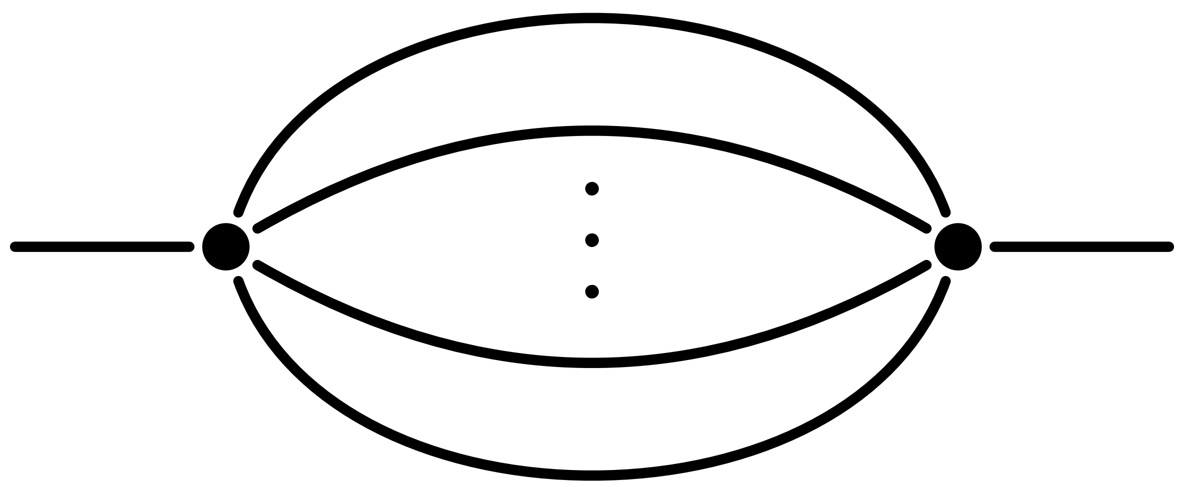
ε-Factorization for Calabi–Yau Integrals

Banana Integrals at three, four, five, six loops, and beyond…

Sebastian Pögel, University of Mainz
QCD meets Gravity 2022, Zürich
15th December 2022

Work in collaboration with Xing Wang and Stefan Weinzierl



`2207.12893` (JHEP 09 (2022) 062)
`2211.04292`
`2212.xxxxx` (to appear next week)

