Towards a Universal Decomposition of Phase-Space Integrands

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Taming Jets' Hair

- Jet differential cross sections are the finest-grained observables at modern colliders
- LO — each jet modeled by a lone parton
- NLO — virtual: same; IR divergences in loop integrals
  — real-emission, some jets modeled by pair,
  IR divergences in phase-space integrals
- NNLO — double-virtual: lone parton, IR divs in loop integrals
  — mixed: mixed divergences
  — double-real: IR divergences in phase-space integrals
- Need to cancel divergences
- Ideally point-by-point in jet phase space

Current Approaches

- Cancel in physical observables (not point by point)
- Slicing — virtual divergences known analytically
  - — separate singular regions (soft, collinear)
  - — integrate analytically
  - — integrate numerically in hard regions

  Giele & Glover; Giele, Glover, & DAK
- Subtraction — singular behavior known analytically
  - — subtract it everywhere, resulting integral finite
  - — integrate singular functions analytically

  Catani & Seymour; Frixione, Kunszt, & Signer; Nagy & Soper; Bevilacqua, Czakon, Kubocz, & Worek
- NNLO — generalizations of subtraction

  Gehrmann-De Ridder, Gehrmann, & Glover; Weinzierl; Del Duca, Duhr, Kardos, Somogyi, Szőr, Trócsányi, & Tulipánt; Czakon; Magnea, Maina, Pelliccioli, Signorile-Signorile, Torrielli, & Uccirati
  - — hybrid schemes

  Stewart, Tackmann, & Waalewijn; Catani & Grazzini
- Complicated and not yet fully general

Framework

- Virtual @ NLO: n partons $\to$ n (proto)jets
- Real Emission @ NLO: $n+1$ partons $\to$ n (proto)jets
- Do phase-space integral exactly in $D = 4 - 2\epsilon$
  - Mixture of analytic and numerical

- Want to align phase spaces, expose analogy
- Reexpress partons in terms of protojets

$$\hat{k}_i = \hat{k}_i(\{k_j\}^{n+1})$$

$$k_r = k_r(\{k_j\}^{n+1})$$

- Factorize phase space (exact)

$$d\,\mathrm{LIPS}^D_{n+1}\left(K;\{k_i\}^{n+1}_{i=1}\right) = d\,\mathrm{LIPS}^D_n\left(\{\hat{k}_i\}\right)\,d\,\mathrm{LIPS}^D_s\left(k_r\right)\ \mathrm{Jac}$$

Vision of the Destination

If you don't know where you're going, you'll end up someplace else. — Yogi Berra

- Master-integral decomposition of multi-emission phase-space integrals

$\Uparrow$

- Master-integral decomposition of single-emission phase-space integrals

$\Uparrow$

- Decomposition of single-emission phase-space integrands: squares of tree amplitudes

$\Uparrow$

A thousand-mile journey begins with a single step

Paraphrase of Laozi (between 2300 – 2500 yrs ago)

- Decomposition of tree amplitudes with one emission
- Suitable mapping to isolate emission
  ~ "theoretical" jet algorithm
- Classification of amplitudes
- Computational algebraic geometry

Decomposition of One-Loop Integrands

- First: integrands with trivial numerators
- Integrand of hexagon
- External momenta strictly in $D = 4$
- Use Gram determinants

$$G\begin{pmatrix} p_1, \ldots, p_m \\ q_1, \ldots, q_m \end{pmatrix} = \det_{i,j}\left(2p_i \cdot q_j\right) .$$

Scalar Hexagon Decomposition

- Six denominators $D_j = \left(\ell - K_{1,j}\right)^2$
- Write a Gram identity

$$0 = G\begin{pmatrix} \ell, & k_1, & \ldots, & k_4 \\ k_5, & k_1, & \ldots, & k_4 \end{pmatrix} = \omega_j D_j + \omega_0$$

- And put it over the denominator

$$\frac{\omega_0}{D_1 D_2 D_3 D_4 D_5 D_6} = -\sum_{j=1}^{6} \frac{\omega_j}{D_1 \cdots \cancel{D_j} \cdots D_6}$$

New Lyrics to an Old Melody

like the Lichtenstein National Anthem

- Look at singularities on both sides of the decomposition

- Both are singular when any D_j vanishes

- What happens when all D_j vanish simultaneously?
  - Left-hand side (6 powers) appears more singular than right-hand side (5 powers)
  - Consistent only if all D_j cannot vanish simultaneously
  - Obstruction must be dependent on external momenta in $D = 4$

Inconsistency of Equations

- Need to show that simultaneous equations

$$D_j = 0 \quad (j = 1, \ldots, 6)\,, \qquad G\begin{pmatrix} \ell, & k_1, & \ldots, & k_4 \\ k_5, & k_1, & \ldots, & k_4 \end{pmatrix} = 0$$

have no solution

- Use computational algebraic geometry

- Show the ideal generated by these polynomials is the unit ideal $\langle 1 \rangle$: compute the Gröbner basis

