Leading and Landau singularities in Feynman integrals

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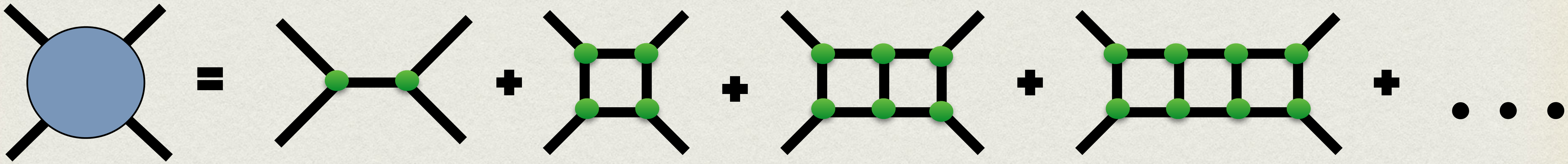
Analytic evaluation of scattering amplitudes



Analyticity & Unitarity of S-matrix

Draw all Feynman diagrams

Generate integrands

Use Integration-By-Parts identities

Profit from Dimensional Regularisation

Evaluate integrals

Numerically

Sector Decomposition

LTD approach

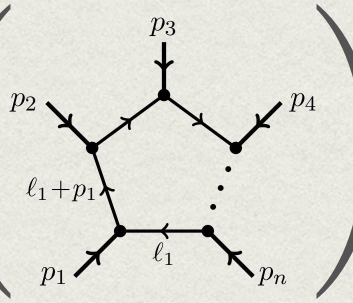
Analytically

Diff. Eqs.

Standard approach @multi-loop level

This talk: we study and prove a connection between *Landau* and *leading* singularities for *N*-point one-loop Feynman integrals

LS (



) $\sim \dfrac{1}{\sqrt{\text{LanS}_n}}$

D=n :: Landau Singularities of first type
D=n+1 :: Landau Singularities of second type

Leading and Landau singularities in Feynman integrals

[2210.09872 hep-th]

Outline

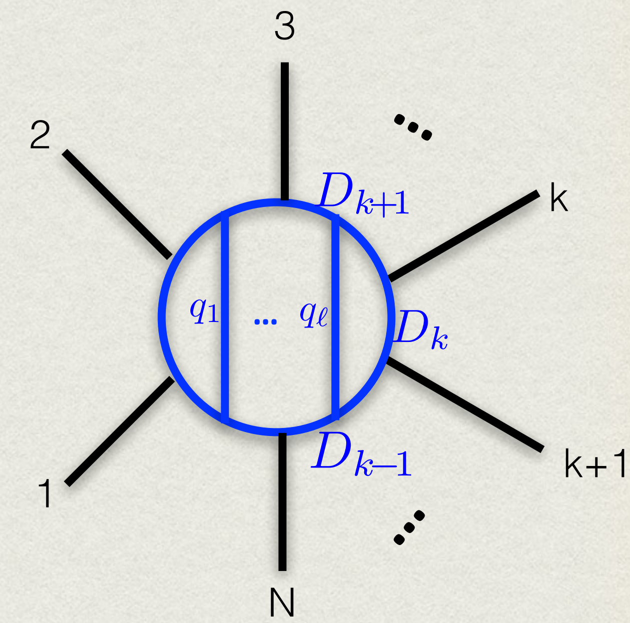
- Motivation
- Space-time dimension in parallel & orthogonal directions
- Landau & Leading singularities
- Differential equations
- Conclusion/Outlook

Preliminary

In loop calculations, one finds

$$J_N^{(L),D}(1,\ldots,n;n+1,\ldots,m) = \int \prod_{i=1}^{L} \frac{d^D \ell_i}{\imath\, \pi^{D/2}} \frac{\prod_{k=n+1}^{m} D_k^{-\nu_k}}{\prod_{j=1}^{n} D_j^{\nu_j}}$$

$$D_i = q_i^2 - m_i^2 + \imath 0$$



What to do?

- Evaluate them?
- Analyse them?

First principles

Get mathematical insights

Everything is connected!