Feynman Integrals

The Ubiquitous

Feynman Integrals

The Ubiquitous

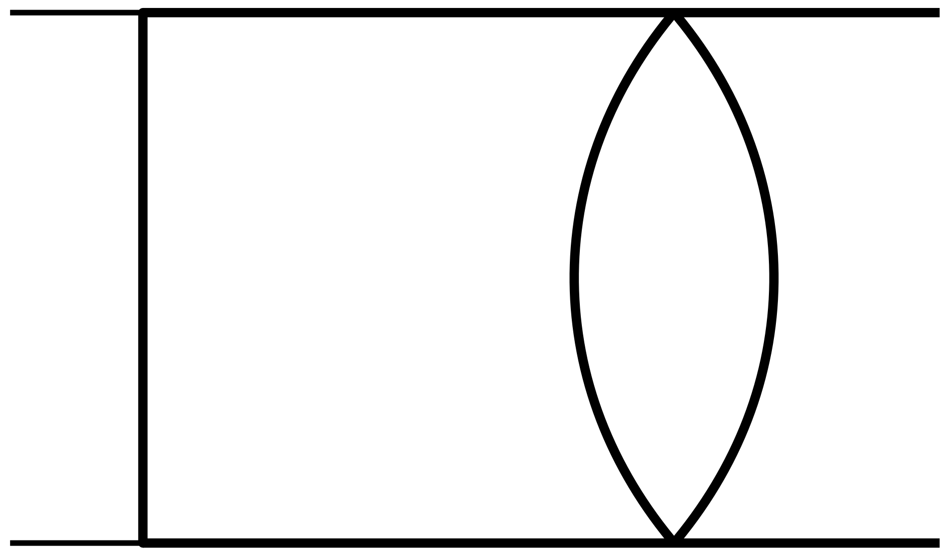
An essential ingredient in both

Feynman Integrals

The Ubiquitous

An essential ingredient in both

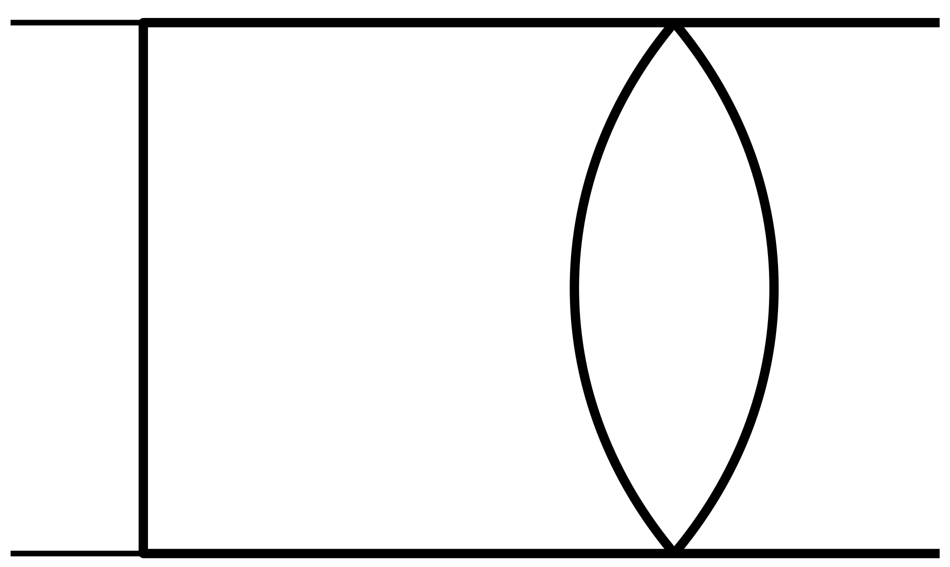
QCD



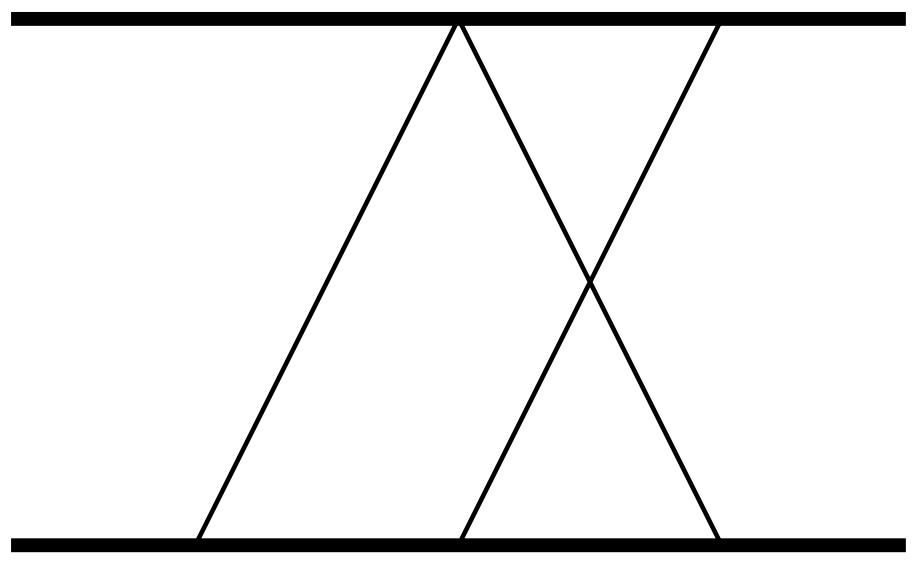
Feynman Integrals
The Ubiquitous

An essential ingredient in both

QCD