- Cofactor matrix would give coefficients $c_j D_j + c_0 G = 1$

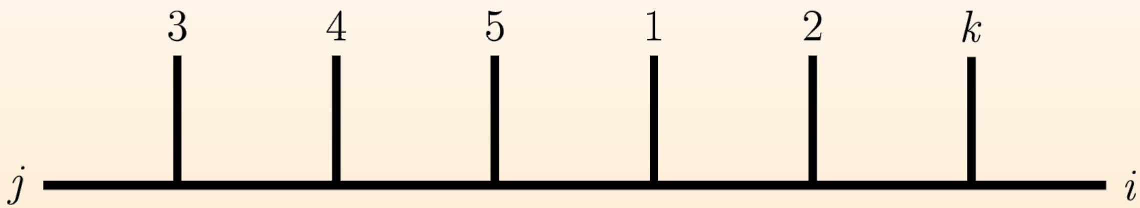
Inverse Antenna Mapping

- Antenna mapping: maps partons → protojets
- Three recombining momenta yielding two massless protojets
- Want to map protojets → partons, so that we can write
  - original partons as f(protojets, real emission)

$$
\begin{aligned}
k_i &= \frac{1}{2}\left(1+\tau(s_{\hat{a}r}, s_{r\hat{b}})w_+(s_{\hat{a}r}, s_{r\hat{b}})\right)k_{\hat{a}} - \frac{1}{2}\left(1+\tau(s_{\hat{a}r}, s_{r\hat{b}})w_\lambda(s_{\hat{a}r}, s_{r\hat{b}})\right)k_r \\
&\quad + \frac{1}{2}\left(1+\tau(s_{\hat{a}r}, s_{r\hat{b}})w_-(s_{\hat{a}r}, s_{r\hat{b}})\right)k_{\hat{b}}\,, \\
k_j &= k_r\,, \\
k_k &= \frac{1}{2}\left(1-\tau(s_{\hat{a}r}, s_{r\hat{b}})w_+(s_{\hat{a}r}, s_{r\hat{b}})\right)k_{\hat{a}} - \frac{1}{2}\left(1-\tau(s_{\hat{a}r}, s_{r\hat{b}})w_\lambda(s_{\hat{a}r}, s_{r\hat{b}})\right)k_r \\
&\quad + \frac{1}{2}\left(1-\tau(s_{\hat{a}r}, s_{r\hat{b}})w_-(s_{\hat{a}r}, s_{r\hat{b}})\right)k_{\hat{b}}
\end{aligned}
$$

- Ultimately, functions of $\tau, \hat{\lambda}, s_{\hat{a}r}, s_{r\hat{b}}, s_{r1}, s_{r2}$

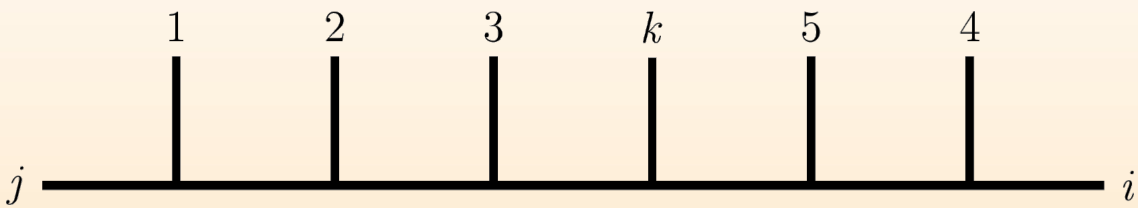
Simple Example



- Contribution $\dfrac{1}{s_{j3}s_{j34}s_{j345}s_{j1345}s_{j12345}}$
- $S_1 = s_{j3}, S_2 = s_{j34}, S_3 = s_{j345}, S_4 = s_{j1345}, S_5 = s_{j12345}$
- Build a Gram $G\begin{pmatrix} k_j, & k_1, & \ldots, & k_4 \\ k_5, & k_1, & \ldots, & k_4 \end{pmatrix}$
- It gives a similar decomposition

$$\frac{\omega_0^r}{S_1 S_2 S_3 S_4 S_5} = -\sum_{j=1}^{5} \frac{\omega_j^r}{S_1 \cdots \cancel{S}_j \cdots S_5}$$

Harder Case



- No simple identity as for the simple example
- Need to add functions imposing mapping constraints

$$0 = \frac{c_0}{T_1 T_2 T_3 T_4 T_5} + \sum_{j=1}^{5} \frac{c_j}{T_1 \cdots \not{T}_j \cdots T_5} + \sum_{j=1}^{n_z} \frac{\hat{c}_j Z_j}{T_1 T_2 T_3 T_4 T_5}$$

- $T_1 = s_{i4}, T_2 = s_{j1}, T_3 = s_{i45}, T_4 = s_{j12}, T_5 = s_{j123}$
- Z_j vanish on physical configurations