Profit from mathematical properties

Keep into account behaviour dictated by physics

Investigate further mathematical formalism

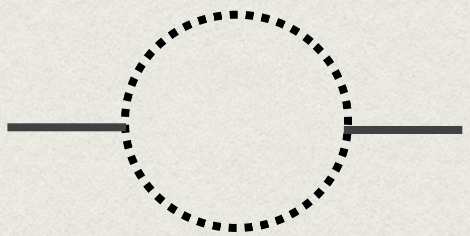
Algorithms for computing Feynman integrals

Standard approach

- DEQ :: Feynman integrals are not independent

$$\partial_x \overrightarrow{J}(x) = A_i(x,\epsilon)\overrightarrow{J}(x)$$

$m^2 = 0$



$$\frac{1}{\epsilon}(-p^2)^{-1-\epsilon}\left(-2+\frac{\pi^2}{6}\epsilon^2+\frac{14}{3}\zeta_3\epsilon^3+\mathcal{O}\left(\epsilon^4\right)\right)$$

Canonical form
Conjecture: there exist a basis of <u>uniform transcendental weight functions</u>

[Henn (2013)]

$$\partial_x \overrightarrow{g}(x) = \epsilon B(x)\overrightarrow{g}(x) \longrightarrow d\,\overrightarrow{g}(x,\epsilon) = \epsilon\left(d\tilde{B}\right)\overrightarrow{g}(x;\epsilon)$$

$$\tilde{B} = \sum_k B_k \log \alpha_k(x)$$

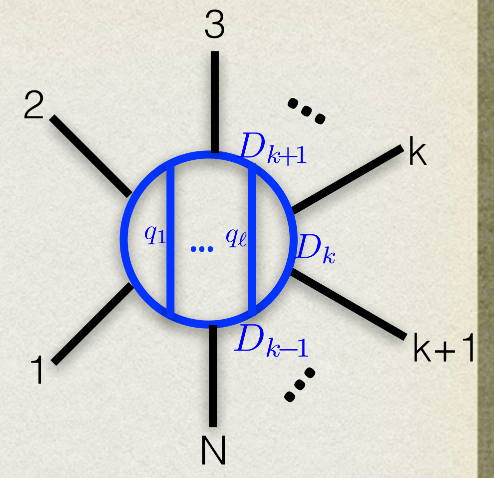
Uniform weight function

- Solution in terms of iterated integrals :: HPL/GPL (PolyLogs)

$$\mathcal{G}\left(a_1,\ldots,a_n;x\right) = \int_0^x dt\frac{1}{t-a_n}\mathcal{G}\left(a_1,\ldots,a_{n-1};t\right)$$

Numerical implementations:
`GinaC`, `HandyG`, `FastGPL`, ...

Feynman integrals in $D = D_{\parallel} + D_{\perp}$



- Decomposition of space-time dimension :: $D = D_{\parallel} + D_{\perp}$

Loop momenta

$$\ell^{\alpha}_{i,[D]} = \ell^{\alpha}_{i,[D_{\parallel}]} + \ell^{\alpha}_{i,[D_{\perp}]}$$

$$\ell^{\alpha}_{i,[D_{\parallel}]} = \sum_{j=1}^{D_{\parallel}} a_{ij}\, p^{\alpha}_{j}$$

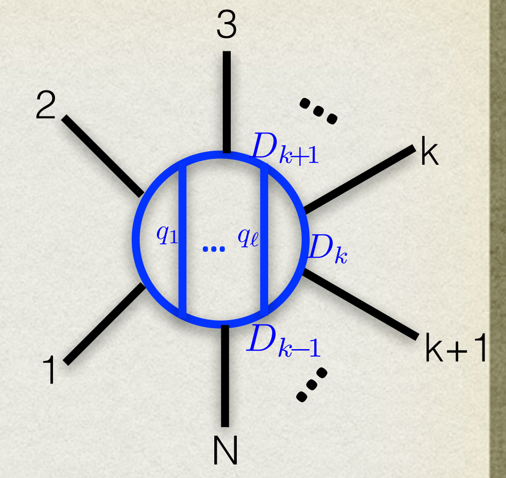
external momenta

$$\ell^{\alpha}_{i,[D_{\perp}]} = \lambda^{\alpha}_{i} = \sum_{j=1}^{D_{\perp}} b_{ij}\, \omega^{\alpha}_{j}$$

orthogonal momenta

$$p_i \cdot \omega_j = 0$$

Feynman integrals in $D = D_{\parallel} + D_{\perp}$



- Decomposition of space-time dimension :: $D = D_{\parallel} + D_{\perp}$

Loop momenta

$$\ell^{\alpha}_{i,[D]} = \ell^{\alpha}_{i,[D_{\parallel}]} + \ell^{\alpha}_{i,[D_{\perp}]}$$