Gravity

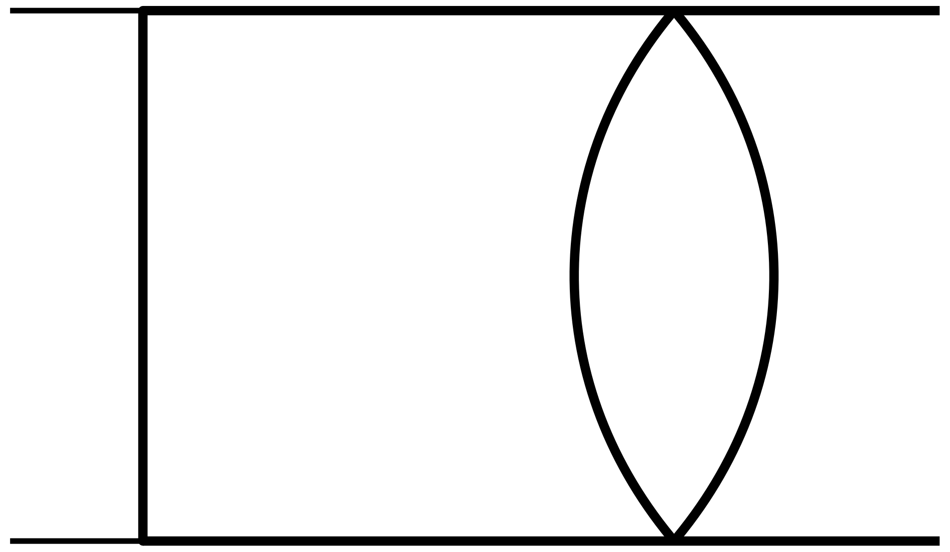


Feynman Integrals

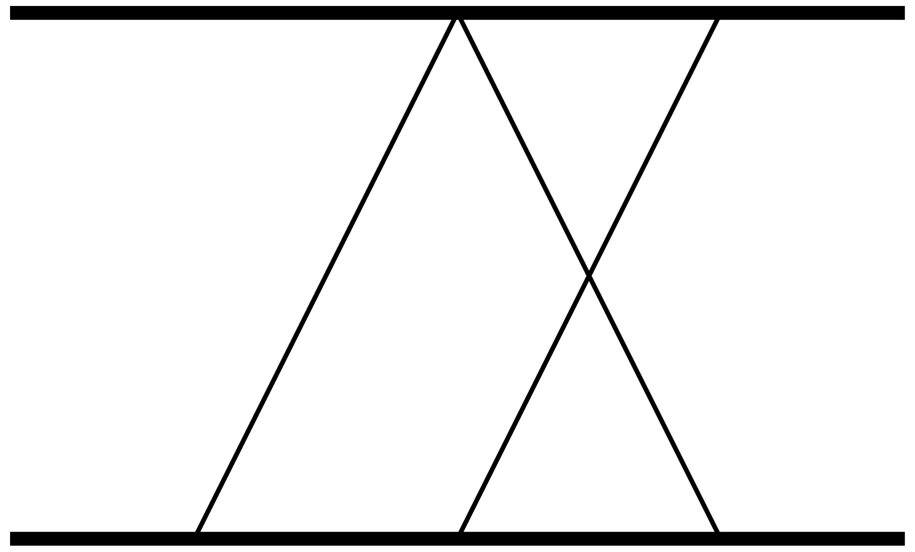
The Ubiquitous

An essential ingredient in both

QCD



Gravity

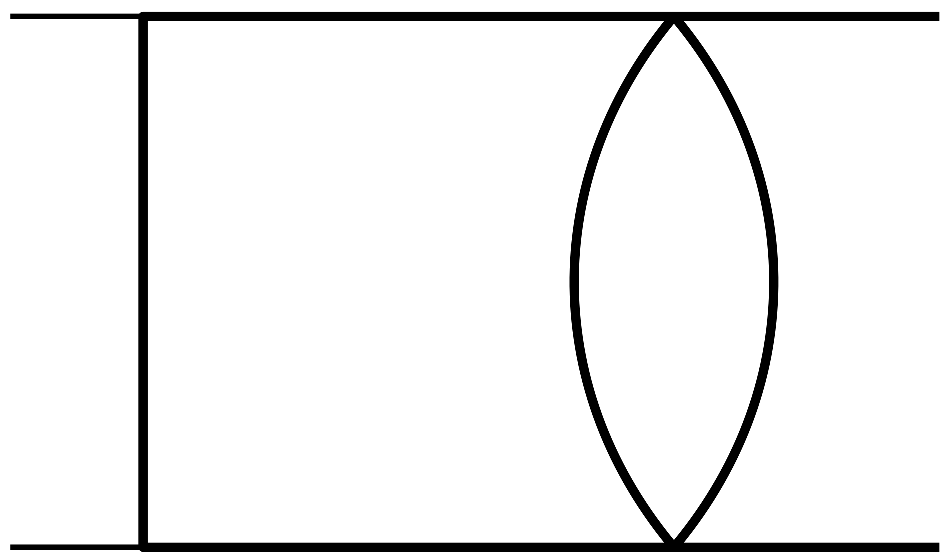


See Christoph's talk

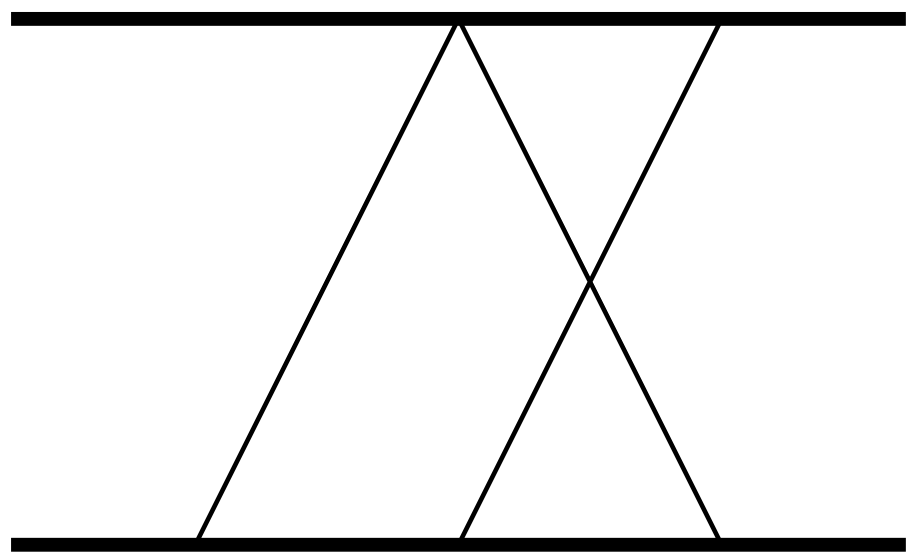
Feynman Integrals
The Ubiquitous

An essential ingredient in both

QCD



Gravity



