Celestial Amplitudes

MAX-PLANCK-INSTITUT
FÜR PHYSIK

Stephan Stieberger, MPP München

QCD Meets Gravity
Universität Zürich

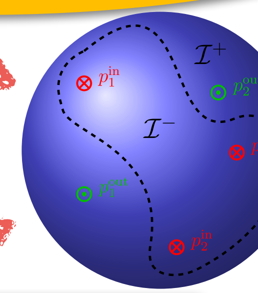
December 12-16, 2022

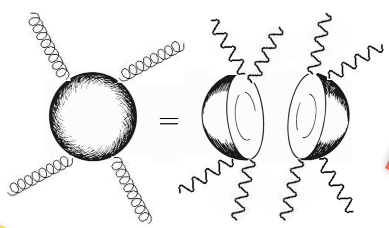
Symmetries, Amplitudes and Geometry

Hidden structure of YM:
- conformal block decomposition
- Liouville description,
- Nekrasov partition function ?

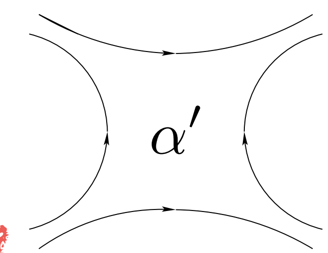
celestial sphere: celestial amplitudes

Asymptotic symmetries:
- symmetries of S-matrix,
- extended super BMS
- IR structure of gravity

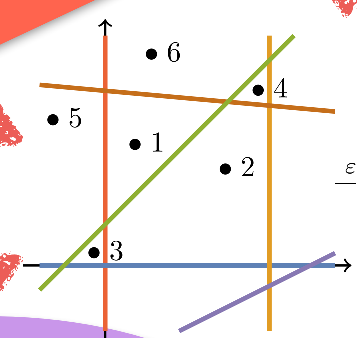




Amplitudes on Riemann surfaces



monodromy of string world-sheet



string amplitudes

Gauge-gravity relations:
- KLT, BCJ, EYM relations
- one-loop EYM
- one-loop KLT

cf. today's arXiv

twisted intersection theory

New theories and double copies:
- catalogue of new twisted forms

cf. talk by P. Mazloumi

Corrections to Einstein:
- massive gravity, bigravity
- massive double copy

cf. talk by C. Markou

based on works 2018 – 2022 with:

Wei Fan, Jiangsu

Angelos Fotopoulos, Northeastern

Tomasz R. Taylor, Northeastern

Bin Zhu, Northeastern & Perimeter

Celestial Liouville theory for Yang-Mills amplitudes #1
Stephan Stieberger (Munich, Max Planck Inst.), Tomasz R. Taylor (Northeastern U.), Bin Zhu (Northeastern U.) (Sep 6, 2022)
Published in: *Phys.Lett.B* 836 (2023) 137588 • e-Print: 2209.02724 [hep-th]
pdf DOI cite claim reference search 4 citations

Celestial Yang-Mills amplitudes and D = 4 conformal blocks #2
Wei Fan (Jiangsu U. Sci. Technol., Zhenjiang), Angelos Fotopoulos (Northeastern U.), Stephan Stieberger (Munich, Max Planck Inst.), Tomasz R. Taylor (Northeastern U.), Bin Zhu (Northeastern U.) (Jun 17, 2022)
Published in: *JHEP* 09 (2022) 182 • e-Print: 2206.08979 [hep-th]
pdf DOI cite claim reference search 3 citations

Elements of celestial conformal field theory #3
Wei Fan (Jiangsu U. Sci. Technol., Zhenjiang), Angelos Fotopoulos (Northeastern U.), Stephan Stieberger (Munich, Max Planck Inst.), Tomasz R. Taylor (Northeastern U.), Bin Zhu (Northeastern U.) (Feb 16, 2022)
Published in: *JHEP* 08 (2022) 213 • e-Print: 2202.08288 [hep-th]
pdf DOI cite claim reference search 14 citations

Conformal blocks from celestial gluon amplitudes. Part II. Single-valued correlators #4
Wei Fan (Jiangsu U. Sci. Technol., Zhenjiang), Angelos Fotopoulos (Northeastern U. and Wentworth Inst. Tech.), Stephan Stieberger (Munich, Max Planck Inst.), Tomasz R. Taylor (Northeastern U.), Bin Zhu (Northeastern U.) (Aug 23, 2021)
Published in: *JHEP* 11 (2021) 179 • e-Print: 2108.10337 [hep-th]
pdf DOI cite claim reference search 22 citations

Conformal blocks from celestial gluon amplitudes #5
Wei Fan (Jiangsu U. Sci. Technol., Zhenjiang), Angelos Fotopoulos (Northeastern U. and Wentworth Inst. Tech.), Stephan Stieberger (Munich, Max Planck Inst.), Tomasz R. Taylor (Northeastern U.), Bin Zhu (Northeastern U.) (Mar 7, 2021)
Published in: *JHEP* 05 (2021) 170 • e-Print: 2103.04420 [hep-th]
pdf DOI cite claim reference search 41 citations

Extended Super BMS Algebra of Celestial CFT #6
Angelos Fotopoulos (Northeastern U. and Wentworth Inst. Tech.), Stephan Stieberger (Munich, Max Planck Inst.), Tomasz R. Taylor (Northeastern U.), Bin Zhu (Northeastern U.) (Jul 13, 2020)
Published in: *JHEP* 09 (2020) 198 • e-Print: 2007.03785 [hep-th]
pdf DOI cite claim reference search 50 citations

