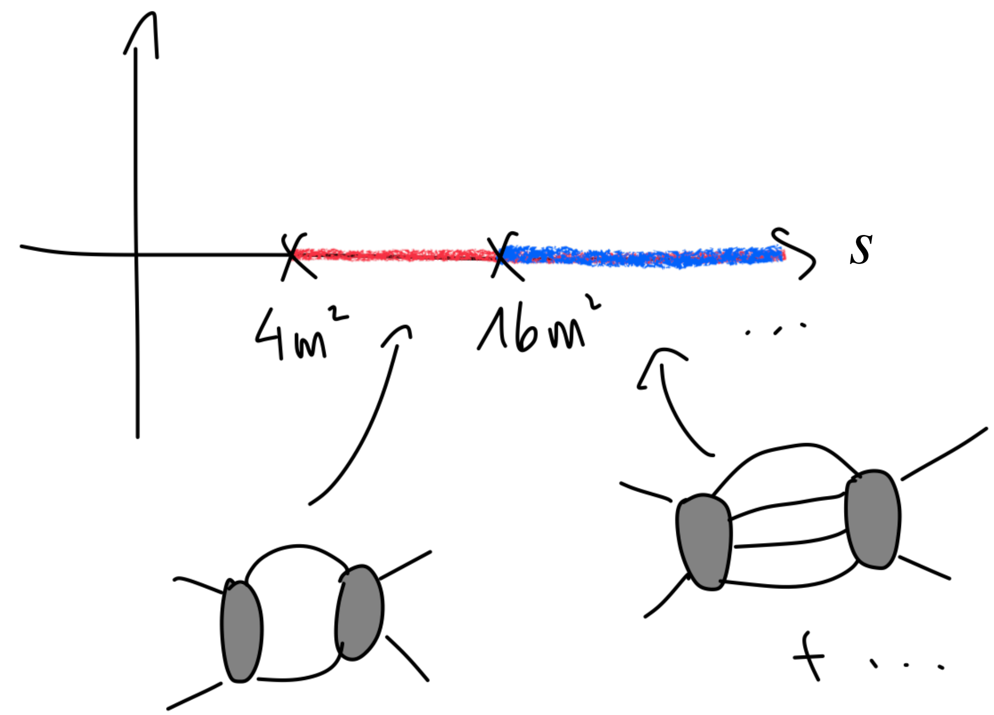


$$2i\,\mathrm{Im}\,T_{2\to2} = |T_{2\to2}|^2 + \sum_{n\geq3} |T_{2\to n}|^2$$

$4m^2 \leq s \leq 16m^2$

$$T_s(s,t) = \frac{(s-4m^2)^{\frac{d-3}{2}}}{8(4\pi)^{d-2}\sqrt{s}} \int_{-1}^{1} dz' \int_{-1}^{1} dz''\, \mathcal{P}_d(z,z',z'')\, T^{(+)}(s,t(z'))\, T^{(-)}(s,t(z''))$$

Not crossing-friendly
→ discontinuity in *s* only in *l.h.s.*
→ physical kinematics

$4m^2 - s < t < 0$

$s > 4m^2$

Elastic unitarity
Non-perturbative equation

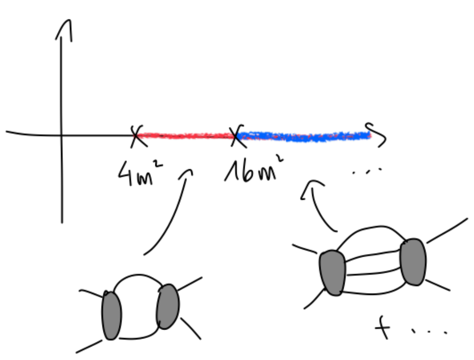
Partial wave unitarity

$$T(s,t) = 16\pi \sum_{J=0}^{\infty} (2J+1) P_J(z) f_J(s) \qquad z = \cos(\theta)$$

$$S_J(s) = 1 + ic(s) f_J(s)$$

↳ diagonalise unitarity:

- $|S_J(s)|^2 = 1, \; s \in [4m^2; 16m^2]$
- $|S_J(s)|^2 \leq 1, \; s \in [16m^2; +\infty[$
- *Straightforward to check, difficult to obtain*

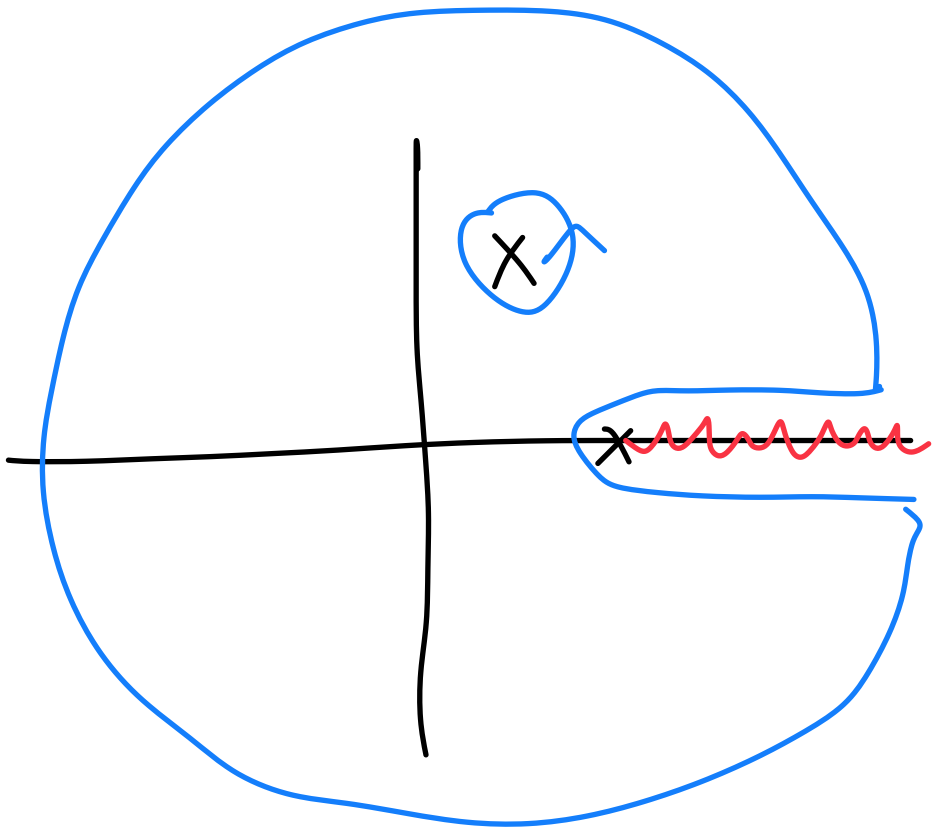


Analytic structure

- Assumption: Lightest particle maximal analyticity (LPMA)
- Mandelstam representation

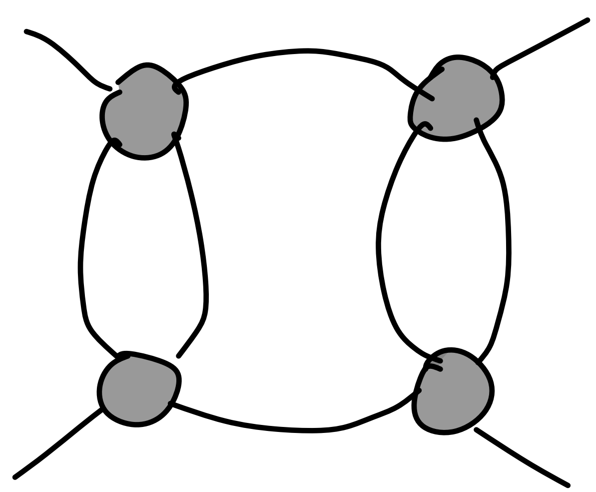
Dispersion relations

$$f(z) = \frac{1}{2i\pi}\int_{z_0}^{\infty}\frac{\mathrm{Disc}\, f(z)}{w-z}dw$$



Double disc

- Typical complex enough graphs might have a double disc.



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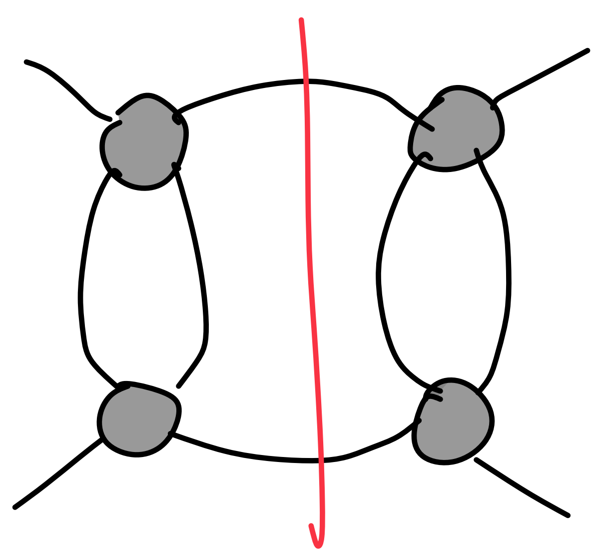

need $s \geq (2m)^2$

Double disc

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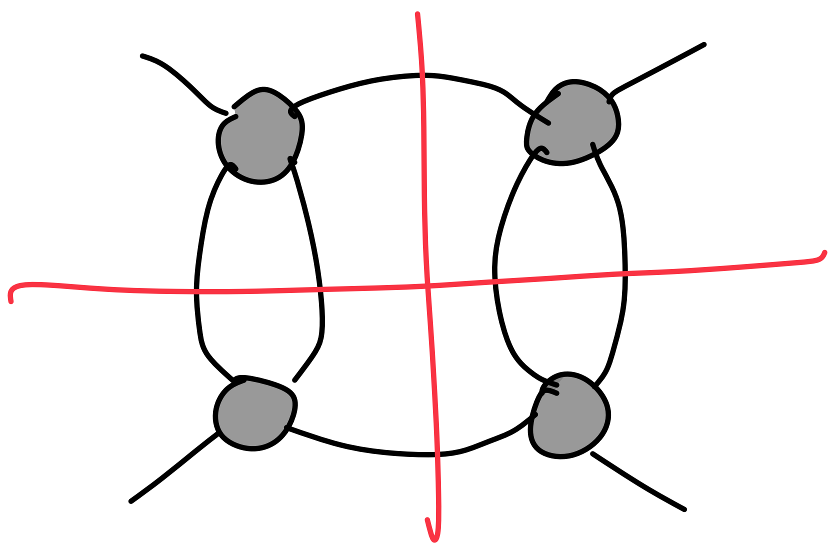


need $s \geq (2m)^2$

need $t \geq (4m)^2$

A proper calculation gives

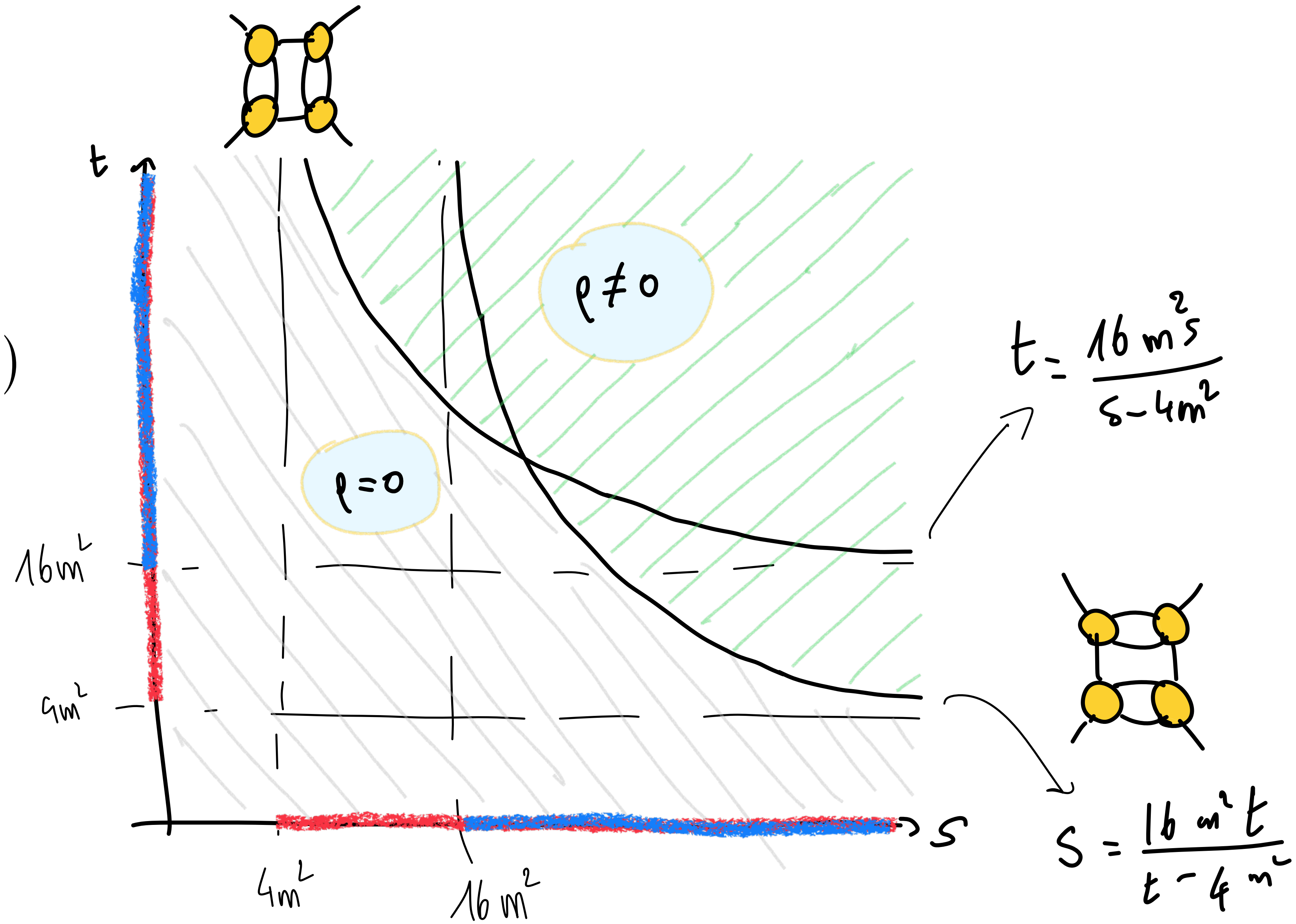
$$t \geq \frac{16s}{s - 4m^2}$$ *Landau curve*

unphysical kinematics

Double disc

see Correira, Sever, Zhiboedov '20

$$\rho(s,t) = \mathrm{Disc}_s \mathrm{Disc}_t \big(T(s,t)\big)$$



Back to elastic unitarity

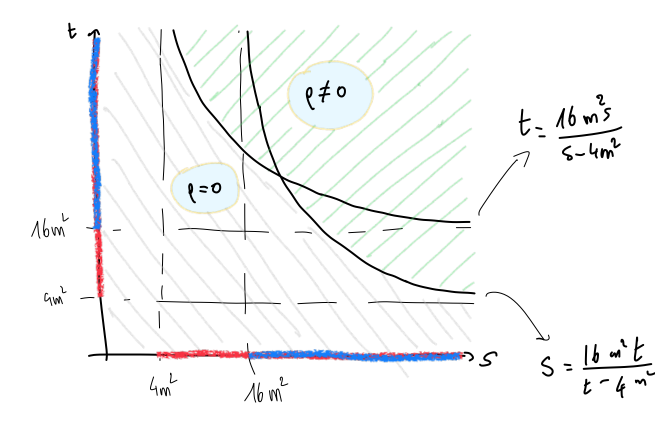
$$T_s(s,t) = \frac{(s-4m^2)^{\frac{d-3}{2}}}{8(4\pi)^{d-2}\sqrt{s}} \int_{-1}^{1} dz' \int_{-1}^{1} dz''\, \mathcal{P}_d(z,z',z'')\, T^{(+)}(s,t(z'))\, T^{(-)}(s,t(z''))$$

take another disc:

$$\rho(s,t) = c(s) \iint_{z_1}^{\infty} T_t(s,t(z'))T_t(s,t(z''))K_d(s,t,z',z'')dz'dz''$$

$$s, t \geq 4m^2$$

Mandelstam equation



Mandelstam representation

$$T(s,t) = \lambda + B(s,t) + B(t,u) + B(u,s)$$

$$B(s,t) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\rho(s')}{s'-s} + \iint_{4m^2}^{\infty} \frac{ds'dt'}{\pi^2} \frac{\rho(s',t')}{(s'-s)(t'-t)}$$