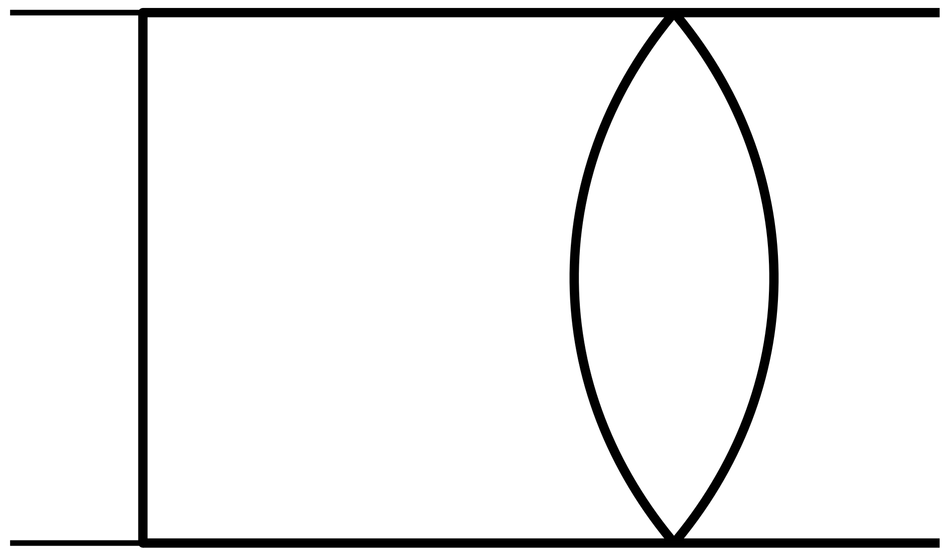
See Christoph's talk

Integrals associated to geometries
Determines function space

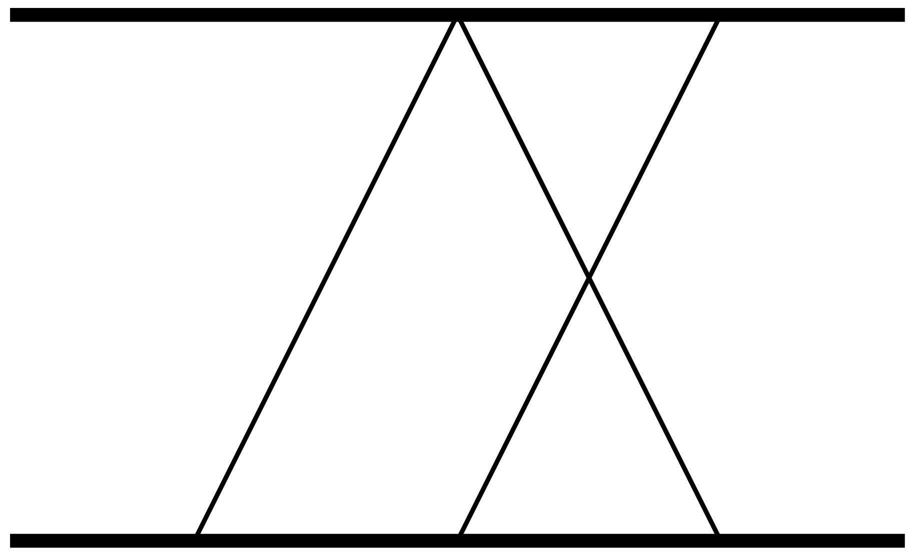
Feynman Integrals
The Ubiquitous

An essential ingredient in both

QCD



Gravity



See Christoph's talk

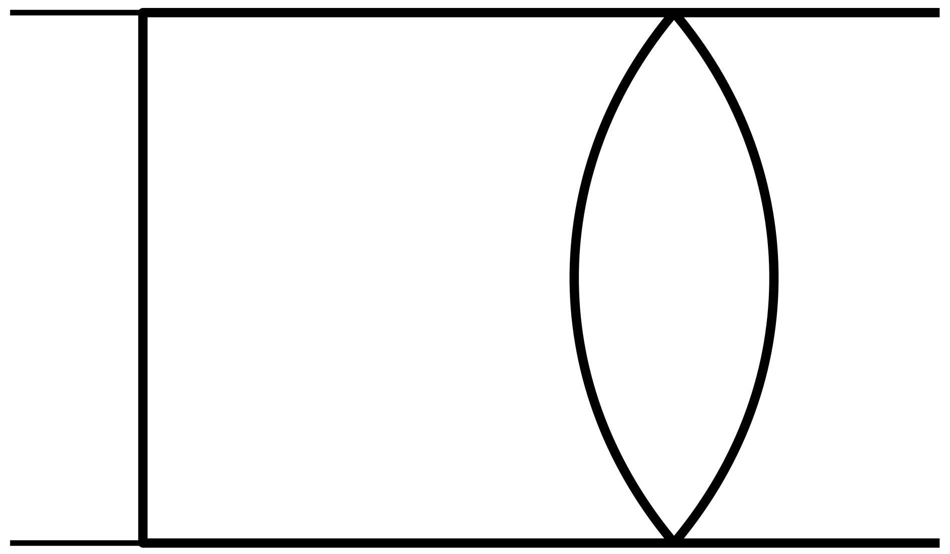
Integrals associated to geometries
Determines function space