On Sugawara construction on Celestial Sphere #7
Wei Fan (Jiangsu U. Sci. Technol., Zhenjiang), Angelos Fotopoulos (Northeastern U. and Wentworth Inst. Tech.), Stephan Stieberger (Munich, Max Planck Inst.), Tomasz R. Taylor (Northeastern U.) (May 21, 2020)
Published in: *JHEP* 09 (2020) 139 • e-Print: 2005.10666 [hep-th]
pdf DOI cite claim reference search 28 citations

Extended BMS Algebra of Celestial CFT #8
Angelos Fotopoulos (Northeastern U. and Wentworth Inst. Tech.), Stephan Stieberger (Munich, Max Planck Inst.), Tomasz R. Taylor (Northeastern U.), Bin Zhu (Northeastern U.) (Dec 23, 2019)
Published in: *JHEP* 03 (2020) 130 • e-Print: 1912.10973 [hep-th]
pdf DOI cite claim reference search 65 citations

Symmetries of Celestial Amplitudes #9
Stephan Stieberger (Munich, Max Planck Inst.), Tomasz R. Taylor (Northeastern U.) (Dec 3, 2018)
Published in: *Phys.Lett.B* 793 (2019) 141-143 • e-Print: 1812.01080 [hep-th]
pdf DOI cite claim reference search 73 citations

Strings on Celestial Sphere #10
Stephan Stieberger (Munich, Max Planck Inst. and Santa Barbara, KITP), Tomasz R. Taylor (Warsaw U. and Northeastern U.) (Jun 14, 2018)
Published in: *Nucl.Phys.B* 935 (2018) 388-411 • e-Print: 1806.05688 [hep-th]
pdf DOI cite claim reference search 60 citations

Lorentz group in $\mathbf{R}^{1,D+1}$ is identical
to Euklidian D-dimensional conformal group SO(1,D+1)

↓

Scattering amplitudes in $\mathbf{R}^{1,D+1}$
interpretation
as Euklidian D-dimensional conformal correlators

↓

D=2: celestial sphere

Lorentz symmetry

$SO(1,3) \simeq SL(2,\mathbf{C})$

$$z_i \to \frac{az_i + b}{cz_i + d}$$

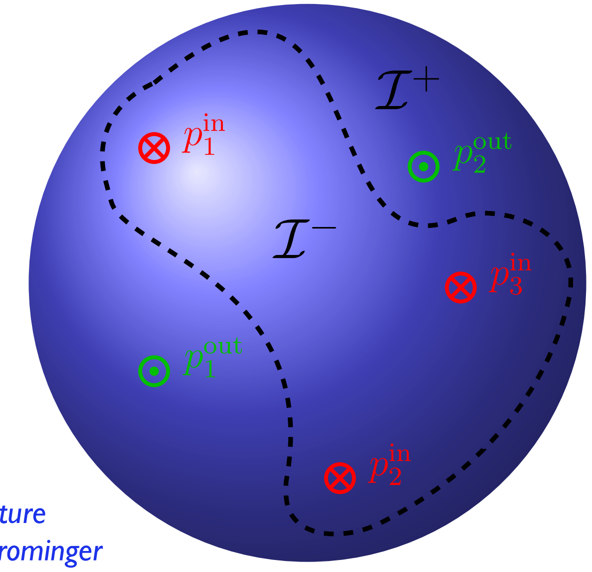
global conformal symmetry on CS^2

D=4 space-time QFT correlators

D=2 Euklidian CFT correlators

Massless particle on celestial sphere

described by
- the point $z \in CS^2$ at which it enters or exits the celestial sphere
- SL(2,C) Lorentz quantum numbers $(h, \bar{h})$



picture from Strominger

celestial *sphere at* $\mathscr{I}^{\pm}$

$$z = \frac{p^1 + ip^2}{p^0 + p^3}$$

$h + \bar{h} = \Delta$ dimension

$h - \bar{h} = J$ spin

$$p^\mu \longrightarrow (\omega, z, \bar{z})$$

$$\omega = p^0 = E$$

$$p_j^\mu = \frac{\omega_j}{1 + |z_j|^2} \left(1 + |z_j|^2, z_j + \bar{z}_j, -i(z_j - \bar{z}_j), 1 - |z_j|^2\right) \; , \; j = 1, \ldots, n$$

provides **connection** between the space of **D=4 kinematics** of n massless particles and that of the position of n points on a **sphere**

Basic Idea

in momentum basis: plane waves with momentum $p^\mu = \omega q^\mu(z)$

in conformal basis: conformal primary wave functions Φ

$$\Phi_{h,\bar{h}}\left(\frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}; \Lambda^\mu{}_\nu X^\nu\right) = (cz+d)^{2h}\, (\bar{c}\bar{z}+\bar{d})^{2\bar{h}}\, \Phi_{h,\bar{h}}(z,\bar{z};X^\mu)$$