$\rho(s,t)$ has support only above the **Landau curves**

This is how we parametrise our amplitudes

Mandelstam representation

$$T(s,t) = \lambda + B(s,t) + B(t,u) + B(u,s)$$

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Let us ignore the single dispersive function for the moment

Equations to solve

$$\rho_{el}(s,t) = \frac{1}{\pi^2}\iint \frac{\mathrm{Disc}_t T(t_1,s)\mathrm{Disc}_t T(t_2,s)}{K(s,t,t_1,t_2)}dt_1 dt_2$$

$$\rho(s,t) = \rho_{el}(s,t) + \rho_{el}(t,s) + v_{inel}(s,t)$$

$$\mathrm{Disc}_t T(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\rho(s,t')}{t'-t}dt'$$

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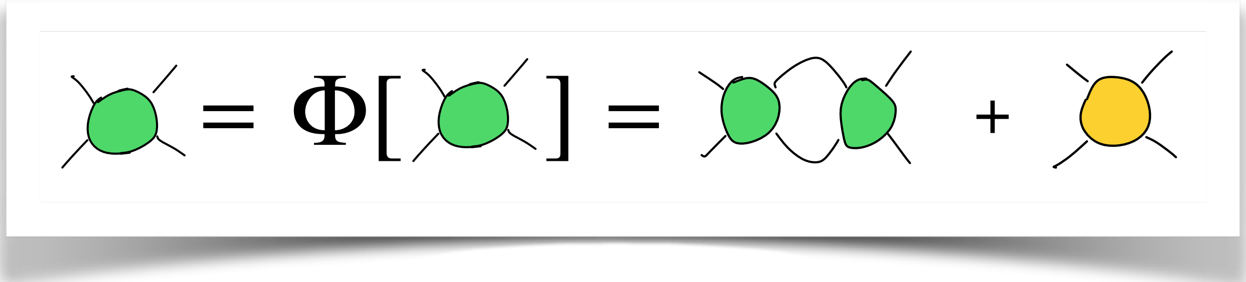


Nucl.Phys. **B15** (1970) 331-331
A Proof of the Existence of Functions That Satisfy Exactly Both Crossing and Unitarity
D. Arkinson

Nucl.Phys. **B15** (1970) 331-331
A Proof of the Existence of Functions That Satisfy Exactly Both Crossing and Unitarity (Ii) Charged Pions. No Subtractions
D. Atkinson

Nucl.Phys. **B13** (1969) 415-436
A Proof of the Existence of Functions That Satisfy Exactly Both Crossing and Unitarity (Iii). Subtractions
D. Atkinson

Nucl.Phys. **B23** (1970) 397-412
A Proof of the Existence of Functions That Satisfy Exactly Both Crossing and Unitarity. Iv.
Nearly Constant Asymptotic Cross-Sections
D. Atkinson

Lecture notes:

S Matrix Construction Project: Existence Theorems, Rigorous Bounds and Models
D. Atkinson

One inconclusive attempt at implementation:

Nucl.Phys. **B72** (1974) 167-188
Numerical Strategies in the Construction of Amplitudes Satisfying Unitarity, Analyticity and Crossing Symmetry. I
J. Boguta

Atkinson's proof

- Start from the map $\Phi : L \mapsto L$ where L is a Banach space of Hölder continuous functions

- Hölder continuity :
  $\forall x, y \in [0;1], |f(x) - f(y)| \leq k|x - y|^\alpha$ for $0 < \alpha < 1$ and $k > 0$

- Let $B = \{f \in L, \|f\| \leq b\}$ an open ball for some $b > 0$

- If $\Phi[B] \subset B$, Leray-Schauder principle $\implies \exists$ fixed point of Φ

- If Φ is *contracting*, i.e. $\|\Phi[f_1 - f_2]\| \leq c\|f_1 - f_2\|$, then the solution is also unique in B.

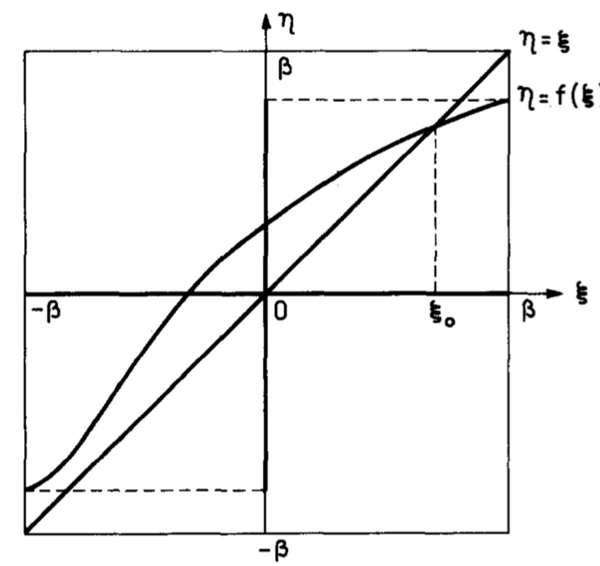


Fig. 1. Illustration of a fixed-point theorem. The image of the interval, $-\beta \leqslant \xi \leqslant \beta$, under the continuous, nonlinear mapping, f, is a subset of the same interval. Therefore the curve $\eta = f(\xi)$ intersects the line $\eta = \xi$ at least once, at a point ξ_0, such that $\xi_0 = f(\xi_0)$.

d=4 and Gribov’s theorem

- Subtle detail about Regge limit (a.k.a. *Gribov’s theorem*)

- R.h.s. of Mandelstam eq. $\rho \sim \iint |T_t|^2 dt_1 dt_2$ produces logs at high energy which need to be tamed, in d=4 this can be done provided that $\rho(s,t) \propto \frac{1}{\log(s)^{1+\epsilon}} f(s,t)$

- Related to the fact that a single Regge pole $T(s,t) \propto s^{\alpha(t)}$ cannot satisfy unitarity

$$\mathrm{Im} f_J \sim \frac{1}{J - \alpha(t)}$$

$$|f_J|^2 = (\mathrm{Re} f_J)^2 + (\mathrm{Im} f_J)^2 \sim \frac{1}{(J - \alpha(t))^2} + (\ldots)^2$$

Implementation, Results

Numerics

- Laptop & grid computations

```
In[27]:= Kernels[] // Length

Out[27]= 48
```

- Mathematica

  convenient for integrals and development

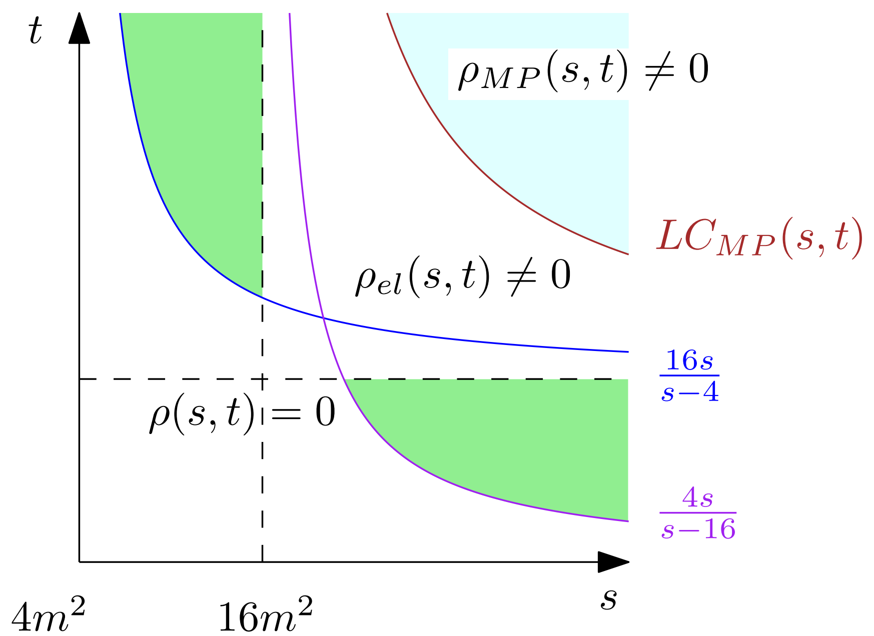
  basic speed of `NIntegrate[]` close to default python `quad()`

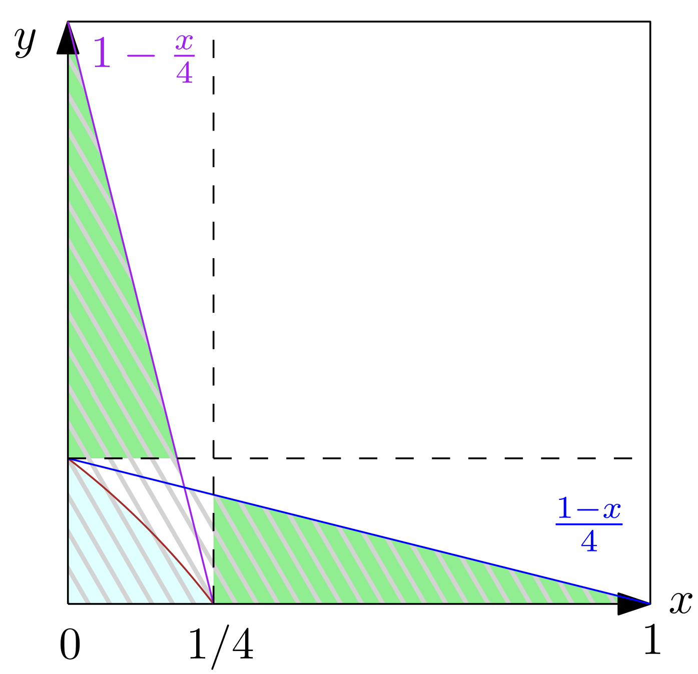
- Needs to be improved to scale up !

Numerical implementation

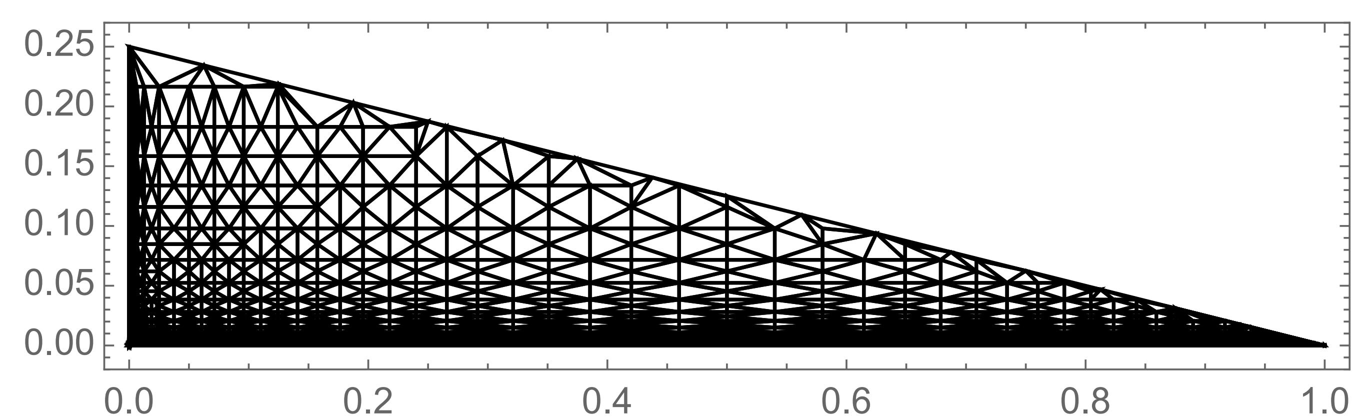
First : map

$$s,t \to x = \frac{4m^2}{s}, y = \frac{4m^2}{t}$$





then:
discretize



`finite element mathematica package`

Algorithm

Start from $\rho_0(s,t)$

Integrate $D_0(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} \rho_0(s',t)\left(\frac{1}{s'-s} + \frac{1}{s'-u}\right) ds'$

Integrate $\rho_{(1),el}(s,t) = \iint \big(D_0(s,t_1)(D_0(s,t_2) + \rho_0(s,t_1)\rho_0(s,t_2)\big) dt_1 dt_2$

Define $\rho_{(1)}(s,t) = \rho_{(1),el}(s,t) + \rho_{(1),el}(t,s) + v_{inel}(s,t)$

Iterate

Algorithm

Start from $\rho_n(s,t)$

Integrate $$D_n(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} \rho_n(s',t)\left(\frac{1}{s'-s} + \frac{1}{s'-u}\right)$$

Integrate $$\rho_{(n+1),el}(s,t) = \iint \big(D_n(s,t_1)(D_n(s,t_2) + \rho_n(s,t_1)\rho_n(s,t_2)\big)\, dt_1 dt_2$$