$$\ell^{\alpha}_{i,[D_{\parallel}]} = \sum_{j=1}^{D_{\parallel}} a_{ij}\, p^{\alpha}_{j}$$

external momenta

$$p_i \cdot \omega_j = 0$$

$$\ell^{\alpha}_{i,[D_{\perp}]} = \lambda^{\alpha}_{i} = \sum_{j=1}^{D_{\perp}} b_{ij}\, \omega^{\alpha}_{j}$$

orthogonal momenta

- @one-loop

$$J^{(1),D}_{N} \sim \mathcal{J}^{(1)}_{[D_{\parallel}]} \int \prod_{i=1}^{D_{\parallel}} da_i\, d\lambda_{11}\, \lambda_{11}^{(D_{\perp}-2)/2} \frac{1}{\prod_{j=1}^{N} D_j}$$

Gram determinant of external momenta

$$\lambda_{ij} = \lambda_i \cdot \lambda_j = \sum_{k=1}^{D_{\perp}} b_{ik} b_{jk}\, \omega_j^2$$

- @multi-loop

$$J^{(L),D} \sim \mathcal{J}^{(L)}_{[D_{\parallel}]} \prod_{i=1}^{L} \prod_{j=1}^{D_{\parallel}} da_{ij} \left[G\left(\lambda_{ij}\right) \right]^{\frac{D_{\perp}-L-1}{2}} \prod_{1\leq i \leq j}^{L} d\lambda_{ij}$$

Directly connected with *Baikov* representation

Gram determinant of λ_i's

Landau singularities

- Landau equations

Feynman integral are many-valued analytic function whose singularities lie on some algebraic varieties — Landau Varieties

$$q_i^2 - m_i^2 = 0 \text{ or } \alpha_i = 0$$

$$\sum \alpha_i \frac{\partial D_i}{\partial k_j} = \sum \alpha_i q_i = 0$$

$$\sum \left(q_i \cdot q_j \right) \alpha_i = 0$$

- Landau singularity

$$\text{LanS} = \det(q_i \cdot q_j) = 0$$

Make use of decomposition of space-time dimension

One-loop Landau singularities in $D = D_\parallel + D_\perp$

- Landau equations $\qquad \mathrm{LanS}_n^{(1)} = \det(q_i \cdot q_j) = \lambda_{11} \det\left((p_i \cdot p_j)_{(n-1)\times(n-1)}\right)$

On-shell conditions!

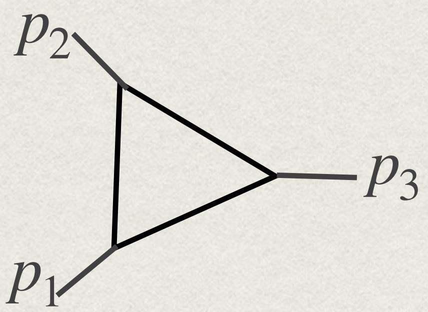
$$D_i - D_{i+1} = -m_i^2 + m_{i+1}^2 - 2\sum_{j=1}^{i-1}\left(p_i \cdot p_j\right) - p_i^2 - 2\sum_{j=1}^{D_\parallel}\left(p_i \cdot p_j\right) a_j = 0$$

solve for a_i's

$$D_1 = \sum_{j,k=1}^{D_\parallel}\left(p_j \cdot p_k\right) a_j a_k - m_1^2 + \lambda_{11} = 0$$

solve for λ_{11}

- One-loop triangle in D=3



$$\ell_1^\alpha = a_1\, p_1^\alpha + a_2\, p_2^\alpha + \lambda_1^\alpha$$
$$\lambda_{11} = m_1^2 - a_1^2 p_1^2 - a_2^2 p_2^2 + a_1 a_2\left(p_1^2 + p_2^2 - p_3^2\right)$$

$$\mathrm{LanS}_3^{(1)} = -\frac{1}{4}\left[\frac{p_1^2\, p_2^2\, p_3^2}{3} - p_3^2\left(\left(m_1^2 - m_2^2\right)\left(m_2^2 - m_3^2\right) + m_2^2\left(p_1^2 + p_2^2 - p_3^2\right)\right)\right] + \text{cycl. perm.}\,.$$