Sphere
↓
MPLs

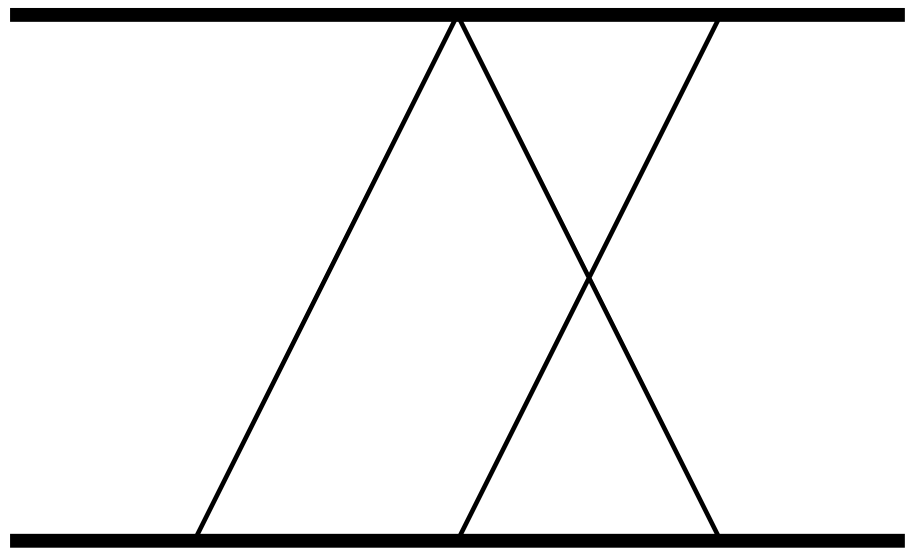
Feynman Integrals
The Ubiquitous

An essential ingredient in both

QCD



Gravity



See Christoph's talk

Integrals associated to geometries

Determines function space

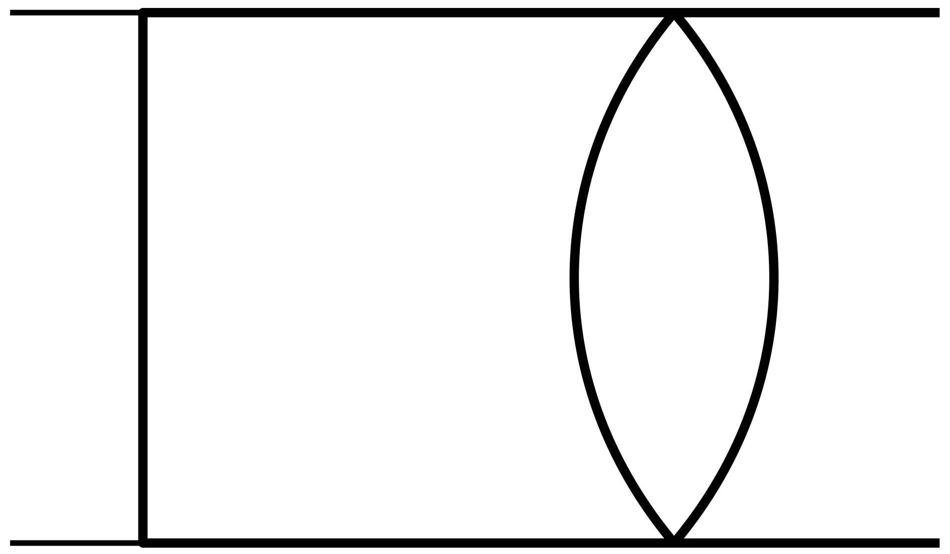
Sphere ↓ **MPLs**

Elliptic curve ↓ **Elliptic Integrals, modular forms, EMPLs**

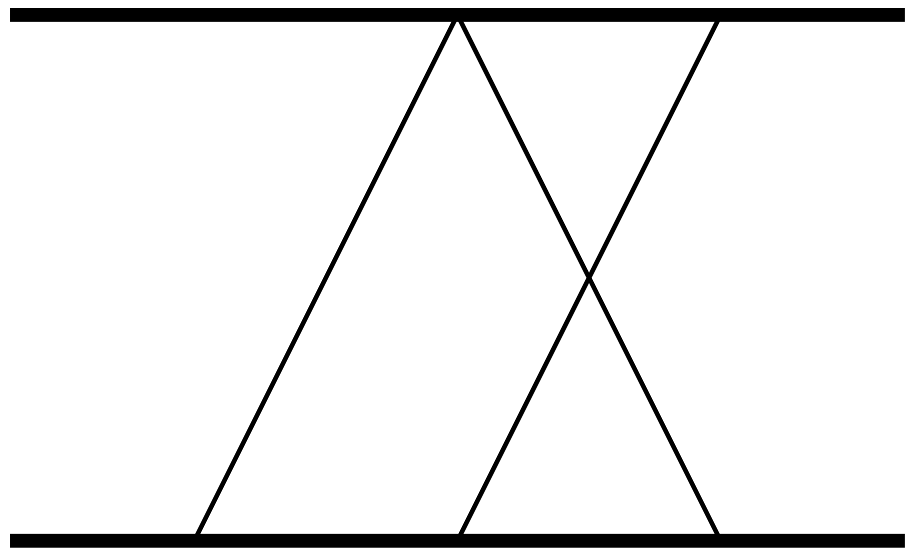
Feynman Integrals
The Ubiquitous

An essential ingredient in both

QCD



Gravity



See Christoph's talk

Integrals associated to geometries
Determines function space

Sphere → **MPLs**

Elliptic curve → **Elliptic Integrals, modular forms, EMPLs**

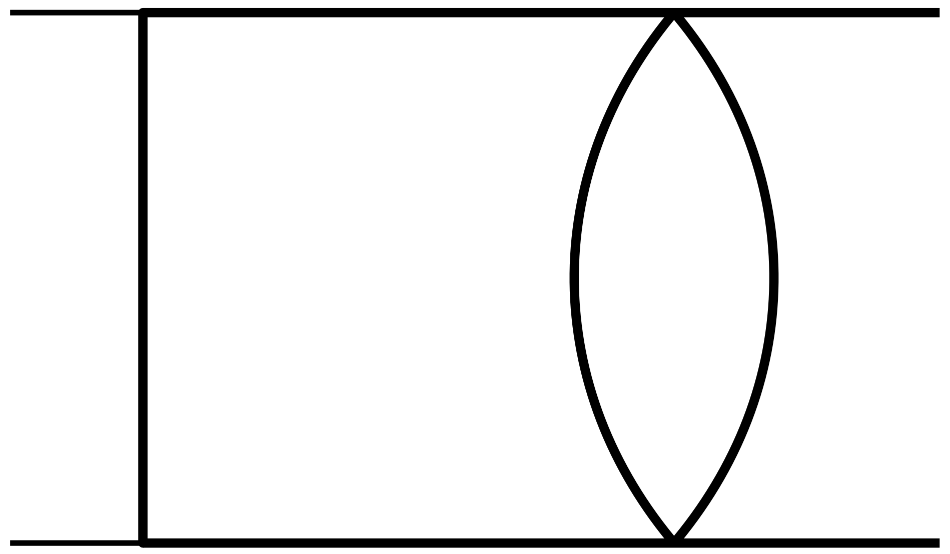
...

What comes beyond elliptics?

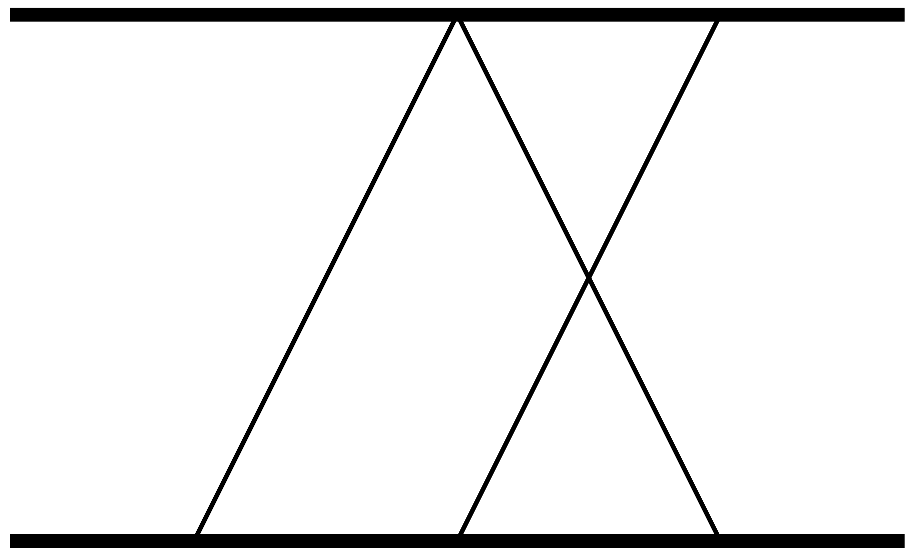
Feynman Integrals
The Ubiquitous

An essential ingredient in both

QCD



Gravity



See Christoph's talk

Integrals associated to geometries
Determines function space

Sphere → **MPLs**

Elliptic curve → **Elliptic Integrals, modular forms, EMPLs**

…

What comes beyond elliptics?

Calabi–Yau geometries
(at least one option)

Fantastic Calabi–Yaus

and where to find them

Fantastic Calabi–Yaus

and where to find them

How do we identify Calabi–Yau integrals?