Define $$\rho_{(n+1)}(s,t) = \rho_{(n+1),el}(s,t) + \rho_{(n+1),el}(t,s) + v_{inel}(s,t)$$

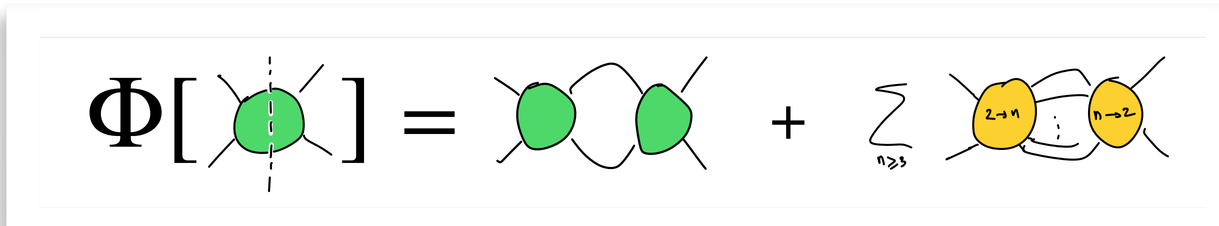
Algorithm

Start from $\rho_n(s,t)$

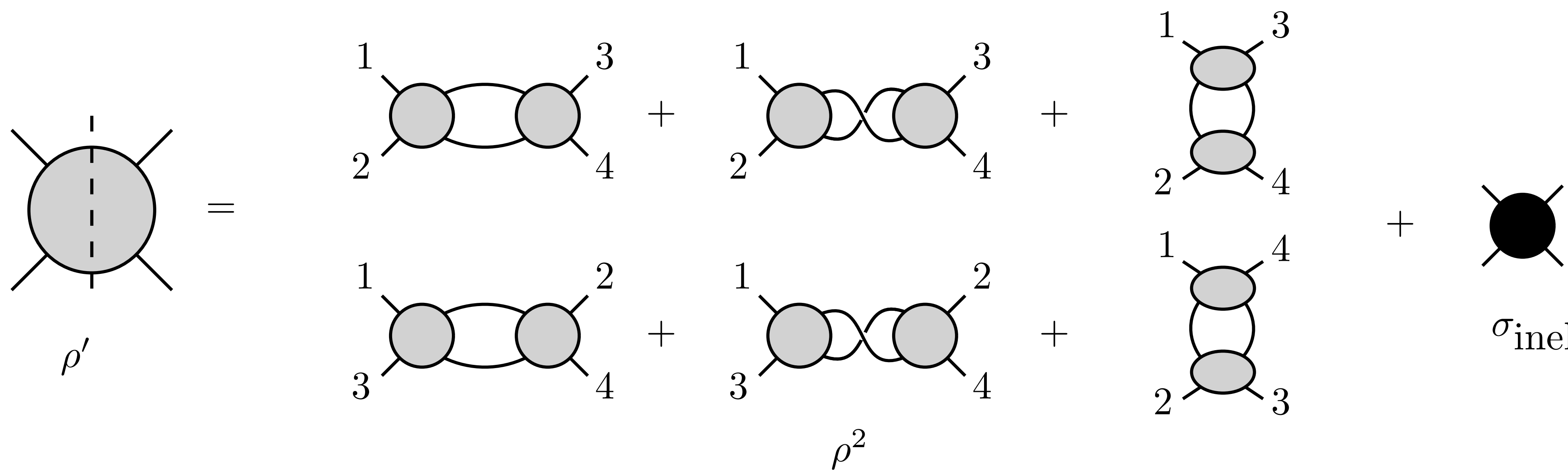
Integrate $D_n(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} \rho_n(s',t)\left(\frac{1}{s'-s} + \frac{1}{s'-u}\right)$

Integrate $\rho_{(n+1),el}(s,t) = \iint \big(D_n(s,t_1)(D_n(s,t_2) + \rho_n(s,t_1)\rho_n(s,t_2)\big)dt_1 dt_2$

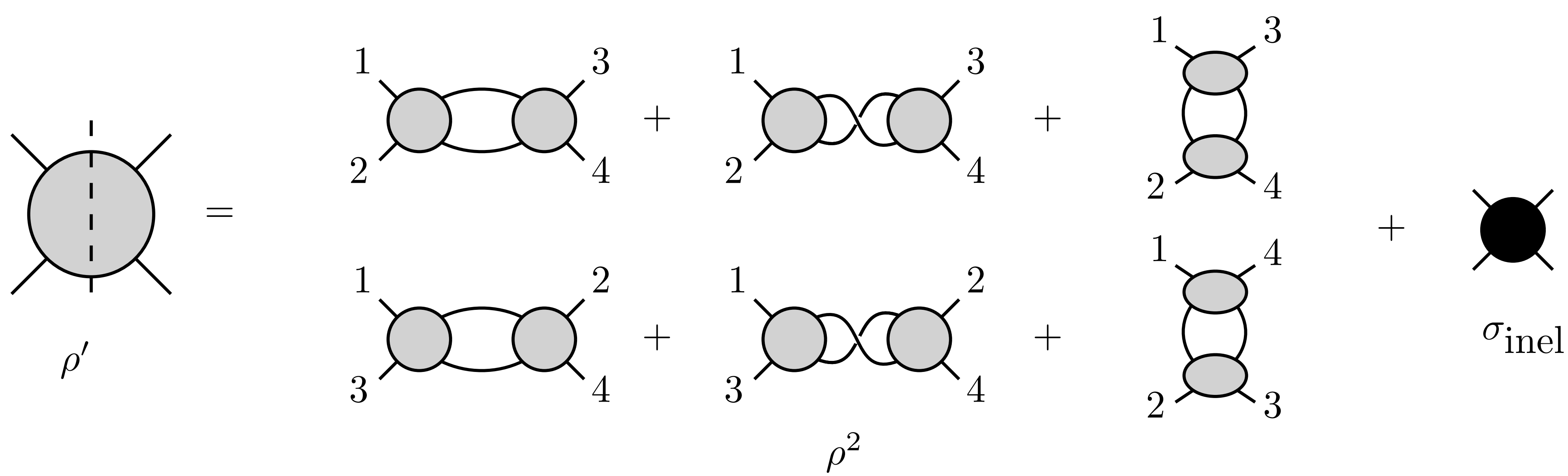
Define $\rho_{(n+1)}(s,t) = \rho_{(n+1),el}(s,t) + \rho_{(n+1),el}(t,s) + v_{inel}(s,t)$



Diagrammatic interpretation



Diagrammatic interpretation



$$N_n = 6N_{n-1}^2 + 1$$

Step $n \leftrightarrow 2^L$ loops

$$N_n \sim 2.55^L$$

finite radius of conv.

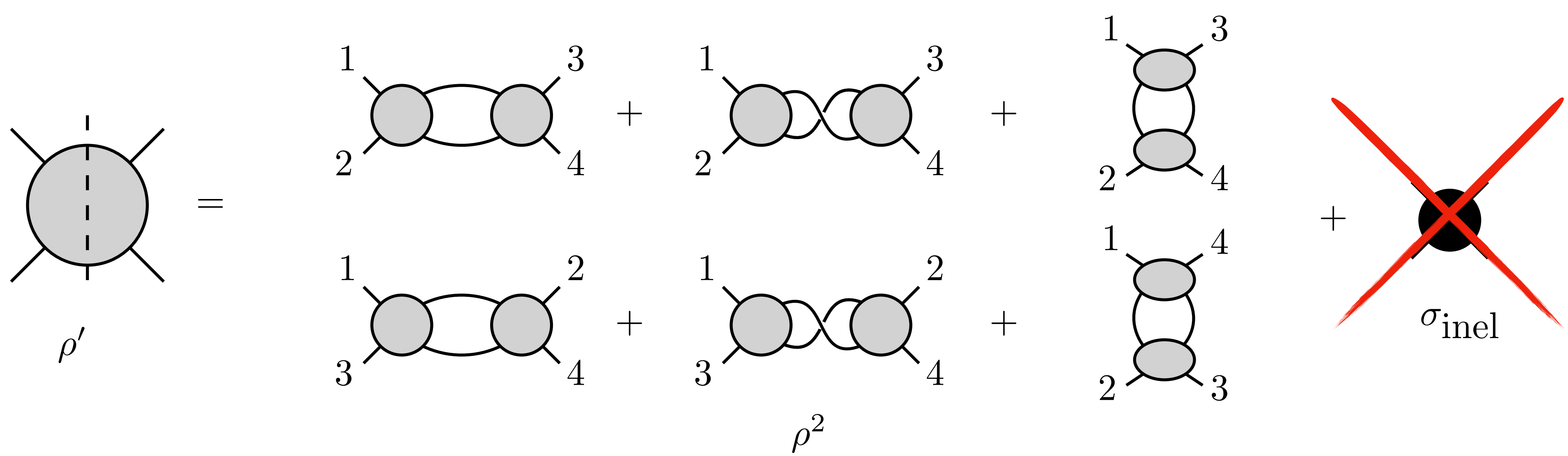
$$\sum_{L=0}^{\infty} (2.55)^L \tilde{A}_L(s,t)$$

vs

asymptotic

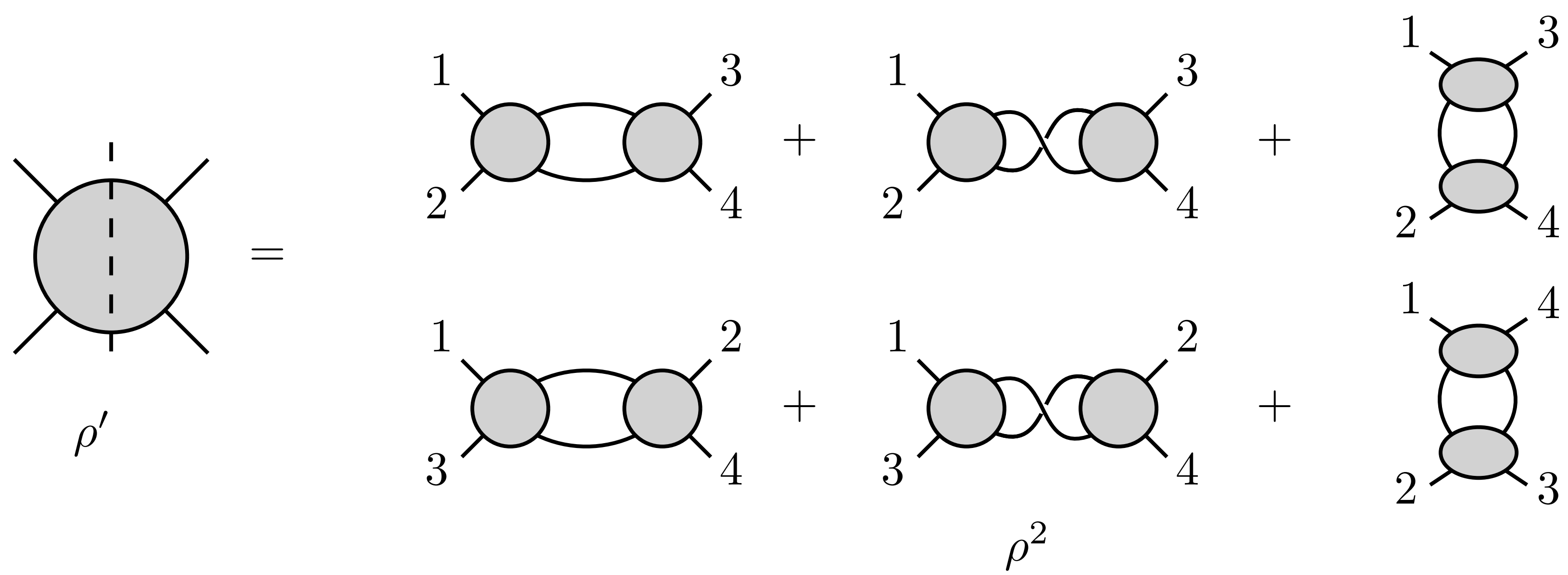
$$\sum_{L=0}^{\infty} (L!) A_L(s,t)$$

Quasi-elastic amplitudes

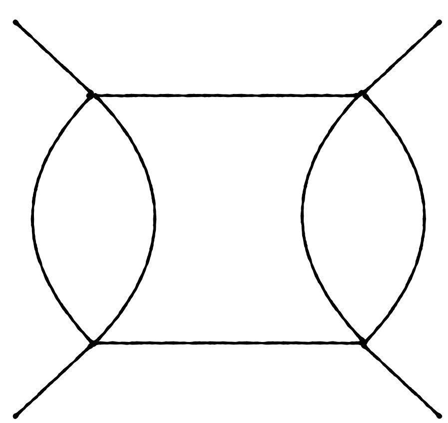


One way to study the question "how small can inelasticity be ?"