E.g.: plane wave $\exp\{\pm i p_\mu x^\mu\}$

$$\phi^{\pm}_{\Delta}(x,z,\bar{z}) = \int_0^\infty d\omega\ \omega^{\Delta-1}\ \exp\left\{\pm i\omega q_\mu x^\mu - \epsilon\omega\right\}$$

$$= \left\{x^\mu q_\mu(z,\bar{z}) \mp i\epsilon\right\}^{-\Delta}$$

solves D=4
Klein-Gordon equation

complete normalisable basis

$\Delta = 1 + i\lambda,\ \lambda \in \mathbf{R}$

scalar: J=0 $\quad h = \bar{h} = \frac{\Delta}{2}$

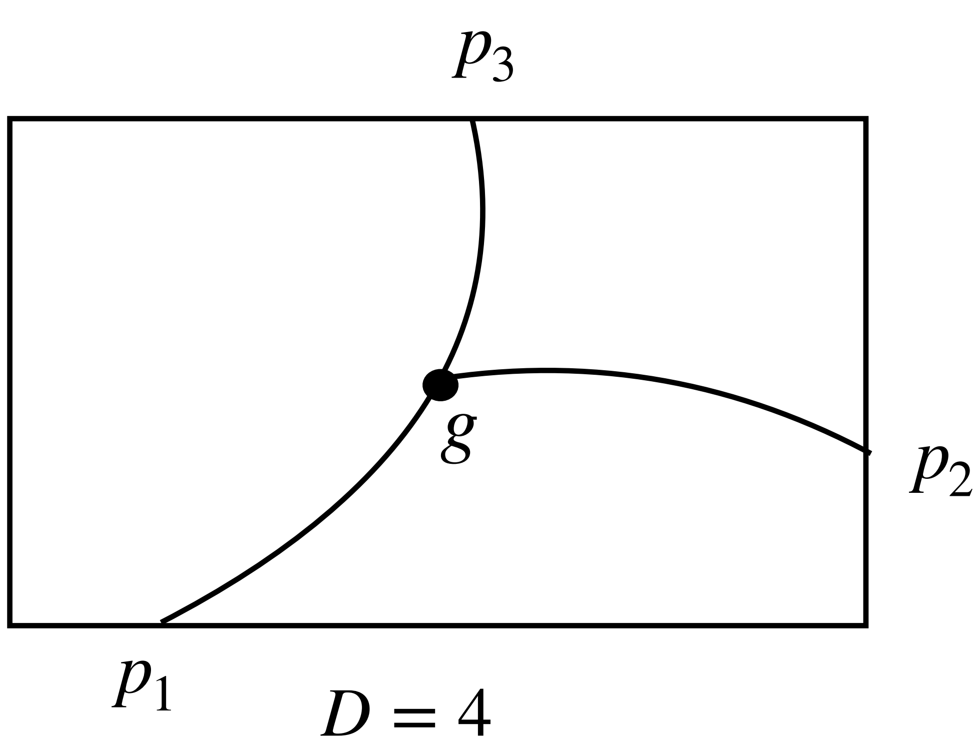
Pasterski, Shao (2017)

state operator correspondence: particles $\Longleftrightarrow$ operators

In the massless case, with or without spin,
transition from momentum space to conformal primary wavefunctions
with conformal dimension Δ
is implemented by Mellin transform:

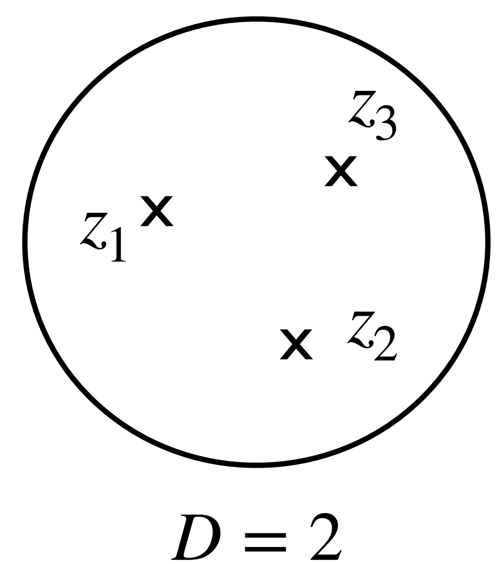
$$|\Delta, z\rangle = \int_0^\infty d\omega\ \omega^{\Delta-1}\ |\omega, z\rangle$$

Amplitudes = conformal correlators of primary fields on celestial sphere



$$z_k = \frac{p_k^1 + ip_k^2}{p_k^0 + p_k^3}$$

$=$



celestial sphere

D=4 space-time QFT correlators

D=2 Euklidian CFT correlators

Celestial Amplitudes

$$\mathscr{A}(\{p_i, \epsilon_j\}) = i(2\pi)^4 \, \delta^{(4)}\left(p_1 + p_2 - \sum_{k=3}^{n} p_k\right) \, A(\{p_i, \epsilon_j\})$$

Celestial amplitudes $\tilde{\mathscr{A}}$ of massless particles are obtained
from momentum-space amplitudes $\mathscr{A}$
by Mellin transforms w.r.t. particle energies $\Delta_j = 1 + i\lambda_j$

Mellin transform converts **plane wave basis** of external wave functions
into **boost eigenstates** characterized by conformal dimension Δ_j

$$\tilde{\mathscr{A}}_{\{\Delta_l\}}(\{z_l, \bar{z}_l\}) = \left(\prod_{l=1}^{n} \int_0^\infty \omega_l^{\Delta_l - 1} \, d\omega_l\right) \delta^{(4)}(\omega_1 q_1 + \omega_2 q_2 - \sum_{k=3}^{N} \omega_k q_k)$$

$$\times A(\{\omega_i, z_i, \bar{z}_i\})$$

D=2 CFT correlators involve conformal wave packets

Why ?