- Too computationally difficult in standard variables

Better Variables & Finite Field Numerics

- Use finite-field momenta for $k_1, \ldots, k_5, k_{\hat{a}}, k_{\hat{b}}$
- Variables $V = \{s_{\hat{a}r}, s_{r\hat{b}}, s_{r1}, s_{r2}, \tau, \hat{\lambda}\}$;
  s_{r3} &c. expressed numerically
- Gram constraints solved explicitly
- Form of $\tau, \hat{\lambda}$ expressed by two constraints $R_\tau, R_{\hat{\lambda}}$

- Ideal $\langle T_1, T_2, T_3, T_4, T_5, R_\tau, R_{\hat{\lambda}} \rangle$

- Compute GröbnerBasis($B; V$) using Singular $\Rightarrow \{1\}$
- No common solutions as desired

Coefficient Simplification

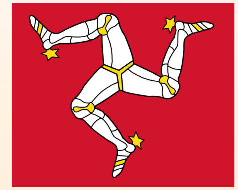
- Coefficients from cofactor matrix $\sum_i \ c_i B_i = 1$

- Not necessarily "simplest"

- Compute syzygies of B
- Compute Gröbner basis of syzygies
- Reduce cofactors against this Gröbner basis

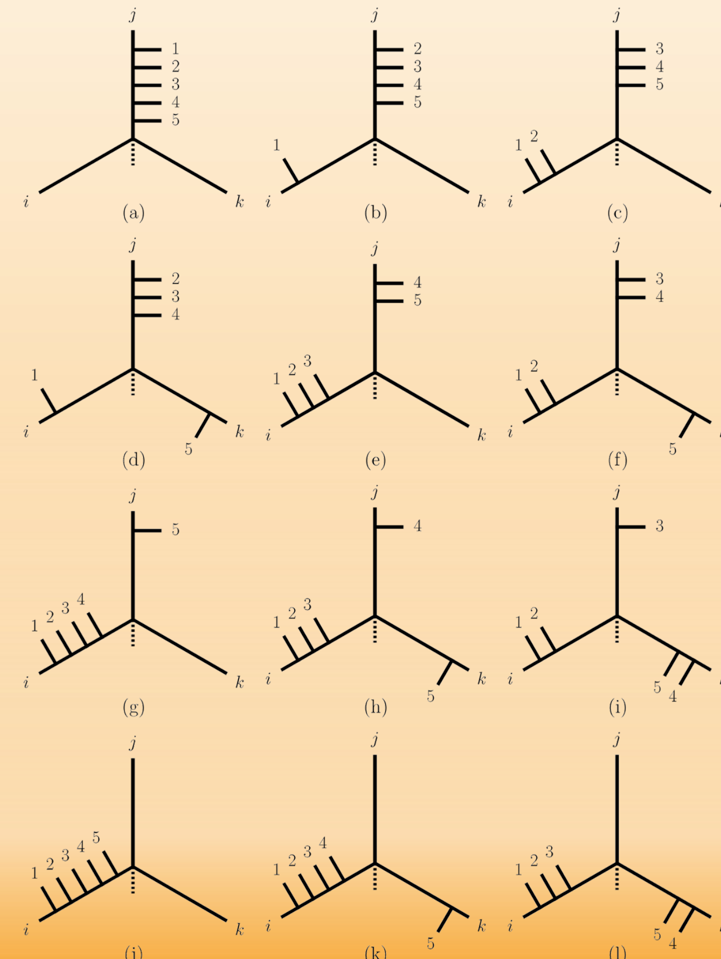
- Can make coefficients independent of k_r, but not $\tau, \hat{\lambda}$

Triskelia



a well-known triskelion

- Classify all possible contributions: focus on all possible arrangements of the three recombining partons
- Lines from each inwards will ultimately meet at a center
- Number and types of legs attached give different triskelia
- Twelve major classes, subclasses depending on masses



Survey

- Consider 1152 triskelia (after symmetries)

- All yield unit Gröbner basis
- In all cases, coefficients can be made independent of k_r
- Results independent of external masses

Numerators

- What about reduction of nontrivial numerators?
- In one-loop integrals, just ordinary partial fractioning
- In CAG,

$$v \quad \text{mod} \;\; \text{GröbnerBasis}\left(\{D_i\}_{i=1}^{5}; W_{\ell:4}\right) = \text{constant} \quad \forall v \in W_{\ell:4}$$

- Analog for tree-level contributions

$$v \quad \text{mod} \;\; \text{GröbnerBasis}\left(\{T_1, T_2, T_3, T_4, R_\tau, R_{\hat{\lambda}}\}; V\right) = \text{Poly}(\tau, \hat{\lambda})$$

$$\forall v \in \{s_{\hat{a}r}, s_{r\hat{b}}, s_{r1}, s_{r2}\}$$

Summary

- First step towards decomposing phase-space integrals into master integrals
- Recast one-loop integral reduction into language of computational algebraic geometry
- Generalizes to allow partial-fractioning of any cubic contribution to a tree-level scattering amplitude
- Finite basis of master integrals