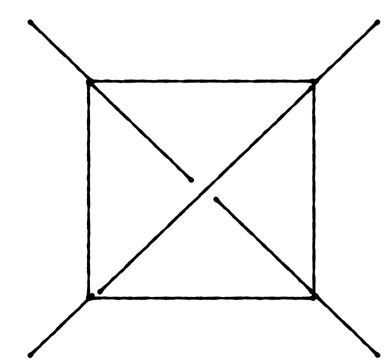
Quasi-elastic amplitudes



Will produce inelastic graphs such as



will never produce graphs which have no 2-pt cuts, such as:



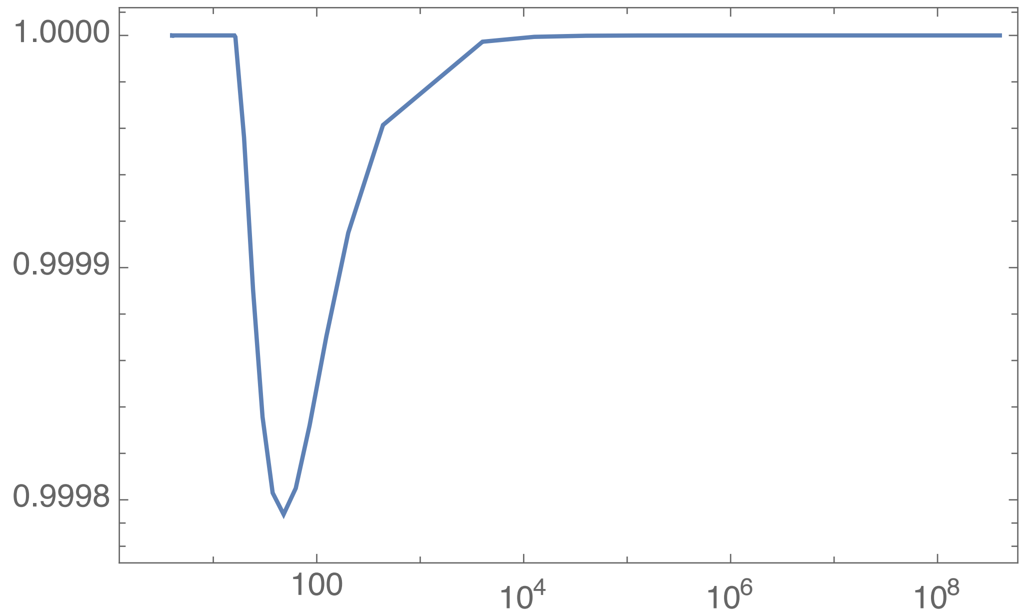
Correira, Sever, Zhiboedeov

Some examples of results in d=3

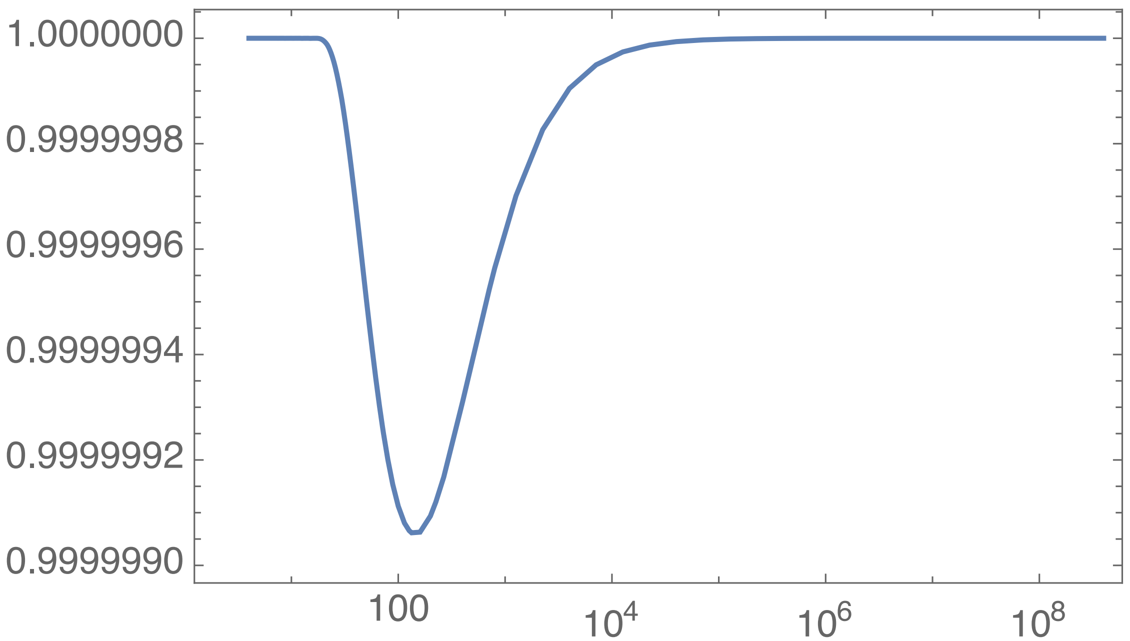
Some examples of results in d=3

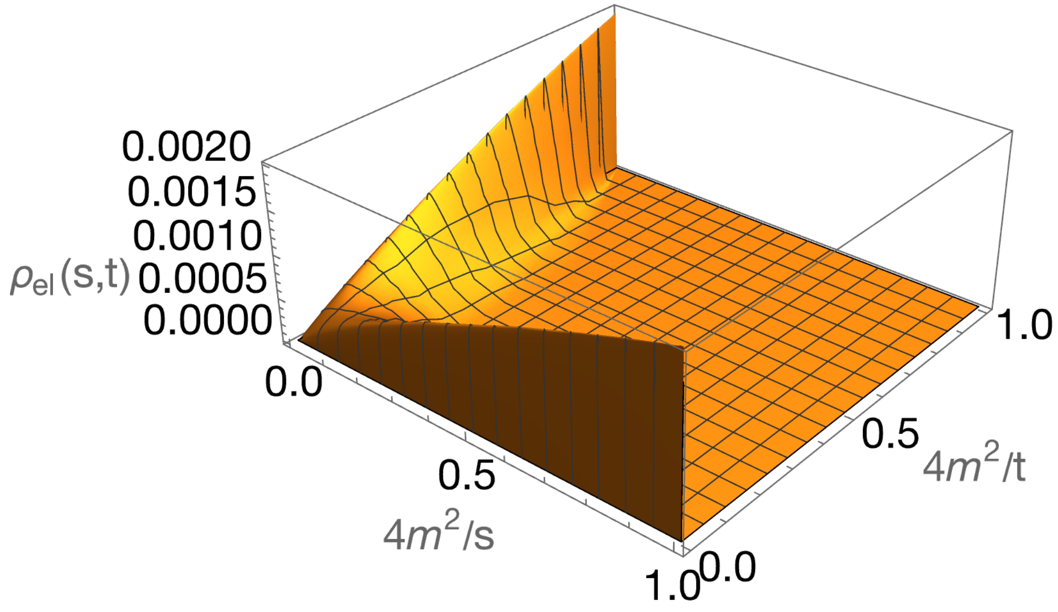
With inelastic input

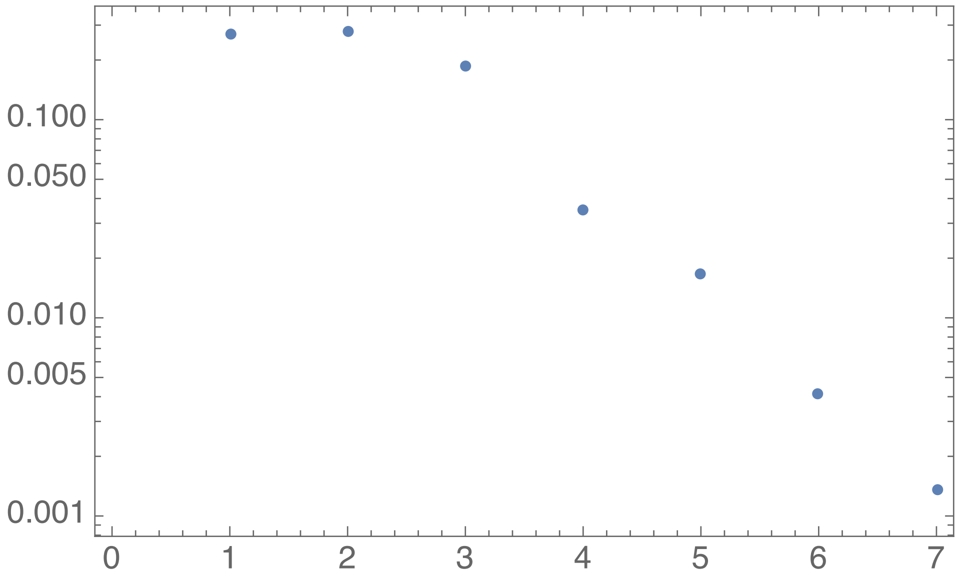
$|S_0|$, for $\lambda = -\pi$



$|S_2|$

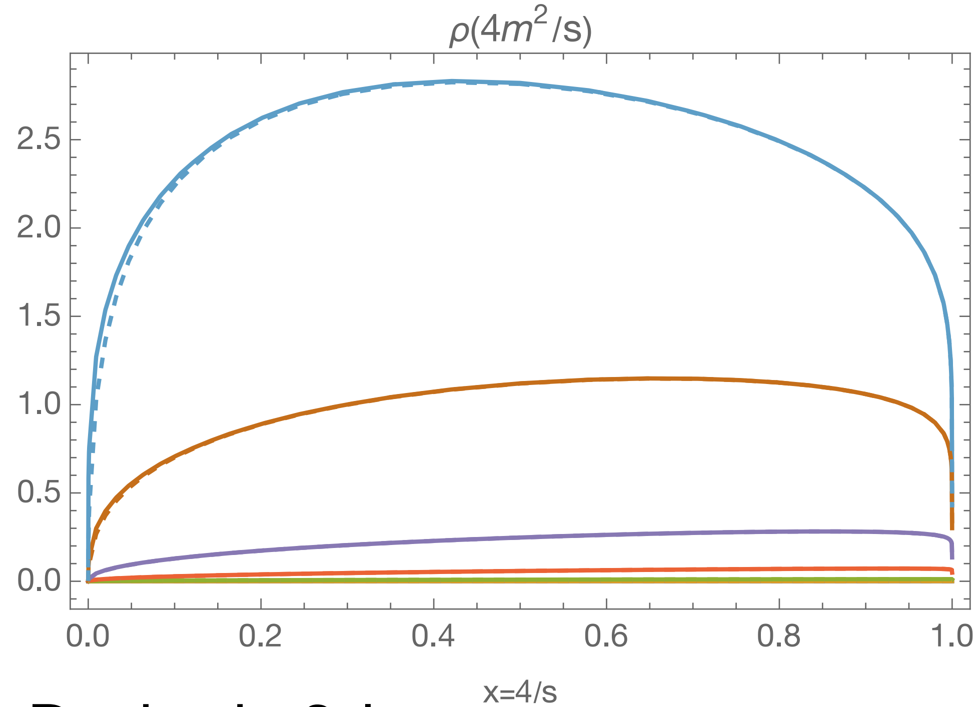


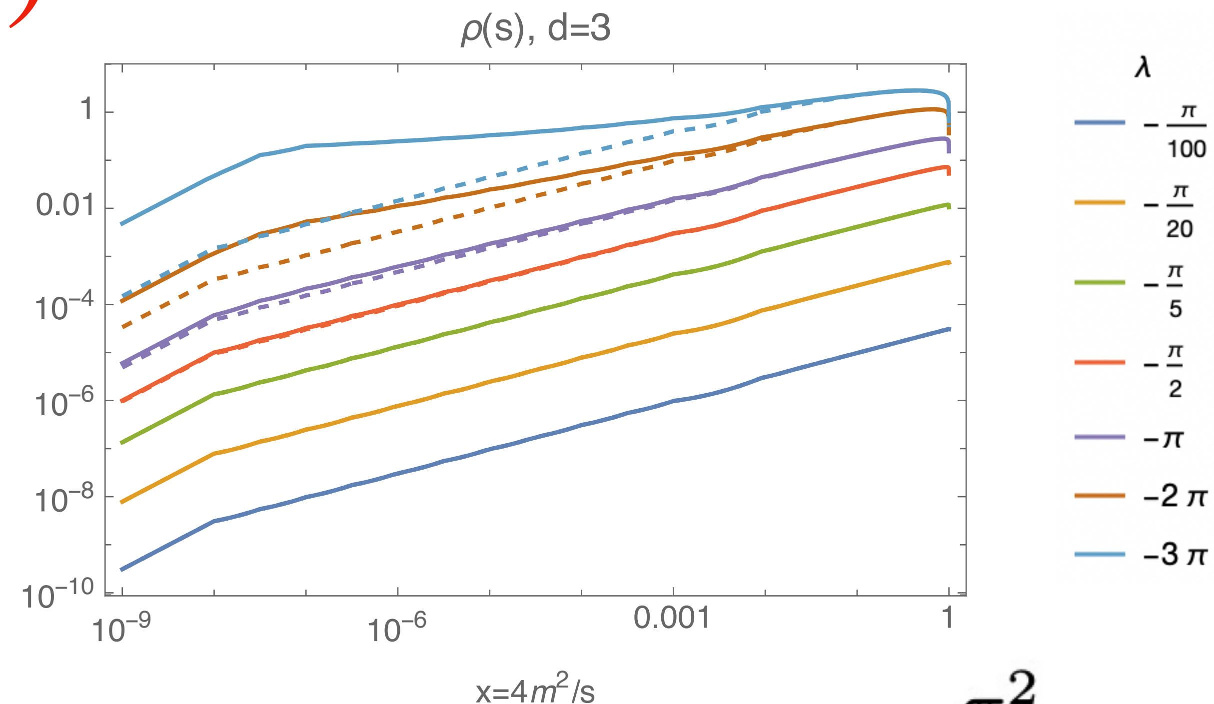




Results in d=3: quasi-elastic amps

$\rho(s)$

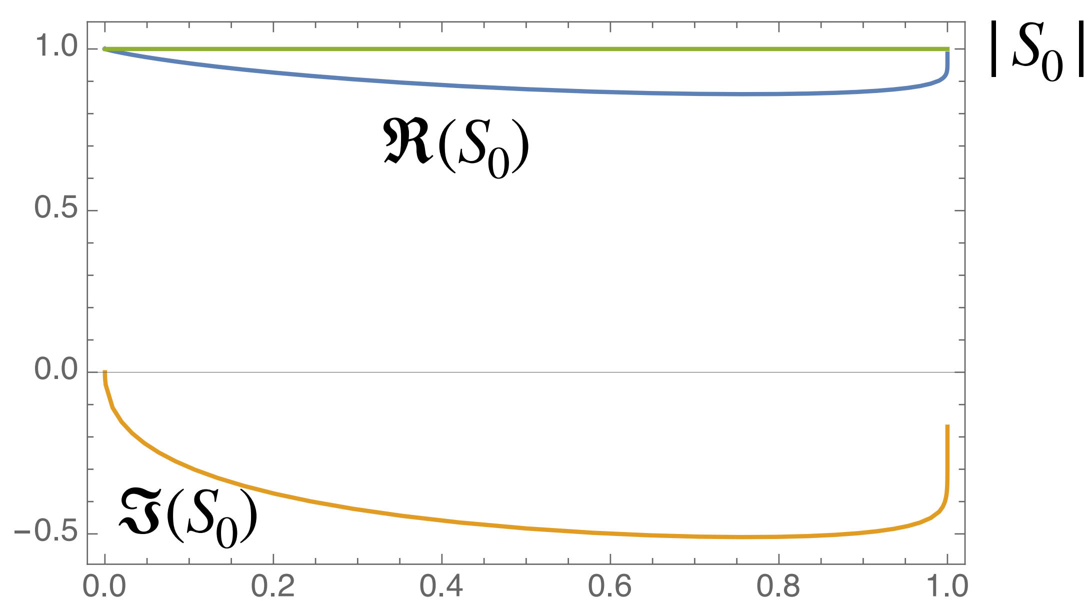


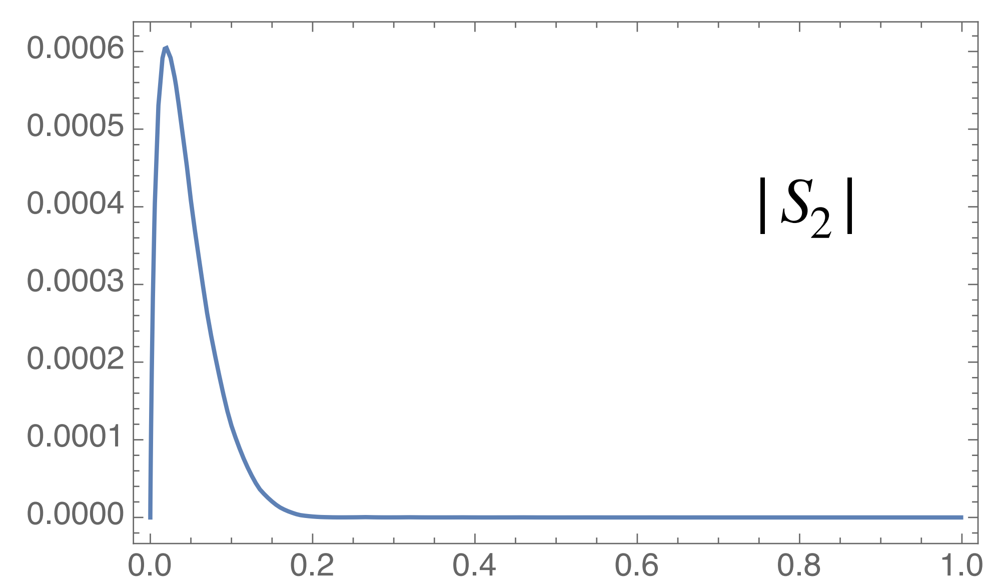