- Constrain S-matrix and understand amplitude relations

- Sensitivity to both UV and IR physics leads to powerful constraints on the analytic structure of amplitudes

Arkani-Hamed, Pate, Raclariu, Strominger (2020)

- Asymptotic symmetries:

Symmetry generating currents arise from conformally soft massless particles $\Delta \to 1,0\ldots$

extended super BMS group on celestial sphere

- New look for new physical principles and mathematical structures to constrain S-matrix

- New way of looking at quantum field theory and quantum gravity

- flat space-time holography

Gauge Amplitudes

four-gluon amplitude:

$$\tilde{\mathcal{A}}_4(-,-,+,+) = 8\pi\,\delta(r-\bar{r})\,\theta(r-1)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\,\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)$$

all four points z_i must lie on a circle

$$\times\, r^{\frac{5}{3}}\,(r-1)^{\frac{2}{3}}\,\delta\left(-4+\sum_{i=1}^{4}\Delta_i\right)$$

$$r = \frac{z_{12}\,z_{34}}{z_{23}\,z_{41}}$$

conformal invariant cross-ratio on CS^2

$$r^{-1} = \sin^2\left(\frac{\theta}{2}\right)$$

Pasterski, Shao, Strominger (2017)

$$h_1 = \tfrac{i}{2}\lambda_1,\;\; h_2 = \tfrac{i}{2}\lambda_2,\;\; h_3 = 1+\tfrac{i}{2}\lambda_3\,,\, h_4 = 1+\tfrac{i}{2}\lambda_4$$

$$\bar{h}_1 = 1+\tfrac{i}{2}\lambda_1,\;\; \bar{h}_2 = 1+\tfrac{i}{2}\lambda_2,\;\; \bar{h}_3 = \tfrac{i}{2}\lambda_3\,,\, \bar{h}_4 = \tfrac{i}{2}\lambda_4$$

higher-point: involve Gaussian hypergeometric functions like string amplitudes

Graviton Amplitudes

four-graviton amplitude:

$$\tilde{\mathscr{A}}_4(--,--,++,++) = 2\pi\,\delta(r-\bar{r})\,\theta(r-1)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\;\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)$$

$$\times\, r^{\frac{11}{3}-\frac{\beta}{3}}\,(r-1)^{-\frac{1}{3}-\frac{\beta}{3}}\,\delta\left(-2+\sum_{i=1}^{4}\Delta_i\right)$$

St.St., Taylor (2018)

$$h_1 = -\frac{1}{2}+\frac{i}{2}\lambda_1,\;\; h_2 = -\frac{1}{2}+\frac{i}{2}\lambda_2,\;\; h_3 = \frac{3}{2}+\frac{i}{2}\lambda_3\;, h_4 = \frac{3}{2}+\frac{i}{2}\lambda_4$$

$$\bar{h}_1 = \frac{3}{2}+\frac{i}{2}\lambda_1,\;\; \bar{h}_2 = \frac{3}{2}+\frac{i}{2}\lambda_2,\;\; \bar{h}_3 = -\frac{1}{2}+\frac{i}{2}\lambda_3\;, \bar{h}_4 = -\frac{1}{2}+\frac{i}{2}\lambda_4$$

$$\beta := 2-\frac{1}{2}\sum_{i=1}^{4}\Delta_i$$

- *first calculation of graviton amplitude in the conformal basis*
- *important for the soft graviton theorems* $\Delta \to 1,0,\ldots$ *in celestial basis*

no holomorphic factorization (due to supertranslation operator P)

Celestial Conformal Field Theory (CCFT)

understand the nature of 2D CFT on celestial sphere,
i.e. spectrum of fields and their interactions

- states, spectrum
- operator products (OPEs)
- energy momentum tensor, Virasoro algebra
- conformal block expansion
- crossing symmetry and conformal bootstrap
- ⋮

recent (tremendous) progress

Operator product expansion

Celestial conformal field theory (CCFT)

$$\begin{aligned}
\mathcal{O}^a_{\Delta_1,-1}(z,\bar z)\mathcal{O}^b_{\Delta_2,+1}(w,\bar w) &= \frac{C_{(-,+)-}(\Delta_1,\Delta_2)}{z-w}\sum_c f^{abc}\ \mathcal{O}^c_{(\Delta_1+\Delta_2-1),-1}(w,\bar w)\\
&+\frac{C_{(-+)+}(\Delta_1,\Delta_2)}{\bar z-\bar w}\sum_c f^{abc}\ \mathcal{O}^c_{(\Delta_1+\Delta_2-1),+1}(w,\bar w)\\
&+C_{(-+)--}(\Delta_1,\Delta_2)\,\frac{\bar z-\bar w}{z-w}\,\delta^{ab}\,\mathcal{O}_{(\Delta_1+\Delta_2),-2}(w,\bar w)\\
&+C_{(-+)++}(\Delta_1,\Delta_2)\,\frac{z-w}{\bar z-\bar w}\,\delta^{ab}\,\mathcal{O}_{(\Delta_1+\Delta_2),+2}(w,\bar w)\ +reg\,.
\end{aligned}$$

Derive from collinear limits of D=4 EYM amplitudes

Fan, Fotopoulos, St.St., Taylor, Zhu (2019)

↕ D=4 S-matrix constrains OPE or vice versa

Derive from first principles and consistency conditions

Pate, Raclariu, Strominger, Yuan (2019)

} extended BMS symmetry

Celestial Double Copy

explicit field realization $\mathcal{O}^a_{\Delta,+1}(z,\bar{z})$ = gluon operator with helicity $+1$

Sugawara construction of the energy momentum tensor

$$T(w) = \frac{1}{2\dim(g)} \lim_{\Delta_1,\Delta_2\to 0} [\Delta_2(\Delta_1+\Delta_2)] \lim_{u\to w} \sum_a \mathcal{O}^a_{\Delta_2,+1}(u,\bar{u})\tilde{\mathcal{O}}^a_{2-\Delta_1,+1}(w,\bar{w})$$

KLT like at the level of operators on CS^2

Fan, Fotopoulos, St.St., Taylor (2020)

cf. also Casali, Puhm (2020)
Casali, Sharma (2020)

$$\frac{1}{2k+C_2} \simeq \frac{1}{2\dim(g)},\quad k=0$$

- inherit (rewrite) D=4 double copy on CS^2
by promoting celestial momenta to operators $\mathcal{P}^\mu_l := \epsilon_l\, q^\mu_l\, e^{\partial_{\Delta_l}}$

- any new gauge-gravity relations ?