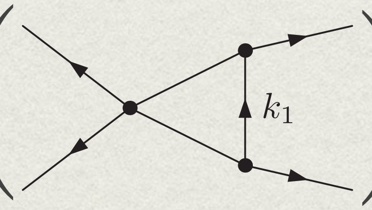
Leading singularities

- One-loop Feynman integrals evaluate to special numbers & functions

$$\zeta_n = \sum_{k\geq 1} \frac{1}{k^n} \qquad \log(x) \qquad \mathrm{Li}_2(x) \qquad G(\{a_1, \ldots, a_n\}; x)$$

- Main motivation —> find integrands :: dx/x

- One-loop massless triangle in D=4



$$\mathcal{I}\left(\cdots \right) = \frac{d^4 k_1}{\left(k_1 - p_1\right)^2 k_1^2 \left(k_1 + p_2\right)^2}$$

$$= \frac{1}{s} d\log\left(k_1 - p_1\right)^2 \, d\log k_1^2 \, d\log\left(k_1 + p_2\right)^2 \, d\log\left(2\, k_1 \cdot e_3\right)$$

leading singularity

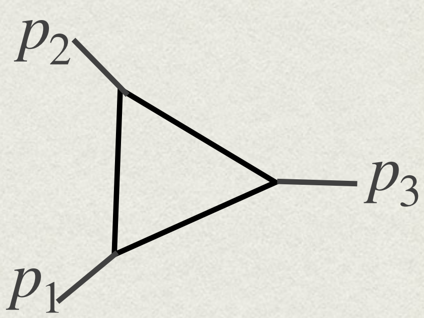
One-loop leading singularities in $D = D_{\parallel} + D_{\perp}$

- One-loop Landau singularities in $D = D_{\parallel} + D_{\perp}$

$$J_N^{(1),D} \sim \mathscr{J}^{(1)}_{[D_{\parallel}]} \int \prod_{i=1}^{D_{\parallel}} da_i \, d\lambda_{11} \, \lambda_{11}^{(D_{\perp}-2)/2} \frac{1}{\prod_{j=1}^{N} D_j} \qquad \lambda_{11} = \sum_{k=1}^{D_{\perp}} b_k^2 \, \omega_j^2$$

arbitrary space-time dimension

- Study various values for $D_{\perp}$

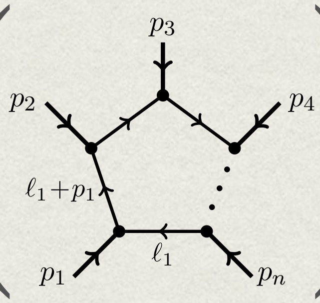


$$J_3^{(1),D=3} \sim \pm \frac{1}{8\sqrt{-\mathrm{LanS}_3^{(1)}}} \int d\log \frac{D_1}{D_{\pm}} \wedge d\log \frac{D_2}{D_{\pm}} \wedge d\log \frac{D_3}{D_{\pm}} \,,$$

$$J_3^{(1),D=4} \sim \pm \frac{1}{4\sqrt{\lambda_{\mathrm{K}}\left(p_1^2, p_2^2, p_3^2\right)}} \int d\log D_1 \wedge d\log D_2 \wedge d\log D_3 \,,$$

$J_3^{(1),D\geq 5} \rightarrow$ No d log representation

Connection between leading & Landau singularities

LS (

) $\sim \dfrac{1}{\sqrt{\text{LanS}_n}}$

D=n :: Landau Singularities of first type
D=n+1 :: Landau Singularities of second type

Theorem 4.1. *The leading singularity of an n-point one-loop Feynman integral in $D = n+1$ space-time dimensions is equal to $\pm 1/\left(2^n\sqrt{-\det\left(p_i\cdot p_j\right)}\right)$, with $i, j \leq n-1$.*

Theorem 4.2. *The leading singularity of an n-point one-loop Feynman integral in $D = n$ space-time dimensions is equal to $\pm 1/\left(2^n\sqrt{(-1)^{D-1}\text{LanS}}\right)$*

- *multi-dimensional theory of residues due to Leray*

Lemma B.4. *The leading singularity of the integrand $\int \prod_{i=1}^{n} da_i\, 1/\left(K_1 K_2 \cdots K_n\right)$ is equal to $1/\det\left(x_{ij}\right)$, where K_i is a linear function in the integration variables, $K_i = x_{i0} + \sum_{j=1}^{n} x_{ij}\, a_j$.*

Dlog representation of one-loop Feynman integrals

Conjecture 1. *An n-point Feynman integral can be written in one of the following forms depending on the space-time dimension D,*