Dashed : 3-loop

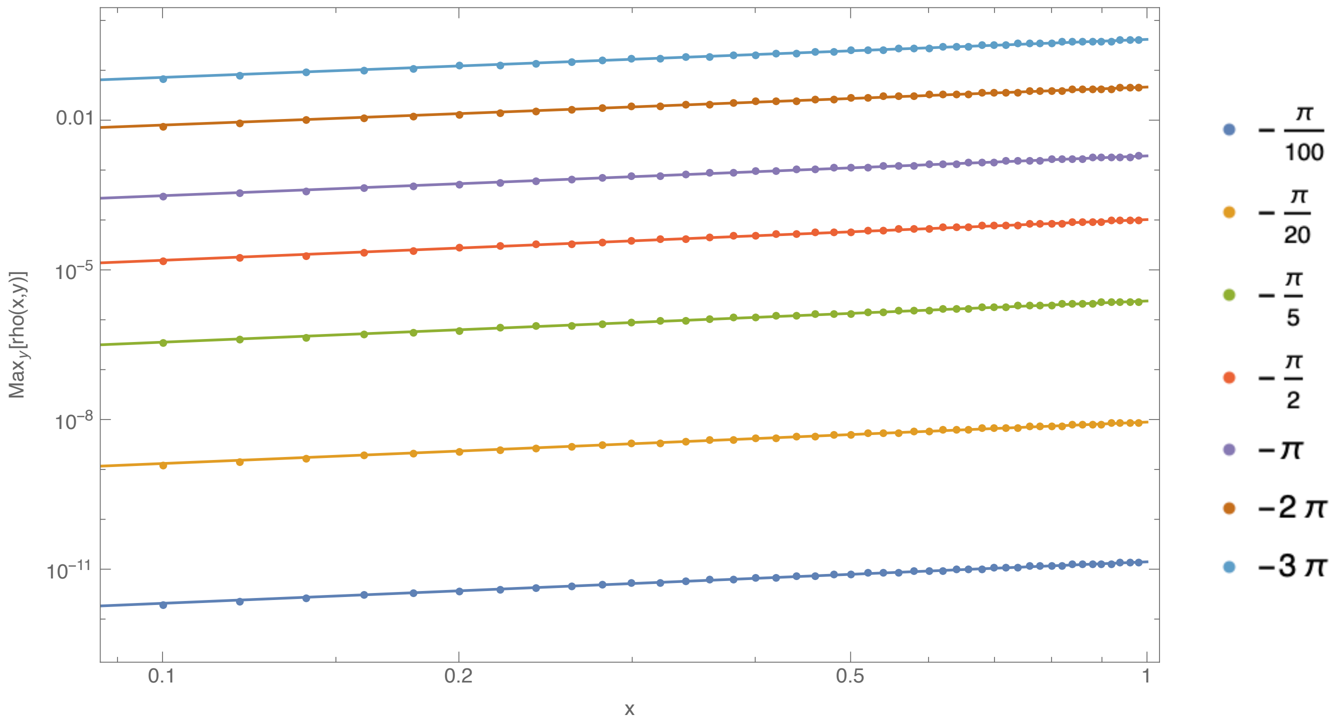
Solid : numerical solution

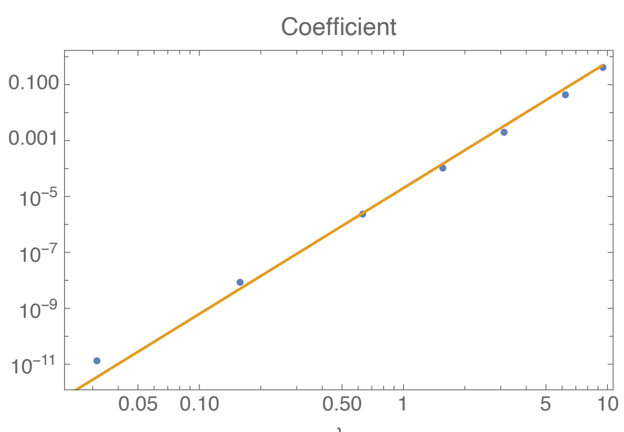
near-threshold $\frac{\pi^2}{\log^2 \sigma_s + \pi^2}$



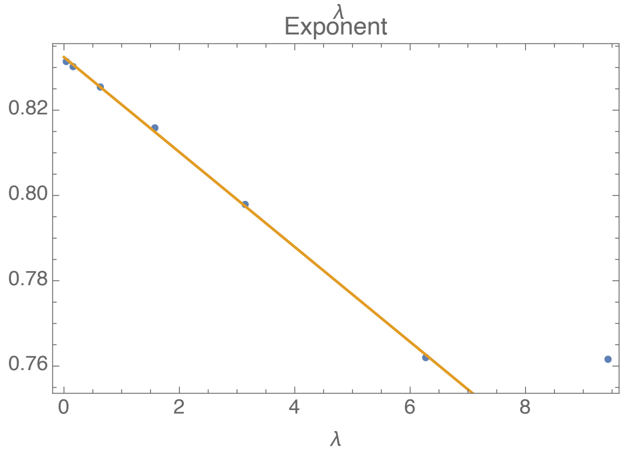


Results in d=3: quasi-elastic amps

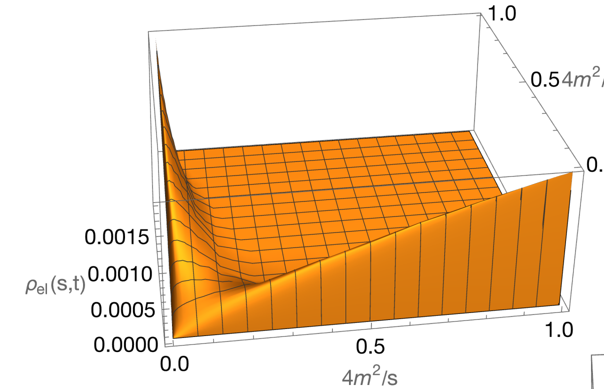


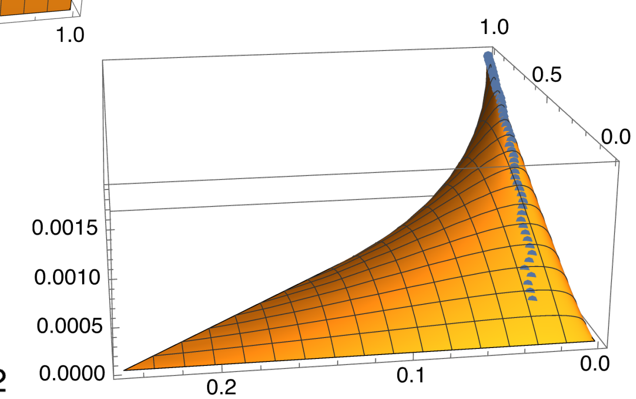


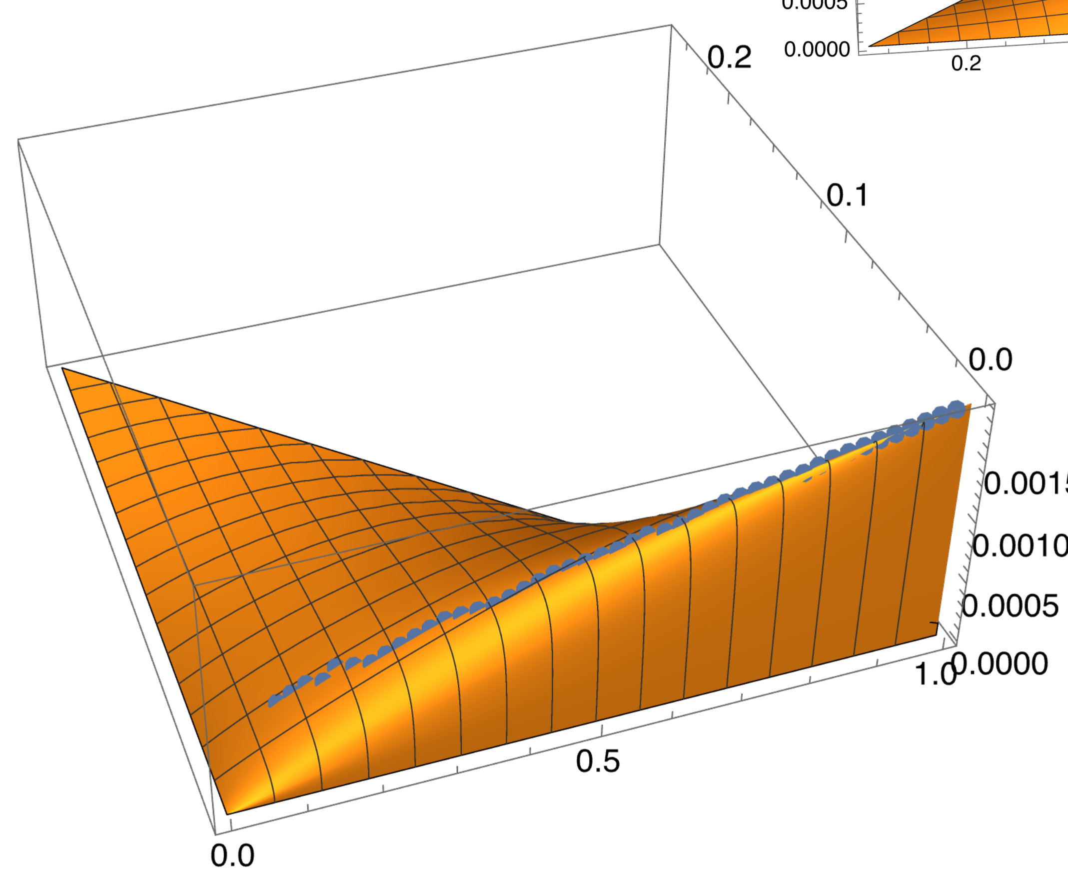
$2.2 \times 10^{-6}\,\lambda^{5.4}$



$0.83 - 0.01\lambda$





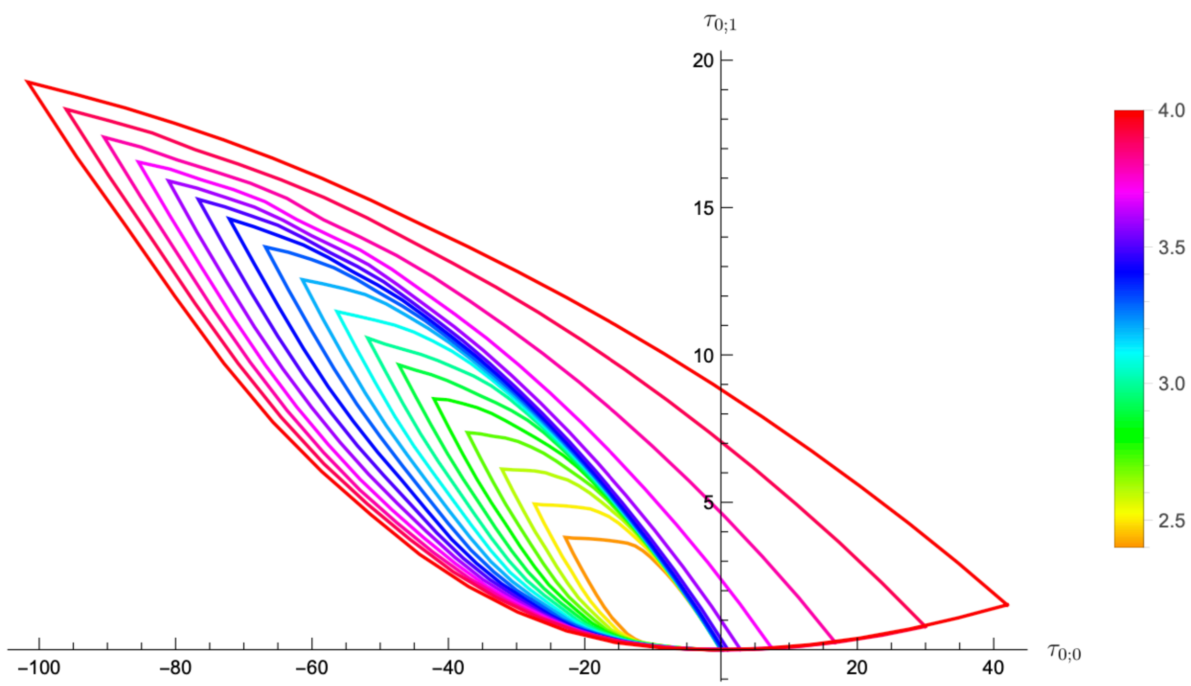


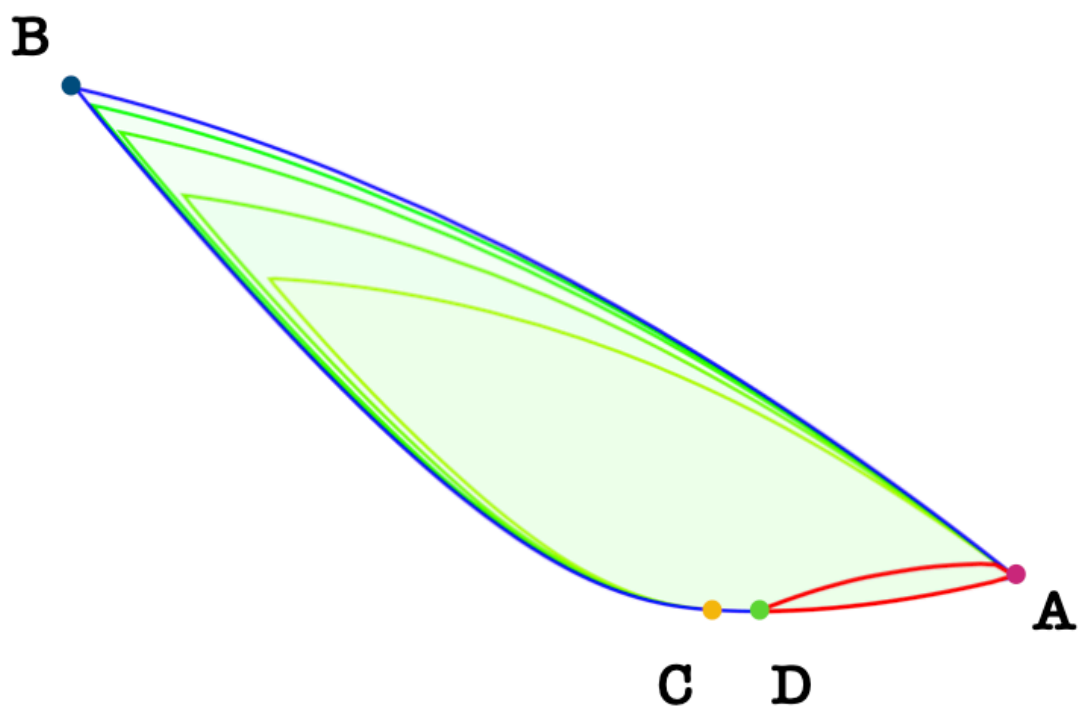
Low energy observables

[arXiv:2207.12448]
Nonperturbative Bounds on Scattering of Massive Scalar Particles in d ≥ 2
H. Chen, A. L. Fitzpatrick, D. Karateev