---

shadow transformation:

$$\tilde{\mathcal{O}}^a_{\tilde{\Delta},\tilde{J}}(z,\bar{z}) = \tilde{\mathcal{O}}^a_{2-\Delta,-J}(z,\bar{z}) = \frac{(\Delta+J-1)}{\pi}\int_{\mathbf{C}} \frac{d^2w}{(z-w)^{2-\Delta-J}(\bar{z}-\bar{w})^{2-\Delta+J}}\ \mathcal{O}^a_{\Delta,J}(w,\bar{w})$$

Ferrara, Grillo, Parisi, Gatto (1972)
Dolan, Osborn (2012)

$\mathcal{O}^a_{\Delta,+1}(z,\bar{z})$

$\Delta \to 1,0,\ldots$

Symmetry generating currents arise from conformally soft massless particles

E.g.: conformally soft modes of gauge bosons generate a Kac-Moody symmetry and correspond to large gauge transformations in D=4

soft-modes of graviton: $\mathcal{O}_{\Delta,\pm 2}(z,\bar{z})$ $\Delta \to 1,0,-1\ldots$

- Lorentz group
- Superrotation
- global space-time translation
- local space-time translation
- supersymmetry
- ⋮

Lorentz group, Superrotation } Virasoro superrotation

global space-time translation, local space-time translation } supertranslation

Asymptotic symmetries of S-matrix at null infinity $\mathcal{I}^{\pm}$

Celestial scattering equations

n points on CS^2

$$z_j = \frac{p_j^1 + i p_j^2}{p_j^0 + p_j^3} = \frac{p_j^0 - p_j^3}{p_j^1 - i p_j^2}\ , \quad j = 1, \ldots .n$$

spinor helicity

$$\langle ij \rangle = 2\ (\omega_i \omega_j)^{1/2}\ (z_i - z_j)$$

$$[ij] = 2\ (\omega_i \omega_j)^{1/2}\ (\bar{z}_i - \bar{z}_j)$$

$$\begin{cases} 2p_i \cdot p_j = \langle ij \rangle [ji] \\ \qquad = 4\omega_i \omega_j |z_{ij}|^2 \end{cases}$$

solve scattering equations

$$\boxed{\sum_{i \neq j}^{n} \frac{p_i \cdot p_j}{z_i - z_j} = 0 \quad , \quad j = 1, \ldots, n}$$

saddle point equations in high energy string theory

→ celestial scattering equations

scattering equations in celestial coordinates:

$$f_i := \sum_{j \neq i} \frac{\epsilon_i \epsilon_j \, \omega_i \omega_j \, q_i q_j}{z_i - z_j} = 0 \quad , \quad i = 1, \ldots, n$$

equations for $z_i \in CS^2$

$$\mathscr{A}_n = \int_{\mathscr{M}_{0,n}} d\mu_n \, \widetilde{\mathscr{F}}(\{\mathscr{P}_i, \tilde{\xi}_i, z_l\}) \, \mathscr{F}(\{\mathscr{P}_l, \xi_l, z_l\}) \prod_{l=1}^{n}{}' \delta(f_l)$$
$$\times 2\pi \, \delta\left(i\left(\sum_{i=1}^{n} \Delta_i - 4\right)\right) F_n(\{\Delta_r, q_r, s_r\})$$

Adamo, Mason Sharma (2019)

$$\mathscr{P}_l^\mu := \epsilon_l \, q_l^\mu \, e^{\partial_{\Delta_l}}$$

$n > 4$: *integrate four* ω_i*–integrals => n-4 integrations left over*

$$F_n(\{\Delta_i, q_i, \epsilon_i\}) = \frac{1}{u} \prod_{l,r} \Theta(u_{lr}) \int_0^1 \prod_{r=5}^{n} \frac{d\xi_r}{\xi_r} \, \xi_r^{\Delta_r} \prod_{l=1}^{4} \left(\sum_{r'=5}^{n} u_{lr'} \, \xi_{r'} \right)^{\Delta_l - 1} \delta\left(1 - \sum_{r''=5}^{n} \xi_{r''}\right)$$

Aomoto-Gelfand hypergeometric functions over Grassmannian Gr(n-4,4)

cf. Schreiber, Volovich, Zlotnikov (2017) Casali, Sharma (2020)

with minors $u = |(1234)|$ and $u_{lr} = -\epsilon_l \epsilon_r \frac{(12\ldots l-1 r l+1\ldots 4)}{(1234)}$

Moduli space of celestial scattering

momentum conservation: $$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \epsilon_i \, \omega_i \, q_i(\bar{z}_l, z_l) = 0$$

gives allowed region $$\bigcup_{i=1}^{n} \{(\epsilon_i, z_i, \bar{z}_i)\} \text{ for } \omega_i > 0$$

Actually, celestial amplitudes are overconstraint by translational invariance or D=4 momentum conservations are too strong:

- *3-pt and 4-pt amplitudes vanish everywhere except for a measure zero hypersurface of celestial coordinates*
- *similar constraints persist for higher-point amplitudes: n-point amplitudes supported only on certain patches of* CS^2 *:*

→ moduli space of open tree-level string theory

$$\mathscr{M}_{n\geq 5} = (\mathbf{CP}^1)^n \backslash \bigcup_{i,j,k,l} \{\Im(r_{ijkl}) = 0\} \Big/ SL(2,\mathbf{C})$$

$$r_{ijkl} = \frac{z_{ij} z_{kl}}{z_{ik} z_{jl}}$$

Mizera, Pasterski (2021)