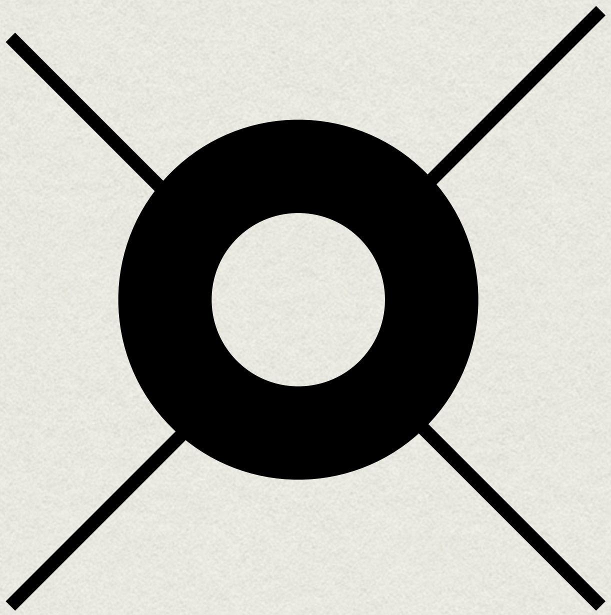
$$J_n^{(1),D=n} \sim \pm \frac{1}{2^n \sqrt{-\mathrm{LanS}_n^{(1)}}} \int d\log \frac{D_1}{D_\pm} \wedge \ldots \wedge d\log \frac{D_n}{D_\pm}\,, \qquad \textit{with } i,j \le n\,, \tag{4.25a}$$

$$J_n^{(1),D=n+1} \sim \pm \frac{1}{2^n \sqrt{-\det\left(p_i \cdot p_j\right)}} \int d\log D_1 \wedge \ldots \wedge d\log D_n\,, \qquad \textit{with } i,j \le n-1\,, \tag{4.25b}$$

$$J_n^{(1),D\ge n+2} \to \textit{ No } d\log \textit{ representation}\,, \tag{4.25c}$$

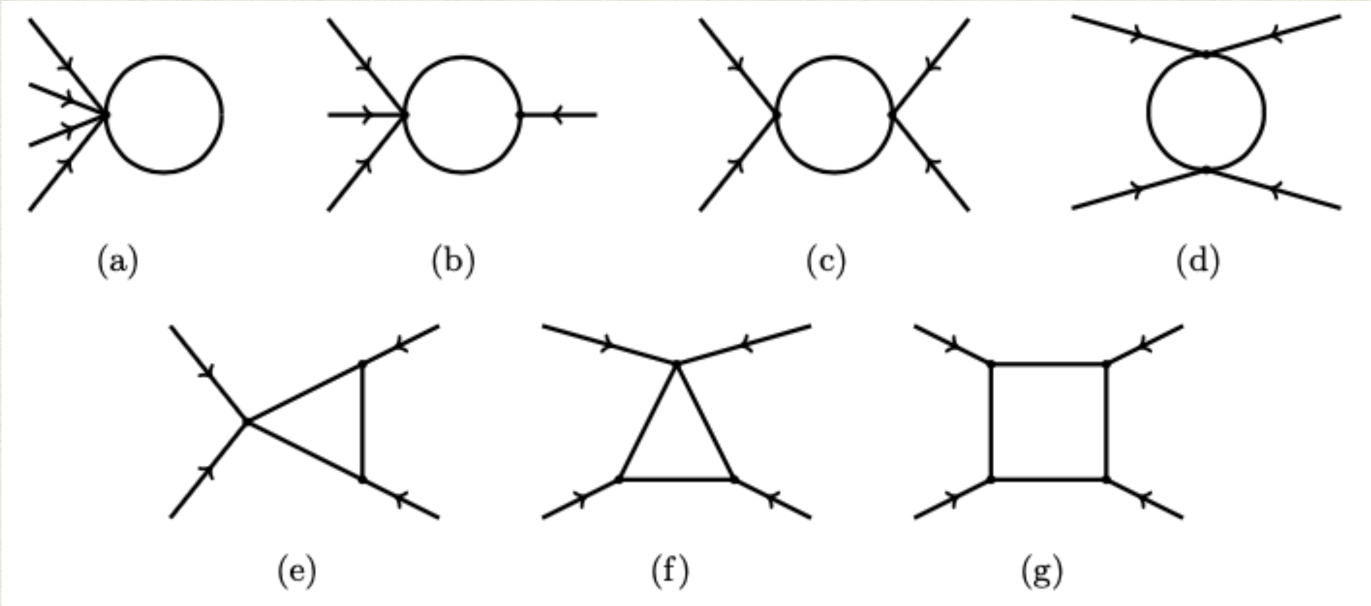
where $\det\left(p_i \cdot p_j\right)$ *can be identified with the Landau singularity of the second type, and* $D_\pm$ *are calculated by imposing the on-shell conditions* $D_1 = D_2 = \ldots = D_n = 0$.