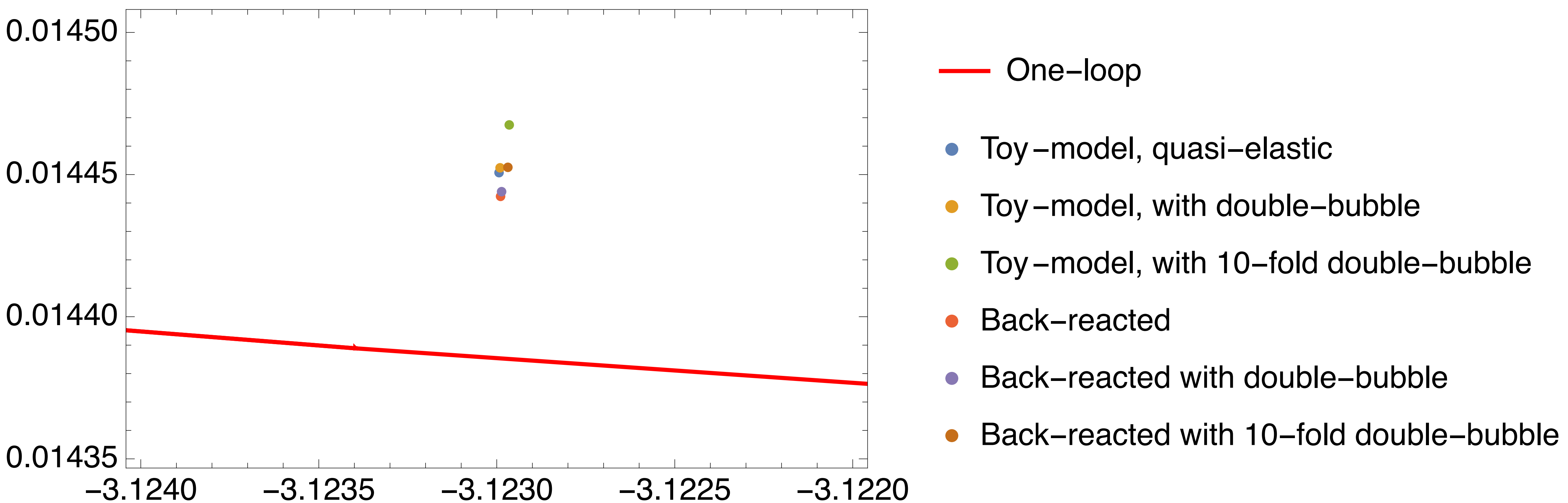
[arXiv:2210.01502]
Bridging Positivity and S-matrix Bootstrap Bounds
J. Elias Miro, A. Guerrieri, M. A. Gumus