Single-valued celestial amplitudes

Celestial gluon amplitude: $\delta(z-\bar{z})$ => no crossing symmetry and no conformal block-decomposition

Note: power of complex analysis
(holomorphy, Cauchy theorem, analytic continuation)
are indispensable for CFT (conformal block decomposition, …)

Way out:

- consider amplitudes involving shadow fields — *Fan, Fotopoulos, St.St., Taylor, Zhu (2021)*
- go to (2,2) signature, but still delta-functions in low-point amplitudes (no power of holomorphy)
- supply background momentum by coupling to dilaton background field — *Fan, Fotopoulos, St.St., Taylor, Zhu (2022)*; *Casali, Melton, Strominger (2022)*
- consider celestial scatterings in non-trivial backgrounds — *de Gioia, Raclariu (2022), Gonzo, McLoughlin, Puhm (2022)*

Note: this is similar to Coulomb gas formulation of minimal models
where correlators also vanish in absence of background charges

our dilaton source (charge) plays the role as the U(1) background charge in Coulomb gas models

Note: In conventional CFT (e.g. minimal models, WZW, …)
two- and three-point correlation functions of primary fields
are determined by conformal invariance

four-point correlators are then constructed from solutions to PDEs

(akin KZ equations)

Note: tree-level string amplitudes are also single-valued in kinematic invariants
and fulfill KZ like equations

Assume: four-point celestial correlators are constructed from solutions to PDEs

our solution:

- single-valued on (entire) CS^2
- hypergeometric function Appell F_4
- conformal block decomposition in D=2
- Coulomb Gas description
- D=4 conformal blocks
- Liouville description

Fan, Fotopoulos, St.St., Taylor, Zhu (2021, 2022)

St.St., Taylor, Zhu (2022)

Celestial KZ-type equations

for positive helicity gluon i we have:

from considering BCFW shifts

$$\left(\lambda_{i-1}^{\alpha}\frac{\partial}{\partial\lambda_i^{\alpha}} - \tilde{\lambda}_i^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_{i-1}^{\dot{\alpha}}}\right)\mathscr{A}_n = \frac{\langle i-1\, i+1\rangle}{\langle i+1\, i\rangle}\,\mathscr{A}_n$$

Hu, Ren, Srikant, Volovich (2021)

in celestial coordinates:

$$\left[-\left(\Delta_i + z_{ij}\frac{\partial}{\partial z_i}\right) + \frac{z_{i-1,j}}{z_{i-1,i}} + \frac{z_{i+1,j}}{z_{i+1,i}} - 1\right]\tilde{\mathscr{A}}_n + \epsilon_i\epsilon_j\left(\Delta_j - J_j - 1 + \bar{z}_{ji}\frac{\partial}{\partial\bar{z}_j}\right)e^{\frac{\partial}{\partial\Delta_i}-\frac{\partial}{\partial\Delta_j}}\,\tilde{\mathscr{A}}_n = 0$$

Banerjee, Ghosh (2020)

from consistency requirements of subleading soft theorem with OPEs of primary fields

(KZ-type equations)

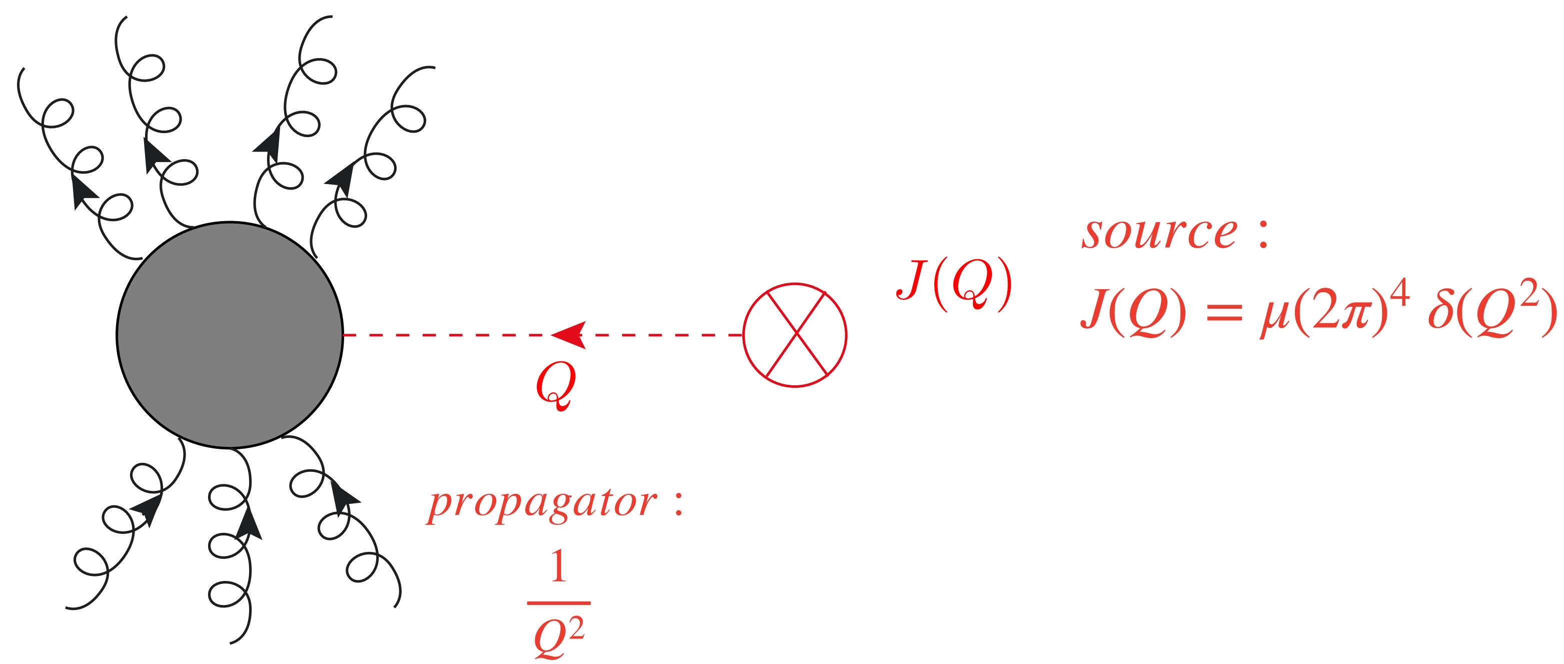
solution:

$$\widetilde{\mathscr{A}}_4 = \left\langle \phi^{a_1,-\epsilon}_{\Delta_1,-}(z_1,\bar{z}_1)\,\phi^{a_2,-\epsilon}_{\Delta_2,-}(z_2,\bar{z}_2)\,\phi^{a_3,+\epsilon}_{\Delta_3,+}(z_3,\bar{z}_3)\,\phi^{a_4,+\epsilon}_{\Delta_4,+}(z_4,\bar{z}_4)\right\rangle$$

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

$$= \underbrace{\left(f^{a_1a_2b}f^{a_3a_4b}\frac{1}{z_{12}z_{23}z_{34}z_{41}} + f^{a_1a_3b}f^{a_2a_4b}\frac{1}{z_{13}z_{32}z_{24}z_{41}}\right)\frac{z_{12}^2}{\bar{z}_{12}^2}}_{\text{gauge sector}}\;\underbrace{\mathcal{S}_4(\{z_i,\bar{z}_i\})}_{\text{Coulomb Gas}}$$

our single-valued amplitudes correspond to (Mellin transforms of) Dilaton-YM gluon amplitudes



dilaton couples at (off-shell) momentum Q

$$\sum_{i=1}^{n} p_i = Q$$

$$\mathscr{A}(1^-, 2^-, 3^+, \ldots, n^+)_{\Phi(Q)} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

Dixon, Glover, Khoze (2004)

Note: dilaton naturally accompanies graviton in closed superstring theory

Comments:

$$\widetilde{\mathcal{A}}_4 = \left\langle \phi^{a_1,-\epsilon}_{\Delta_1,-}(z_1,\bar{z}_1)\,\phi^{a_2,-\epsilon}_{\Delta_2,-}(z_2,\bar{z}_2)\,\phi^{a_3,+\epsilon}_{\Delta_3,+}(z_3,\bar{z}_3)\,\phi^{a_4,+\epsilon}_{\Delta_4,+}(z_4,\bar{z}_4)\right\rangle_{\Phi(Q)}$$

$$= \left(f^{a_1a_2b}f^{a_3a_4b}\,\frac{1}{z_{12}z_{23}z_{34}z_{41}} + f^{a_1a_3b}f^{a_2a_4b}\,\frac{1}{z_{13}z_{32}z_{24}z_{41}} \right) \frac{z_{12}^2}{\bar{z}_{12}^2}\,\mathcal{S}_4(\{z_i,\bar{z}_i\})$$

- solution factorizes into a current (soft) and scalar part:

$$\widetilde{\mathcal{A}}_n(z_1,\bar{z}_1,\ldots,z_n,\bar{z}_n\,|\,\Delta_1,\ldots,\Delta_n) = \mathcal{J}_n(z_i)\;\mathcal{S}_n(z_i,\bar{z}_i)$$

- $\mathcal{S}_4$ can be written as single D=4 conformal block, should be related to D=4 conformal symmetry of SYM

- gauge (soft) factor: group-dependent part of WZW correlator of n currents

- scalar part: complex integral like Dotsenko-Fateev (Coulomb gas)

→ infinite central charge limit of Liouville theory

Conformal blocks

<u>*standard CFT:*</u> *conformal block decomposition of correlation functions*

→ comprises full spectrum → information on all states

$$G^{21}_{34}(x,\bar{x}) = \lim_{z_1,\bar{z}_1\to\infty} z_1^{2h_1}\bar{z}_1^{2\bar{h}_1}\left\langle \phi^{a_1}_{\Delta_1,+}(z_1,\bar{z}_1)\phi^{a_2}_{\Delta_2,-}(1,1)\underbrace{\phi^{a_3}_{\Delta_3,+}(x,\bar{x})\phi^{a_4}_{\Delta_4,+}(0,0)}_{\textit{OPE}}\right\rangle$$

D=2 channel $(12 \rightleftharpoons 34)_2$

Di Francesco, Mathieu, Senechal (1997)