Differential equations of one-loop Feynman integrals



$s = (p_1 + p_2)^2\,,\ t = (p_2 + p_3)^2\,,\ p_i^2 = m^2\,.$

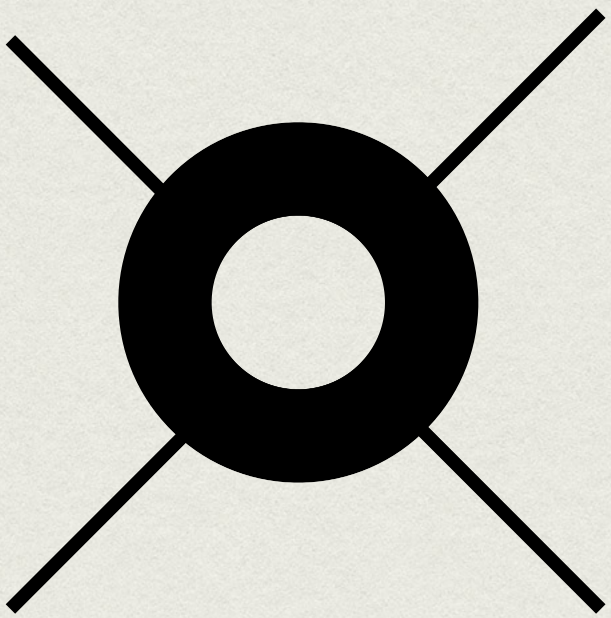
Master integrals (Laporta basis)



- $\partial_x \overrightarrow{J} = M(x,\epsilon)\overrightarrow{J}$ —> $\partial_x \overrightarrow{g} = \epsilon A_x \overrightarrow{g}$

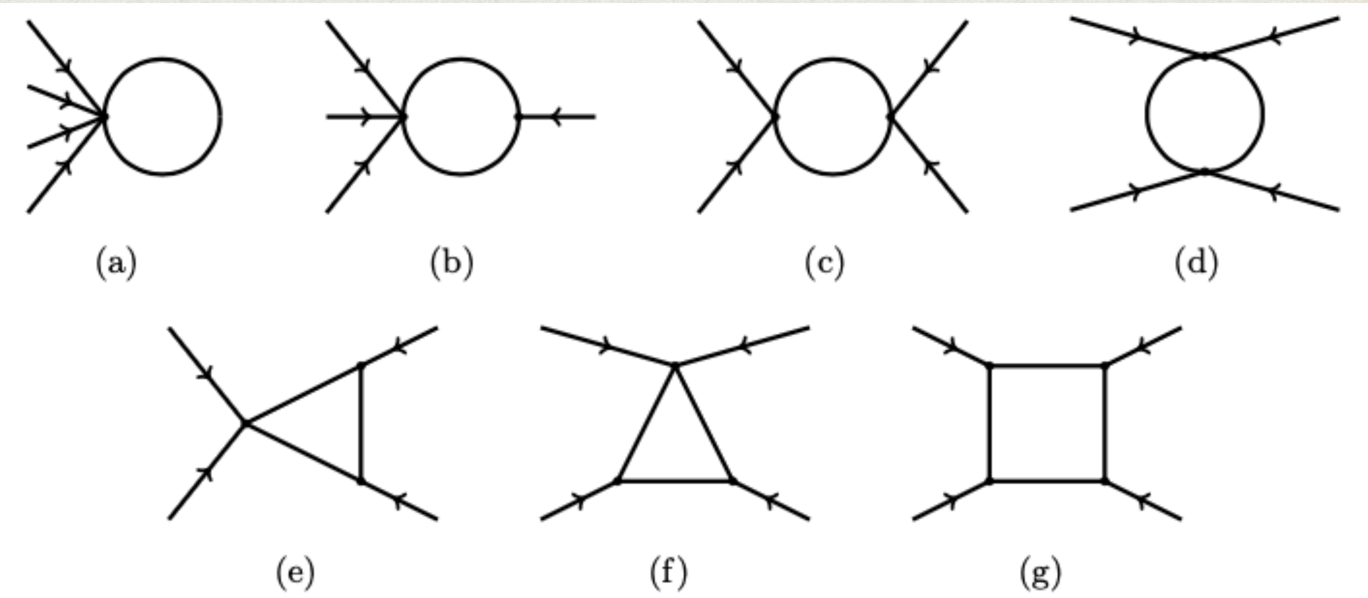
Transform differential equation into *canonical form*