$$\tau_{j;k} \equiv \frac{m^{d-4+2k}}{k!} \partial_s^k \mathcal{T}_j(2m^2)$$

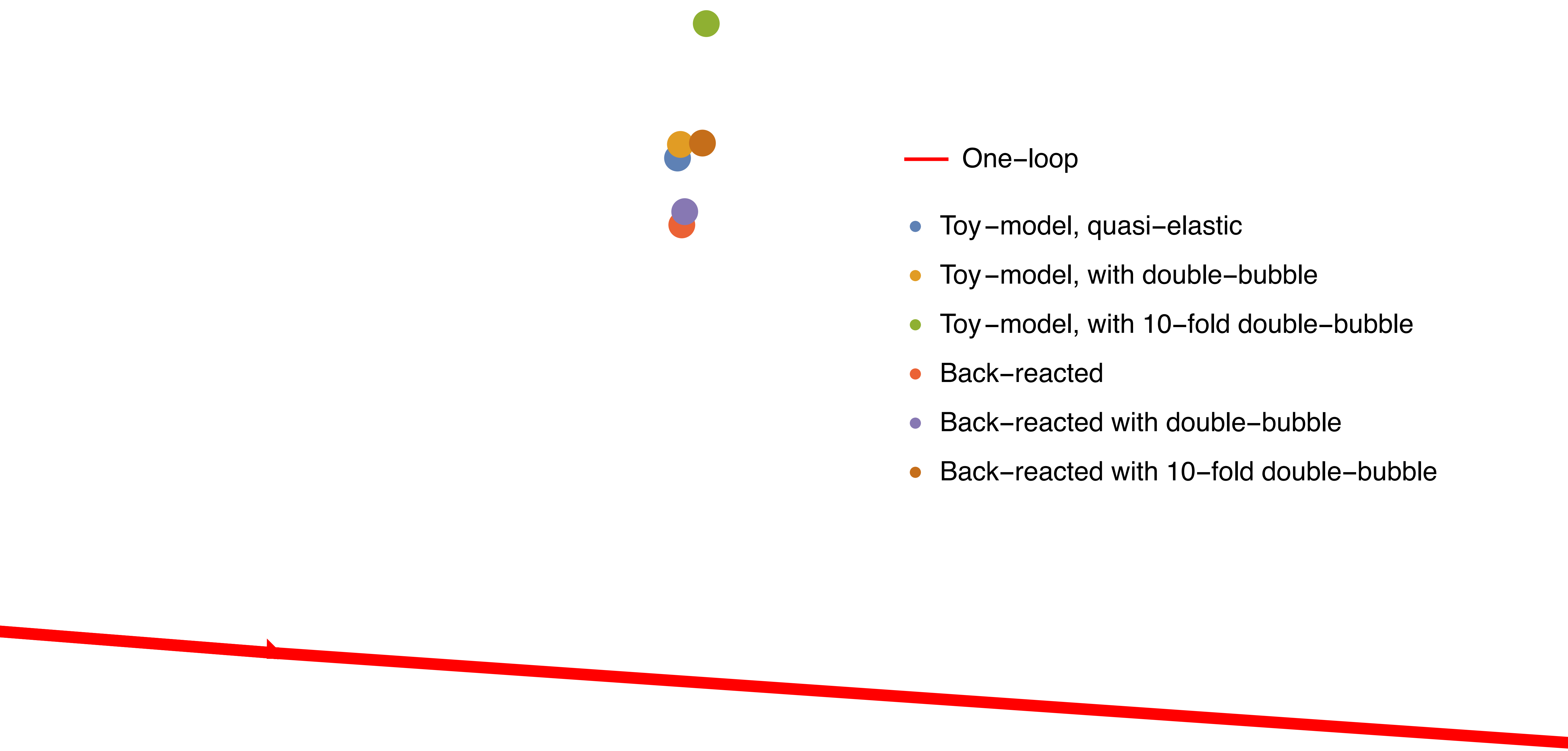




Low energy observables



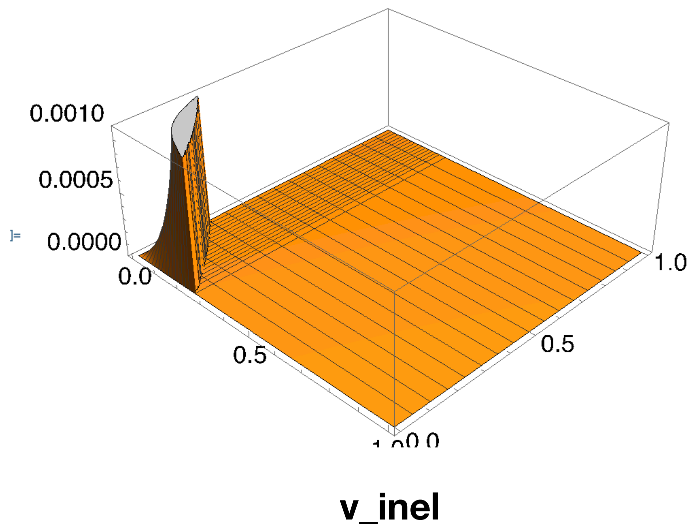
Low energy observables

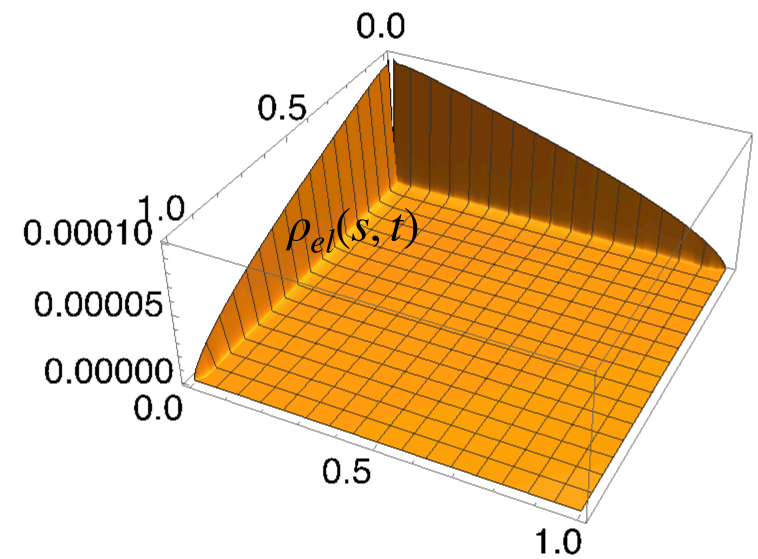


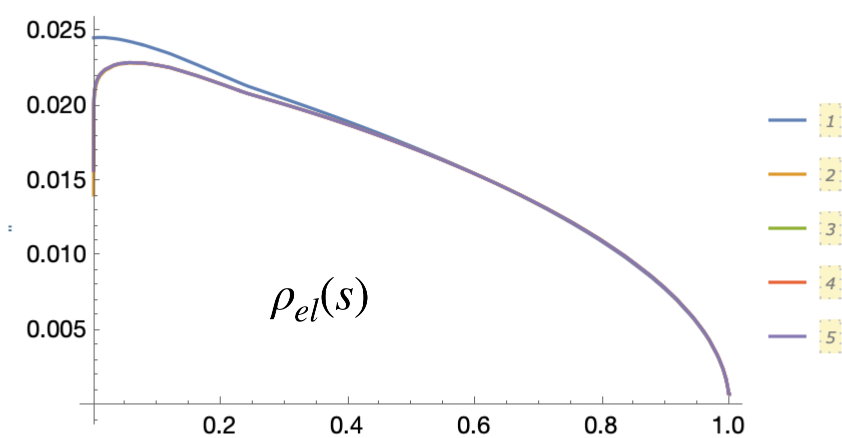
Some examples of results in d=4

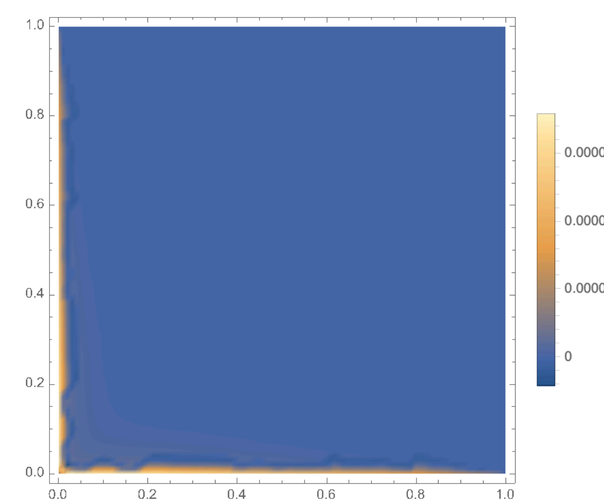
Some examples of results in d=4

$$\lambda = \pi/2$$





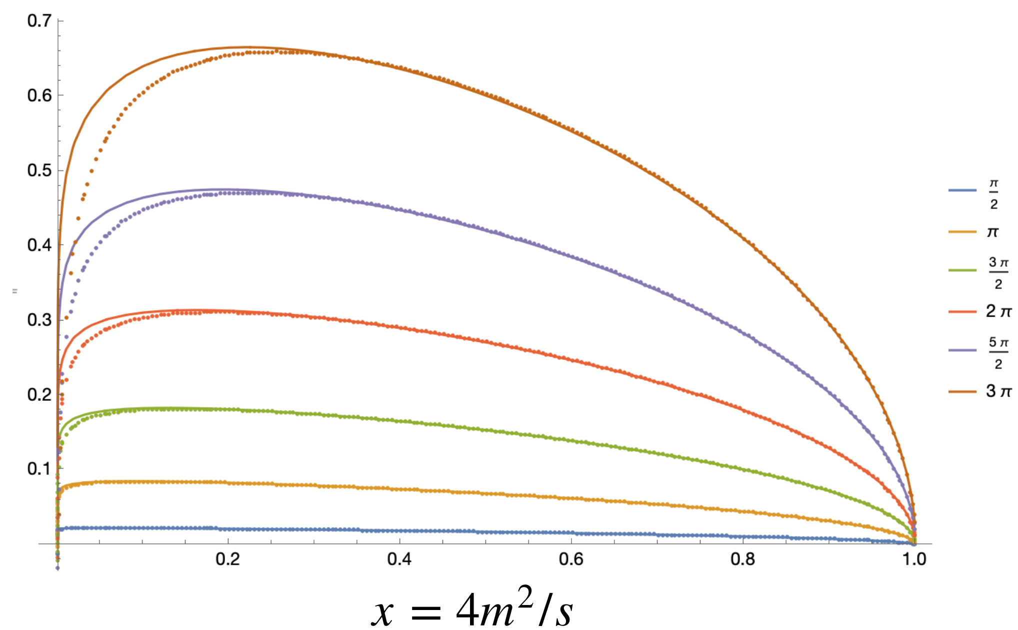




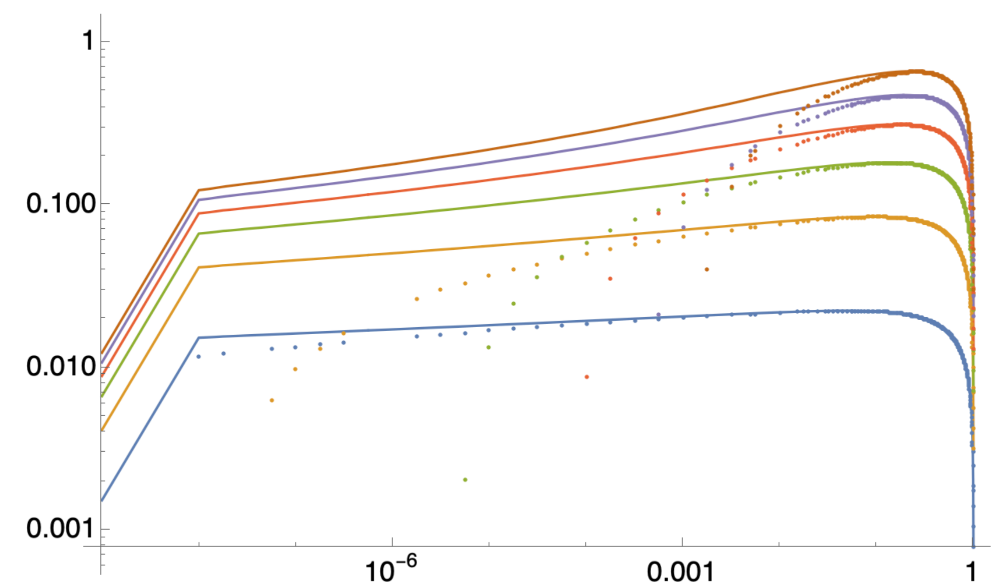
3-loop vs reggeisation

$\rho(s)$

perfect match at low energies



Reggeisation at high energies

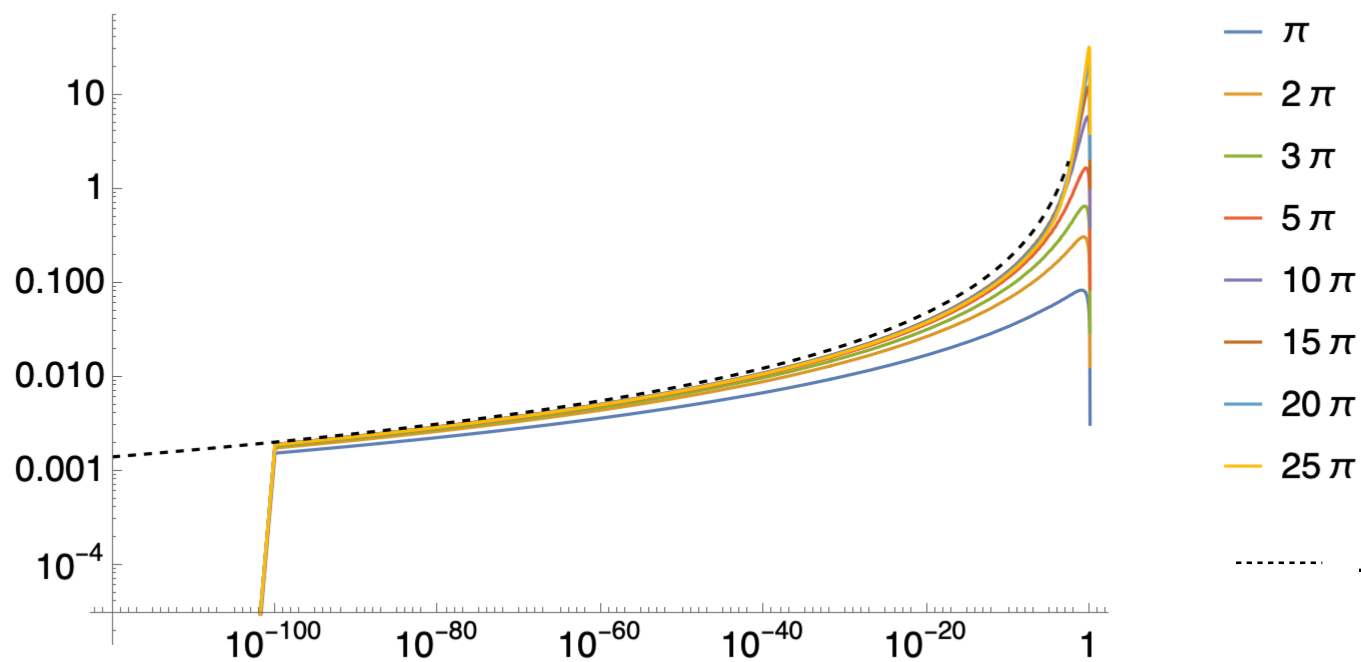


dots = 3-loop (analytic)

joined = numerics

same but log scale

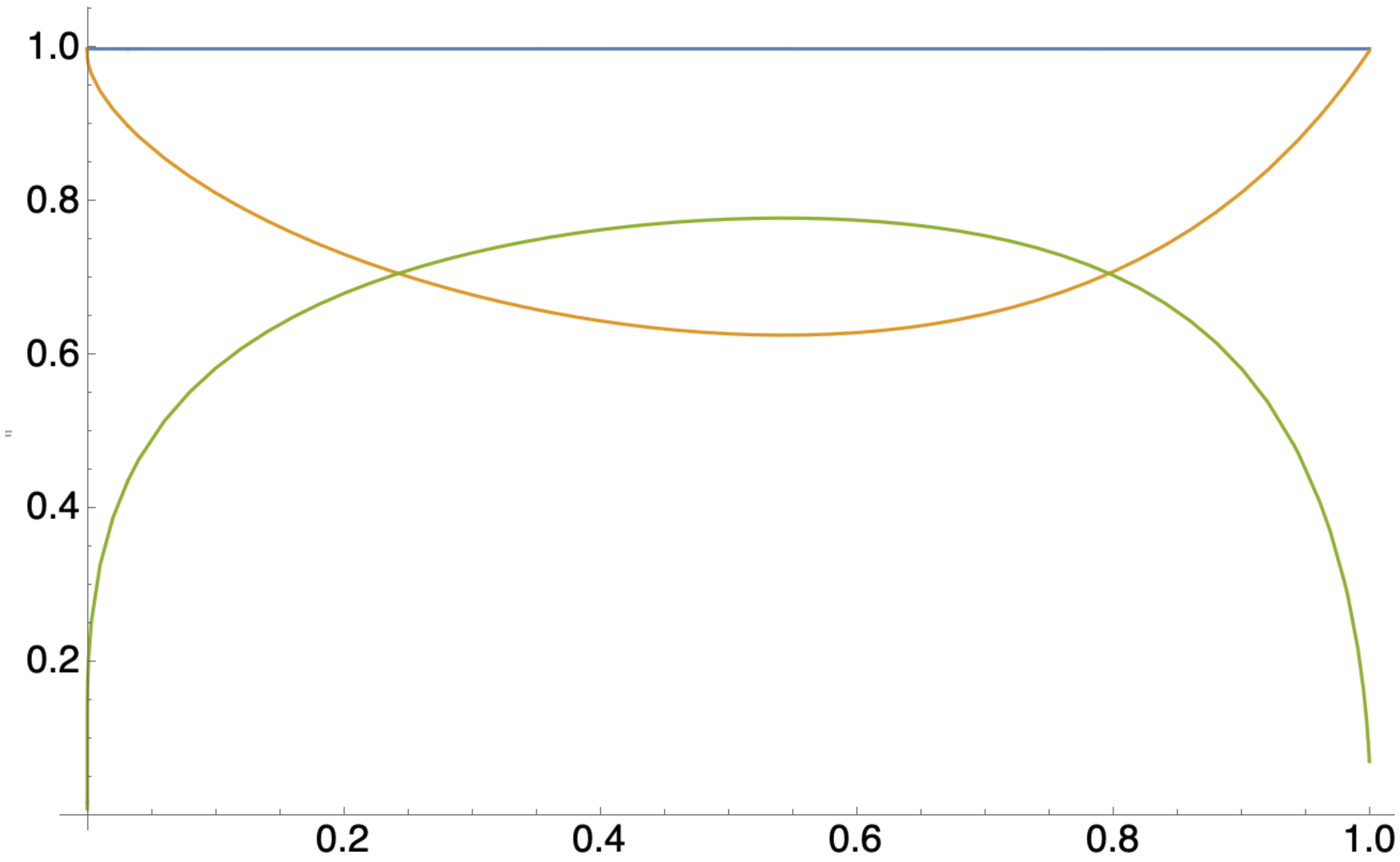
Regge behaviour / toy-model

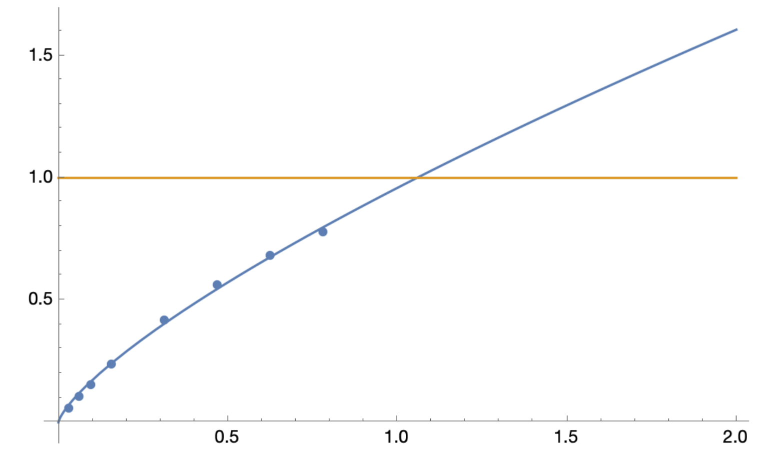


$$\frac{c_0}{\left(\log(s-4m^2)\right)^2}$$ Universal fit

how to get 1/log ? $$\frac{\lambda^2}{1+\lambda^2(\log(s-4m^2))^2}$$

Amplitudes, maximal coupling ?





Prediction:

$$\lambda = 35.2\pi$$

gives $\lambda = 1.1$

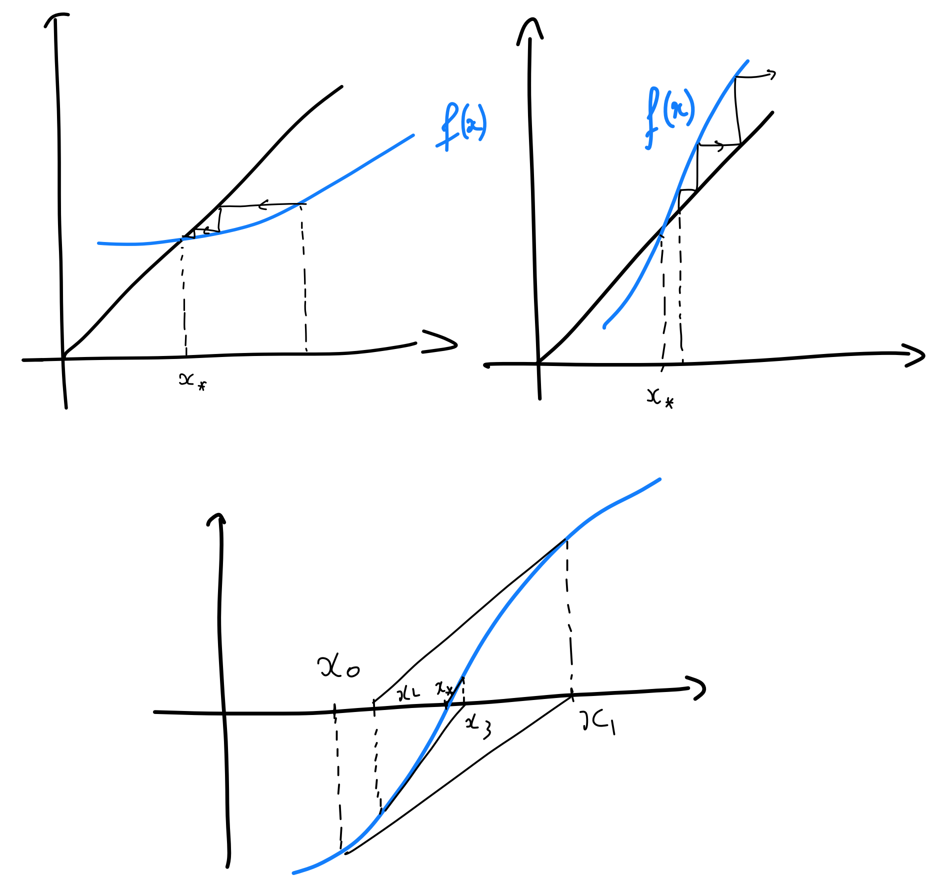
in PPTvRV's units

$$\lambda \equiv \frac{1}{32\pi} M\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$$

Summary

- We have constructed scattering amplitudes which satisfy fully unitarity axioms.
- We have introduced quasi-elastic amplitudes, which are fully unitary modifications of ϕ^4 and could lie at boundary of space of allowed functions
- In 3d we control the whole amplitude
- In 4d we converge with a cut-off, still have to check if we can control it to infinity

Numerics : prospects



- Bottlenecks: double integrals in Mandelstam equation and f_0 reconstruction

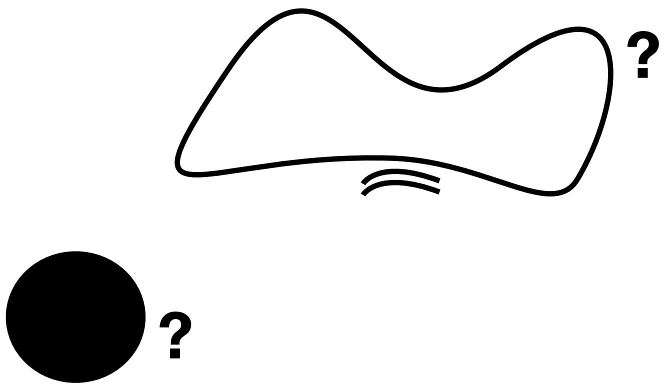
- Gradient descent (Newton-Raphson) would also be faster than fixed-point

$$\Psi[\rho] = \rho - \Phi[\rho]$$

$$\rho_{n+1} = \rho_n - \frac{1}{\Phi'(\rho_n)}\Phi(\rho_n)$$

Perspectives

- Improve numerics to use gradient-descent and enlarge space in which convergence is achieved
- Produce amplitudes that saturate Froissart bound
- Pion S-matrix
- Apply to gravity S-matrix in d>4, d=4



thank you!

extras

Toy model for Regge

- Based on empirical observation that $\rho(s,t)$ is small and adding $v_{inel}(s,t)$ always give a small effect on rho(s).