Fantastic Calabi–Yaus

and where to find them

How do we identify Calabi–Yau integrals?

Graph Polynomial

$$I \sim \int \prod \mathrm{d}\alpha_i \, \alpha_i^{\nu_i - 1} \frac{U^{\nu-(\ell+1)D/2}}{F^{\nu-\ell D/2}}$$

U/F **define projective variety**

Fantastic Calabi–Yaus

and where to find them

How do we identify Calabi–Yau integrals?

Graph Polynomial

$$I \sim \int \prod \mathrm{d}\alpha_i \, \alpha_i^{\nu_i - 1} \frac{U^{\nu - (\ell+1)D/2}}{F^{\nu - \ell D/2}}$$

U/F **define projective variety**

Extracting geometry is hard!

Fantastic Calabi–Yaus

and where to find them

How do we identify Calabi–Yau integrals?

Graph Polynomial

$$I \sim \int \prod \mathrm{d}\alpha_i \, \alpha_i^{\nu_i - 1} \frac{U^{\nu-(\ell+1)D/2}}{F^{\nu-\ell D/2}}$$

U/F **define projective variety**

Extracting geometry is hard!

Fantastic Calabi–Yaus

and where to find them

How do we identify Calabi–Yau integrals?

Graph Polynomial

$$I \sim \int \prod \mathrm{d}\alpha_i \, \alpha_i^{\nu_i - 1} \frac{U^{\nu-(\ell+1)D/2}}{F^{\nu-\ell D/2}}$$

U/F **define projective variety**

Extracting geometry is hard!

Maximal Cuts

Fantastic Calabi–Yaus

and where to find them

How do we identify Calabi–Yau integrals?

Graph Polynomial

$$I \sim \int \prod \mathrm{d}\alpha_i \, \alpha_i^{\nu_i - 1} \frac{U^{\nu-(\ell+1)D/2}}{F^{\nu-\ell D/2}}$$

U/F **define projective variety**

Extracting geometry is hard!

Maximal Cuts

$$\mathrm{MaxCut}\, I \sim \int \frac{\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_n}{\sqrt{P(\alpha_1, \ldots, \alpha_n)}}$$

Hypersurface in weighted projective space

[Bourjaily, McLeod, Vergu, Volk, von Hippel, Wilhelm, '20]

$$[1 : \alpha_1 : \ldots : \alpha_n : y] \in \mathbb{WP}^{1,1,\ldots,1,(n+1)}$$

$$y^2 = P(\alpha_1, \ldots, \alpha_n) \quad \text{with} \quad \deg P = 2(n+1)$$

Codimension 1 = Dimension n

Calabi–Yaus: “A (bounded) bestiary”

[Bourjaily, McLeod, von Hippel, Wilhelm, ’19]

Calabi–Yaus: “A (bounded) bestiary”

[Bourjaily, McLeod, von Hippel, Wilhelm, ’19]

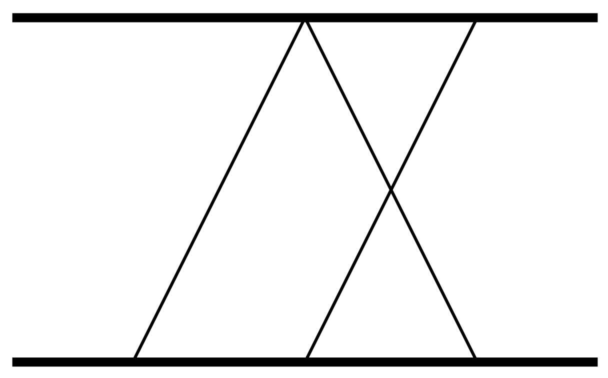
$n =$ dimension of manifold

Calabi–Yaus: “A (bounded) bestiary”

[Bourjaily, McLeod, von Hippel, Wilhelm, ’19]

n = dimension of manifold

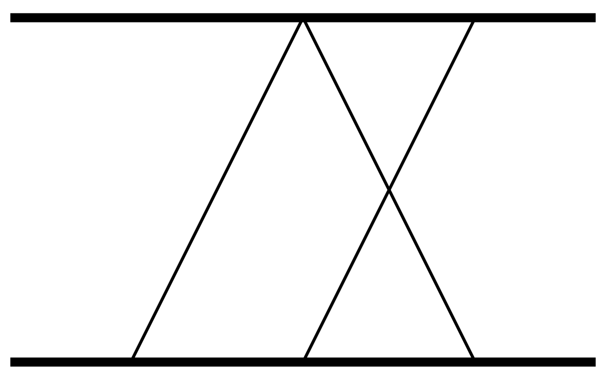
$n = 2$

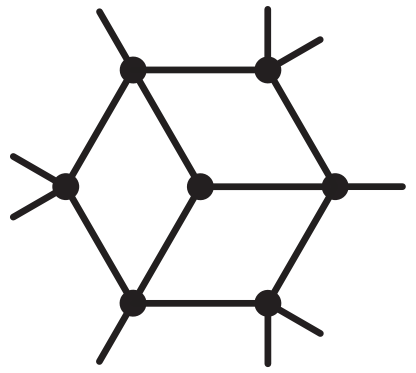


Calabi–Yaus: “A (bounded) bestiary”

[Bourjaily, McLeod, von Hippel, Wilhelm, ’19]

$n =$ dimension of manifold

$n = 2$ 

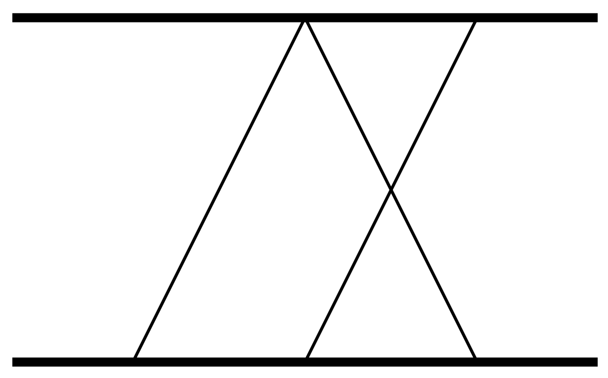
$n = 3$ 

Calabi–Yaus: "A (bounded) bestiary"