Differential equations of one-loop Feynman integrals



$s = (p_1 + p_2)^2\,,\; t = (p_2 + p_3)^2\,,\; p_i^2 = m^2\,.$

Master integrals (Laporta basis)

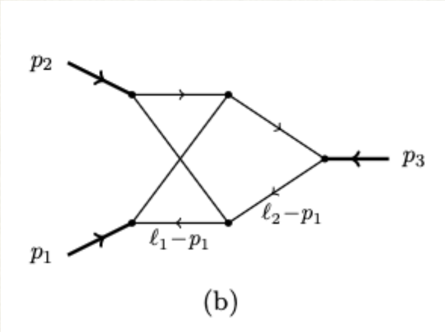


- Canonical differential equations ($\partial_x \vec{g} = \epsilon A_x \vec{g}$) in even & odd dimensions

$$
\begin{aligned}
g_1 &= \epsilon\, J_1^{(1),D=2-2\epsilon}(1)\,,\\
g_2 &= \epsilon\, m^2\, J_2^{(1),D=2-2\epsilon}(1,4)\,,\\
g_3 &= \epsilon\, \sqrt{-s\,(4m^2-s)}\, J_2^{(1),D=2-2\epsilon}(1,3)\,,\\
g_4 &= \epsilon\, \sqrt{-t\,(4m^2-t)}\, J_2^{(1),D=2-2\epsilon}(2,4)\,,\\
g_5 &= \epsilon^2\, \sqrt{-s\,(4m^2-s)}\, J_3^{(1),D=4-2\epsilon}(1,2,4)\,,\\
g_6 &= \epsilon^2\, \sqrt{-t\,(4m^2-t)}\, J_3^{(1),D=4-2\epsilon}(1,2,3)\,,\\
g_7 &= \epsilon^2\, \sqrt{st\,(12m^4-4m^2\,(s+t)+st)}\, J_4^{(1),D=4-2\epsilon}(1,2,3,4)\,.
\end{aligned}
$$

$$
\begin{aligned}
g_1 &= \epsilon\, \sqrt{m^2}\, J_1^{(1),D=1-2\epsilon}(1)\,,\\
g_2 &= \epsilon^2\, \sqrt{m^2}\, J_2^{(1),D=3-2\epsilon}(1,4)\,,\\
g_3 &= \epsilon^2\, \sqrt{-s}\, J_2^{(1),D=3-2\epsilon}(1,3)\,,\\
g_4 &= \epsilon^2\, \sqrt{-t}\, J_2^{(1),D=3-2\epsilon}(2,4)\,,\\
g_5 &= \epsilon^2\, \sqrt{-sm^2\,(3m^2-s)}\, J_3^{(1),D=3-2\epsilon}(1,2,4)\,,\\
g_6 &= \epsilon^2\, \sqrt{-tm^2\,(3m^2-t)}\, J_3^{(1),D=3-2\epsilon}(1,2,3)\,,\\
g_7 &= \epsilon^3\, \sqrt{-st\,(4m^2-s-t)}\, J_4^{(1),D=5-2\epsilon}(1,2,3,4)\,.
\end{aligned}
$$

Multi-loop leading singularities :: Leray's residues



$$= \int \frac{d^4\ell_1}{\imath\pi^2}\frac{d^4\ell_2}{\imath\pi^2}\frac{1}{\ell_1^2\left(\ell_1-p_1\right)^2\left(\ell_2-p_1\right)^2\left(\ell_2+p_2\right)^2\left(\ell_1-\ell_2\right)^2\left(\ell_1-\ell_2-p_2\right)^2}\,.$$

Parametrisation of loop momenta $\ell_i^\alpha = a_{i1}\,p_1^\alpha + a_{i2}\,p_2^\alpha + \lambda_i^\alpha\,,$