- Remove double-disc

$$
\begin{aligned}
T(s,t) &= \lambda + B(s,t) + B(s,u) + B(t,u), \\
B(s,t) &= \frac{s-s_0}{2}\int_{4m^2}^{\infty}\frac{ds'}{\pi}\frac{\rho(s')}{(s'-s)(s'-s_0)} + \frac{t-t_0}{2}\int_{4m^2}^{\infty}\frac{dt'}{\pi}\frac{\rho(t')}{(t'-t)(t'-t_0)} \\
&\quad + (s-s_0)(t-t_0)\int_{4m^2}^{\infty}\frac{ds'dt'}{\pi^2}\frac{\rho(s',t')}{(s'-s)(t'-t)(s'-s_0)(t'-t_0)}
\end{aligned}
$$

Toy model for Regge

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\end{aligned}
$$

Toy model for Regge

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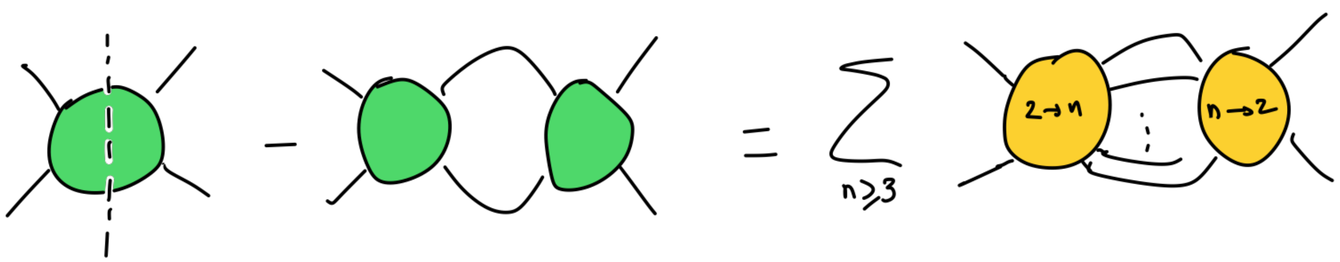
- Remove double-disc

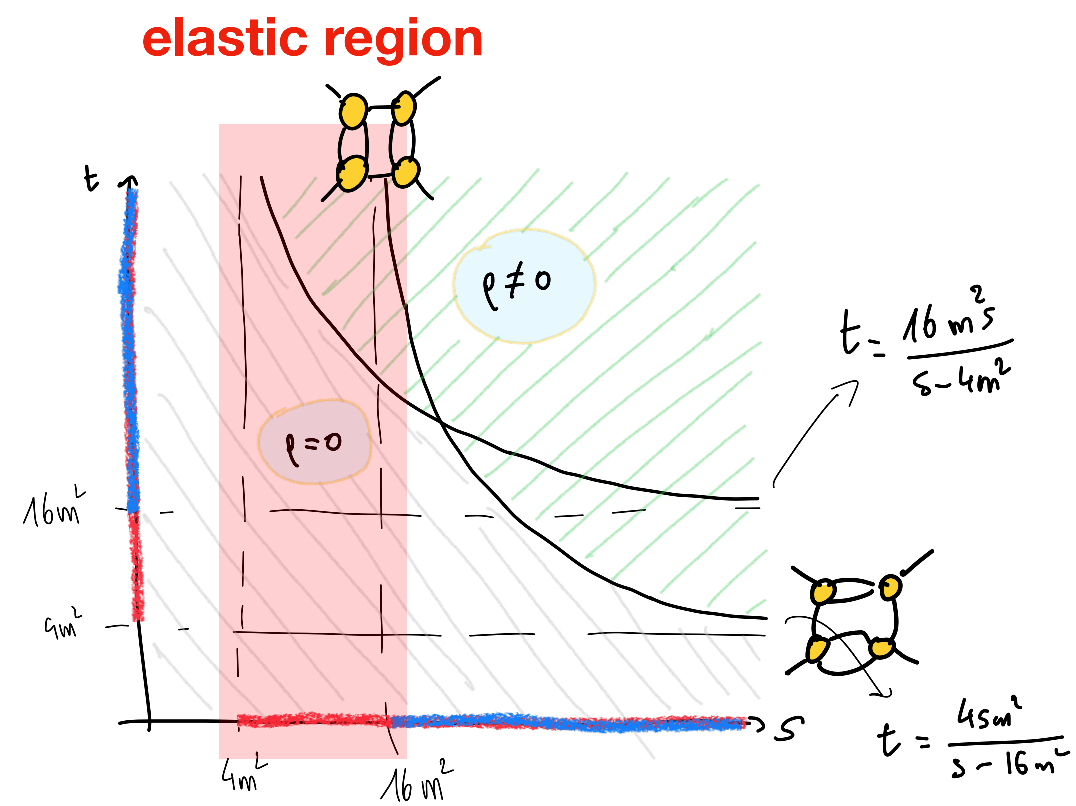
$$T(s,t) = \lambda + \int \frac{ds'}{\pi} \frac{\rho(s')}{s'-s_0} \left(\frac{s-s_0}{s'-s} + \frac{t-s_0}{s'-t} + \frac{u-s_0}{s'-u} \right)$$

- just solve
  $|S_0(s)|^2 = 1$

Aks theorem

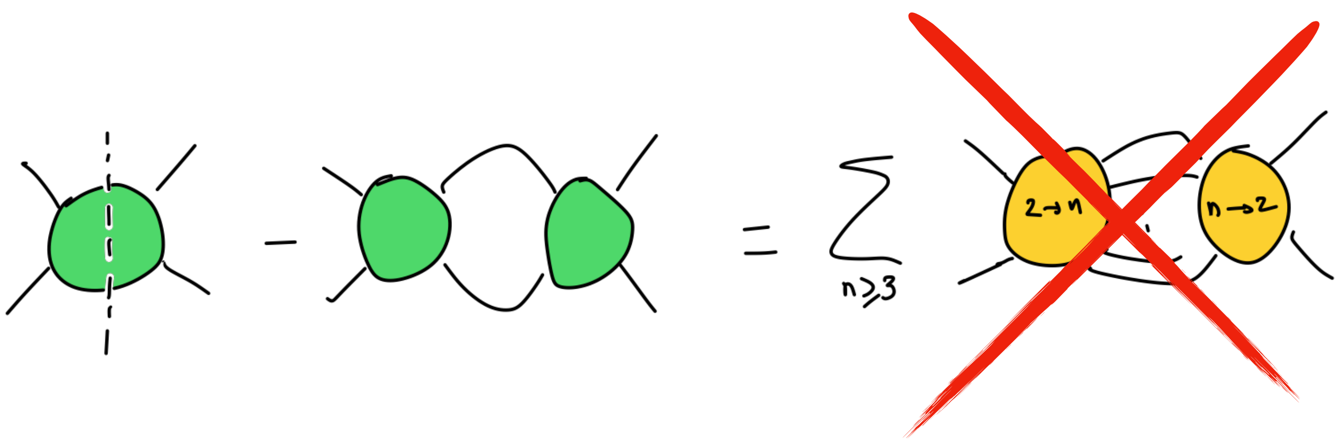
Elastic unitarity in 4d

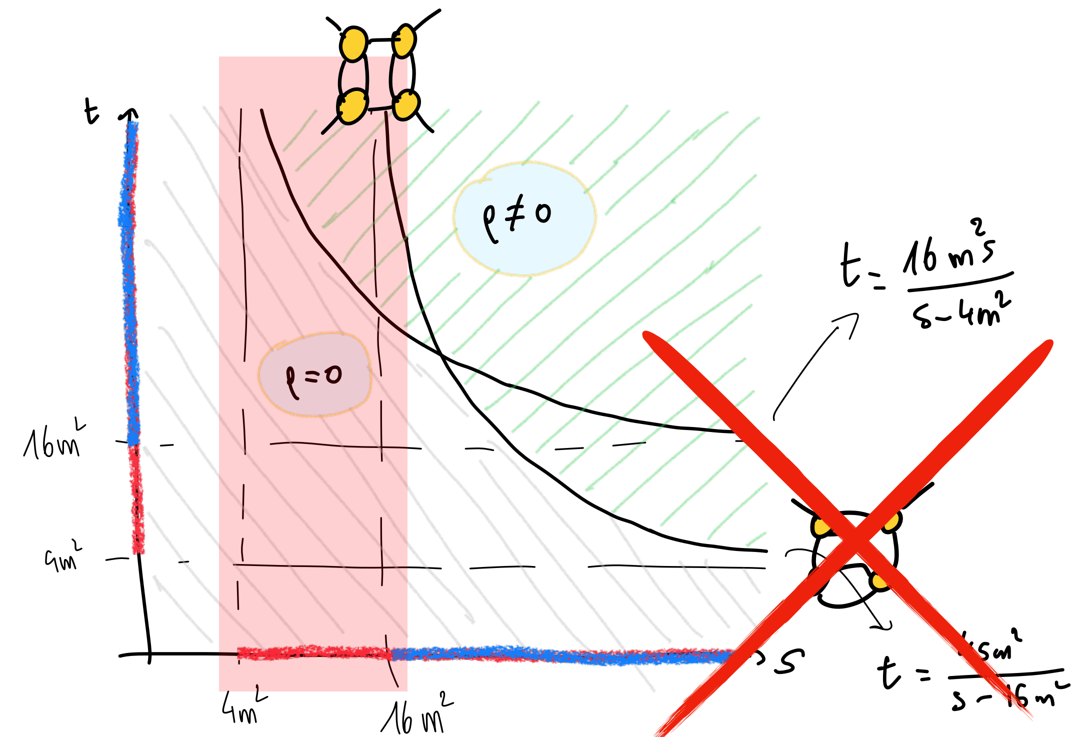




Green hashed: Support of double disc in (s,t)-plane

Elastic unitarity in 4d





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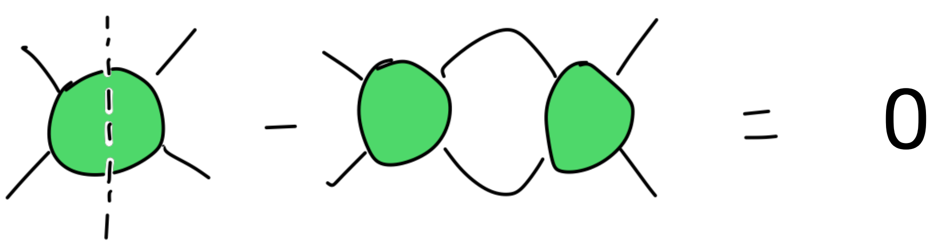
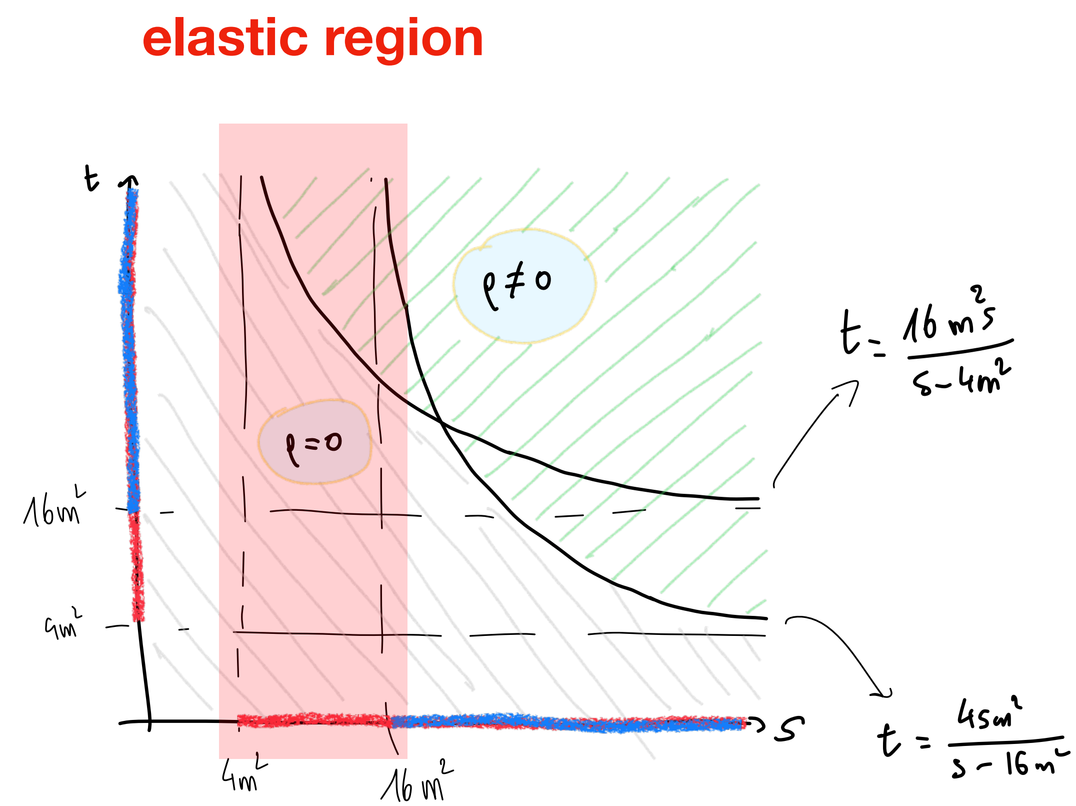
Elastic unitarity in 4d



fully non-perturbative equation



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