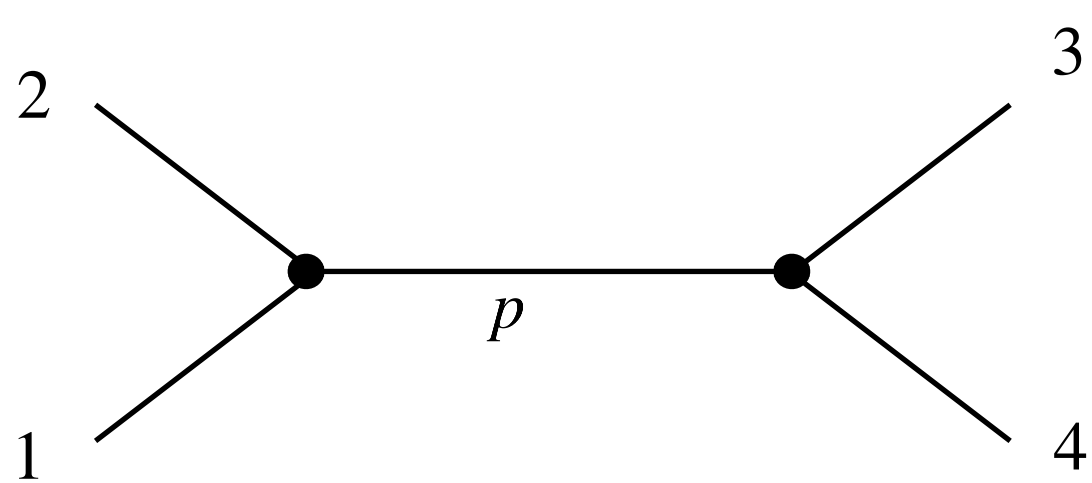
$$\phi_3(x,\bar{x})\phi_4(0,0) = \sum_p C^p_{34}\, x^{h_p-h_3-h_4}\, \bar{x}^{\bar{h}_p-\bar{h}_3-\bar{h}_4}\, \Psi_p(x,\bar{x}\,|\,0,0)$$

$$\stackrel{!}{=} \sum_p C^p_{34} C^p_{12}\, A^{21}_{34}(p;x,\bar{x}) = \sum_p C^p_{34} C^p_{12}\, \underbrace{\mathscr{F}^{21}_{34}(p;x)\times\bar{\mathscr{F}}^{21}_{34}(p;\bar{x})}_{\text{conformal blocks}}$$

conformal blocks

(holomorphic factorization in x and $\bar{x}$)

$A^{21}_{34}(p;x,\bar{x}) =$



$= \mathscr{F}^{21}_{34}(p;x)\times\bar{\mathscr{F}}^{21}_{34}(p;\bar{x})$

s-channel representation

Explicit expression for conformal blocks is *not known* in general:

$$\mathscr{F}^{12}_{34}(p;x) = x^{h_p-h_3-h_4} \sum_{K=0}^{\infty} \mathscr{F}_K \; x^K$$

we want to understand this on CS^2

however, if we deal only
with primary fields with conformal dimensions $(h,\bar{h})$
we have:

conformal block: $$\mathscr{F}^{12}_{34}(h;x) = x^{h-h_3-h_4} \; {}_2F_1\left[\begin{matrix} h-h_{12}, h+h_{34} \\ 2h \end{matrix}; x\right]$$

$$K^{21}_{34}[h,\bar{h}] := \mathscr{F}^{12}_{34}(\bar{h};\bar{x}) \times \mathscr{F}^{12}_{34}(h;x)$$

Osborn (2012)

<u>Recall:</u> $\widetilde{\mathscr{A}}_4 = \left\langle \phi^{a_1,-\epsilon}_{\Delta_1,-}(z_1,\bar{z}_1)\,\phi^{a_2,-\epsilon}_{\Delta_2,-}(z_2,\bar{z}_2)\,\phi^{a_3,+\epsilon}_{\Delta_3,+}(z_3,\bar{z}_3)\,\phi^{a_4,+\epsilon}_{\Delta_4,+}(z_4,\bar{z}_4)\right\rangle_{\Phi(Q)}$

$$= \left(f^{a_1a_2b}f^{a_3a_4b}\,\frac{1}{z_{12}z_{23}z_{34}z_{41}} + f^{a_1a_3b}f^{a_2a_4b}\,\frac{1}{z_{13}z_{32}z_{24}z_{41}} \right)\frac{z_{12}^2}{\bar{z}_{12}^2}\,\mathcal{S}_4(\{z_i,\bar{z}_i\})$$

$$G_{\mathcal{S}}(x,\bar{x}) = \lim_{z_1,\bar{z}_1\to\infty} z_1^{i\lambda_1}\bar{z}_1^{i\lambda_1}\,\mathcal{S}_4(z_1; z_2=1; z_3=x; z_4=0)$$

$$= \sum_{n=0}^{\infty}\Big\{ a_n K^{21}_{34}\left[n+\frac{i\lambda_3+i\lambda_4}{2}, n+\frac{i\lambda_3+i\lambda_4}{2}\right](x,\bar{x})$$

$$+ b_n K^{21}_{34}\left[n+2-\frac{i\lambda_3+i\lambda_4}{2}, n+2-\frac{i\lambda_3+i\lambda_4}{2}\right](x,\bar{x})\Big\}$$

two sets of primaries contributing

$$h_n = \bar{h}_n = n + \frac{i\lambda_3+i\lambda_4}{2}$$

$$h_n = \bar{h}_n = n + 2 - \frac{i\lambda_3+i\lambda_4}{2}$$

spin $J=0$ gauge singlets

infinite tower of primary fields

infinite number of symmetries

CCFT symmetries go far beyond BMS

cf. also Guevara, Himwich, Pate, Strominger (2021)

appearance of W algebras in celestial symmetry

infinite tower of tree-level soft graviton symmetries can be organized into $w_{1+\infty}$ symmetry algebra

Strominger (2022)

Further Directions

- understand better celestial amplitudes
  - *what are the fundamental symmetries of nature ?*
  - *yield further symmetry constraints on amplitudes !*
  - *celestial double copy*

- clarify impact of Coulomb gas formalism and relations to string theory
  - *Aomoto-Gelfand hypergeometric structure*
  - *moduli space*