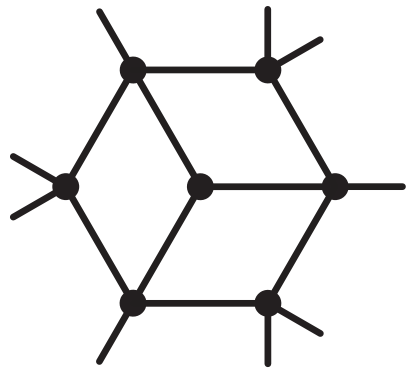
[Bourjaily, McLeod, von Hippel, Wilhelm, '19]

$n =$ dimension of manifold

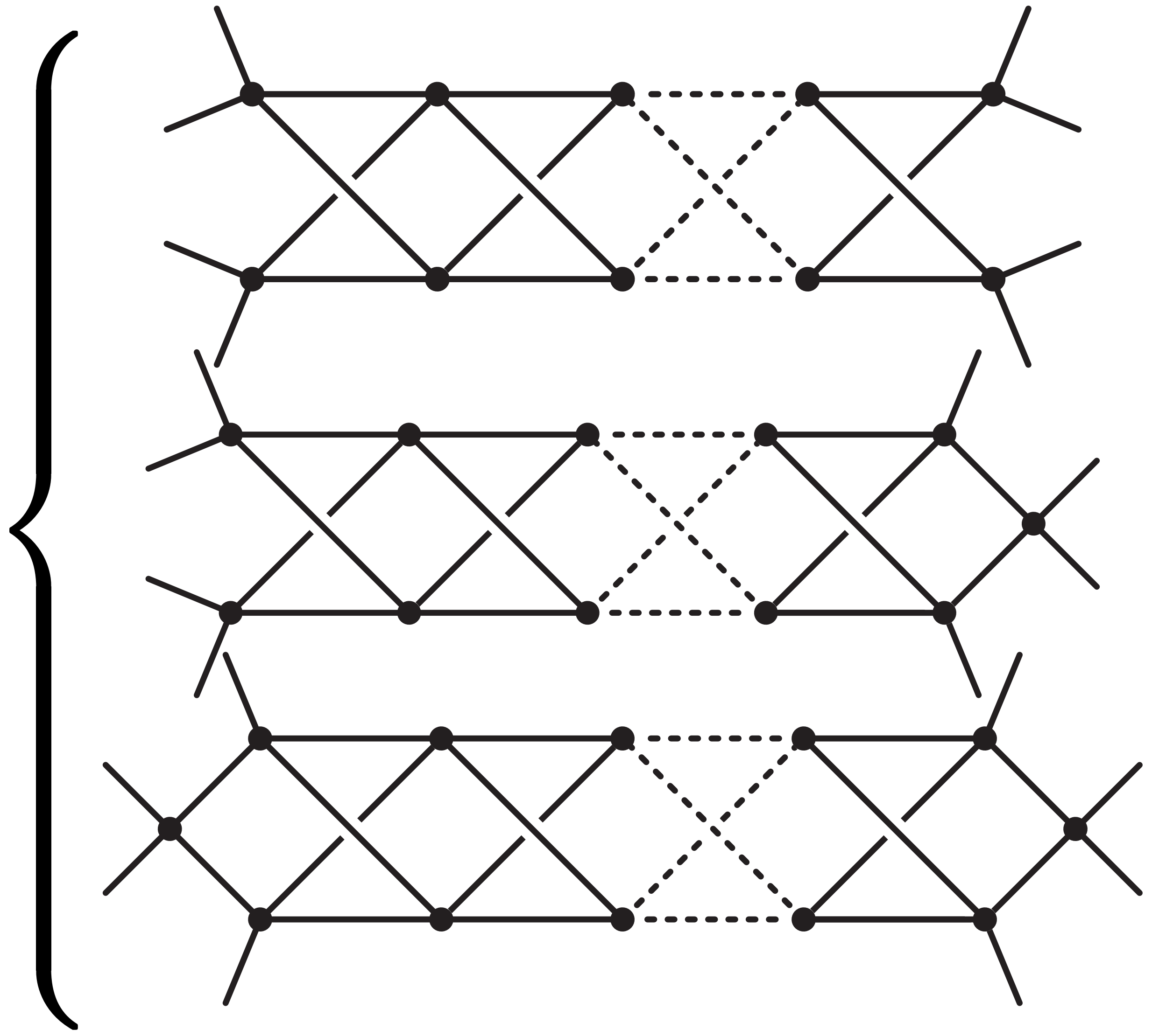
$n = 2$



$n = 3$



$n = 2(\ell - 1)$

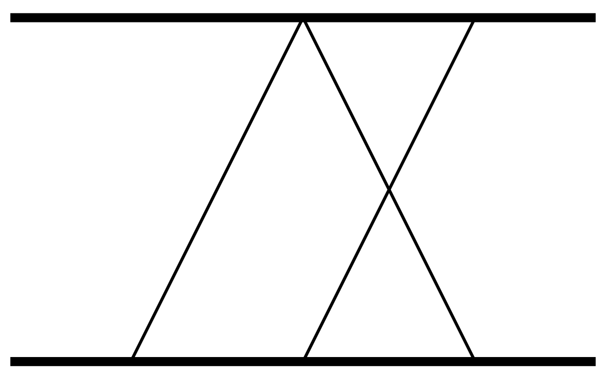


Calabi–Yaus: "A (bounded) bestiary"