Compute residue :: Leray's residue

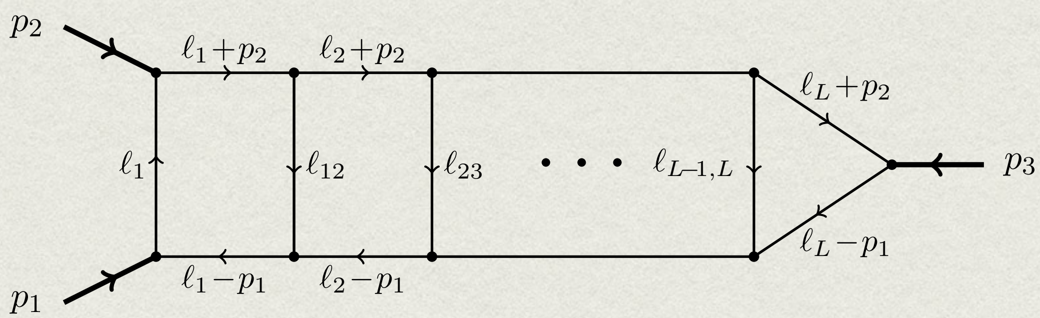
$$J^{(2),D=4}_{\mathrm{NP;3\text{-}pt}} \sim \pm\frac{1}{16\lambda_{\mathrm{K}}\left(p_1^2,p_2^2,p_3^2\right)}\frac{\sqrt{\lambda_{11}\lambda_{22}-\lambda_{12}^2}}{\lambda_{11}\left(a_{21}-a_{22}-1\right)-\lambda_{12}\left(a_{11}-a_{12}-a_{22}-1\right)-a_{12}\lambda_{22}}$$

Solve for integration variables $\sqrt{\lambda_{11}\lambda_{22}-\lambda_{12}^2} = \ell_1^2 = \left(\ell_1-p_1\right)^2 = \cdots = \left(\ell_1-\ell_2-p_2\right)^2 = 0$

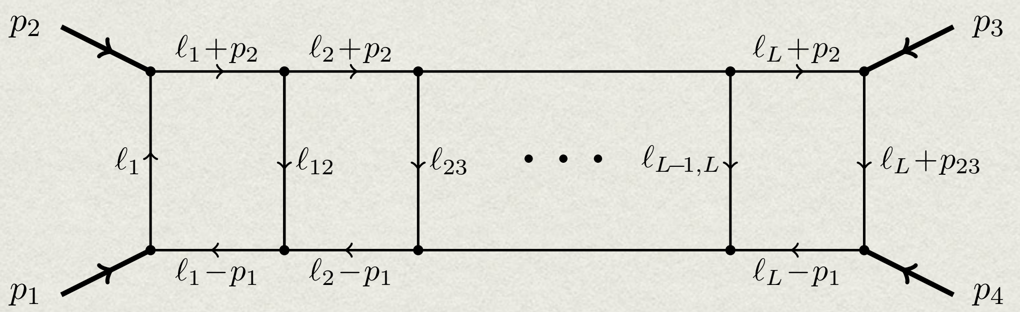
$$J^{(2),D=4}_{\mathrm{NP;3\text{-}pt}} \sim \pm\frac{1}{16\lambda_{\mathrm{K}}\left(p_1^2,p_2^2,p_3^2\right)}$$

Multi-loop leading singularities :: loop-by-loop

- integrate out loop-by-loop in D=4



$$\longrightarrow \frac{1}{(p_3^2)^L \sqrt{\lambda_K\left(1,\frac{p_1^2}{p_3^2},\frac{p_2^2}{p_3^2}\right)}}$$



$$\longrightarrow \frac{1}{s^L t \sqrt{\lambda_K\left(1,\frac{p_1^2 p_3^2}{st},\frac{p_2^2 p_4^2}{st}\right)}}$$

- Integrals in top sector needed to generate canonical differential equations

INITIAL algorithm <<Dlapa

Analytic evaluations

- We have reached:

- ☑ Connection between leading & Landau singularities
- ☑ Extensive use of the multivariate theory of residues :: Leray
- ☑ Useful input for the generation of differential equations in canonical form

- Open questions & future directions

- ☐ classification of Landau varieties for 2 —> 2 processes (curves) and beyond
- ☐ relation between Landau and leading singularities for multi-loop integrals
- ☐ connection between alphabet and Landau singularities

Analytic evaluations

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☑ Connection between leading & Landau singularities

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Thanks