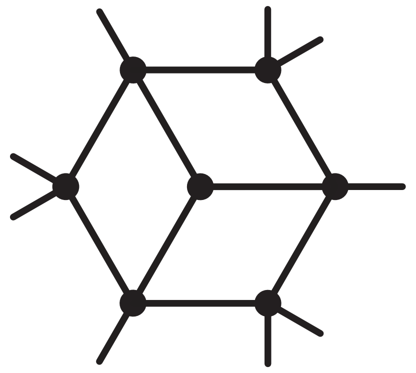
[Bourjaily, McLeod, von Hippel, Wilhelm, '19]

$n =$ dimension of manifold

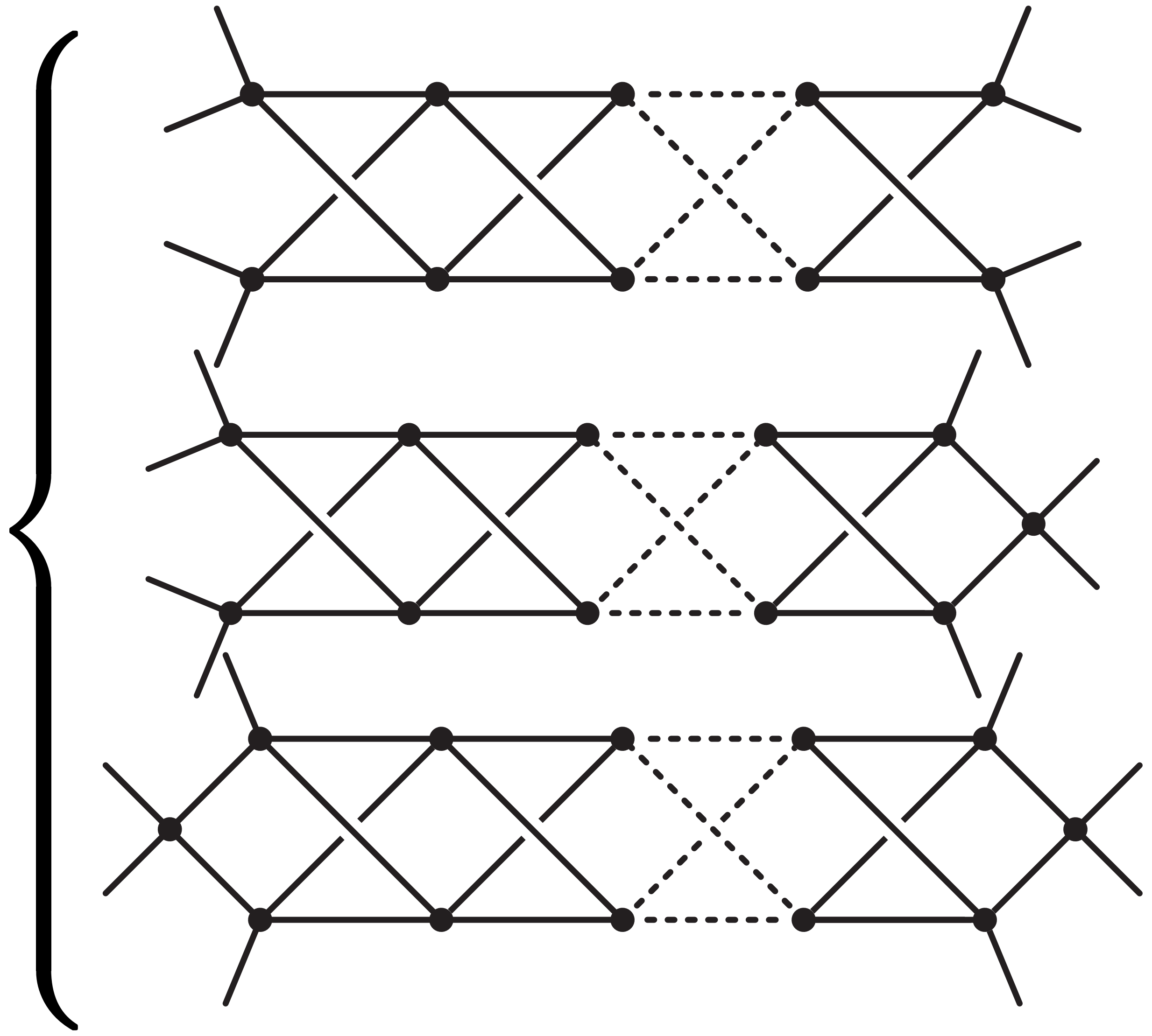
$n = 2$



$n = 3$



$n = 2(\ell - 1)$



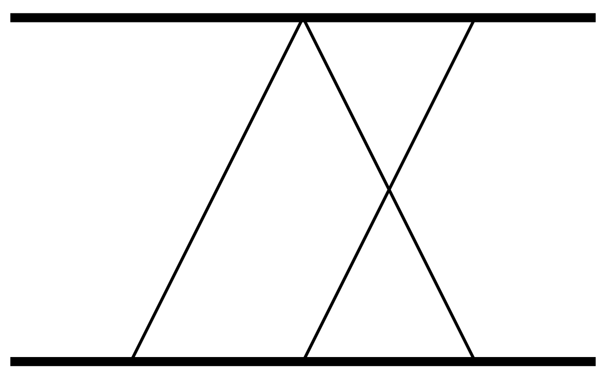
and more…

Calabi–Yaus: “A (bounded) bestiary”

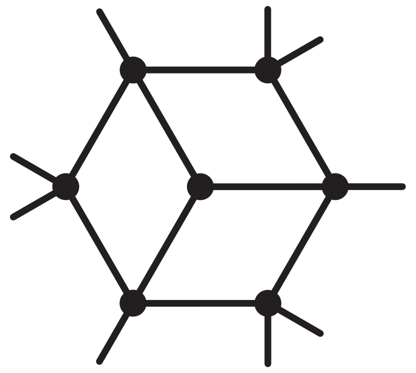
[Bourjaily, McLeod, von Hippel, Wilhelm, ’19]

$n =$ dimension of manifold

$n = 2$

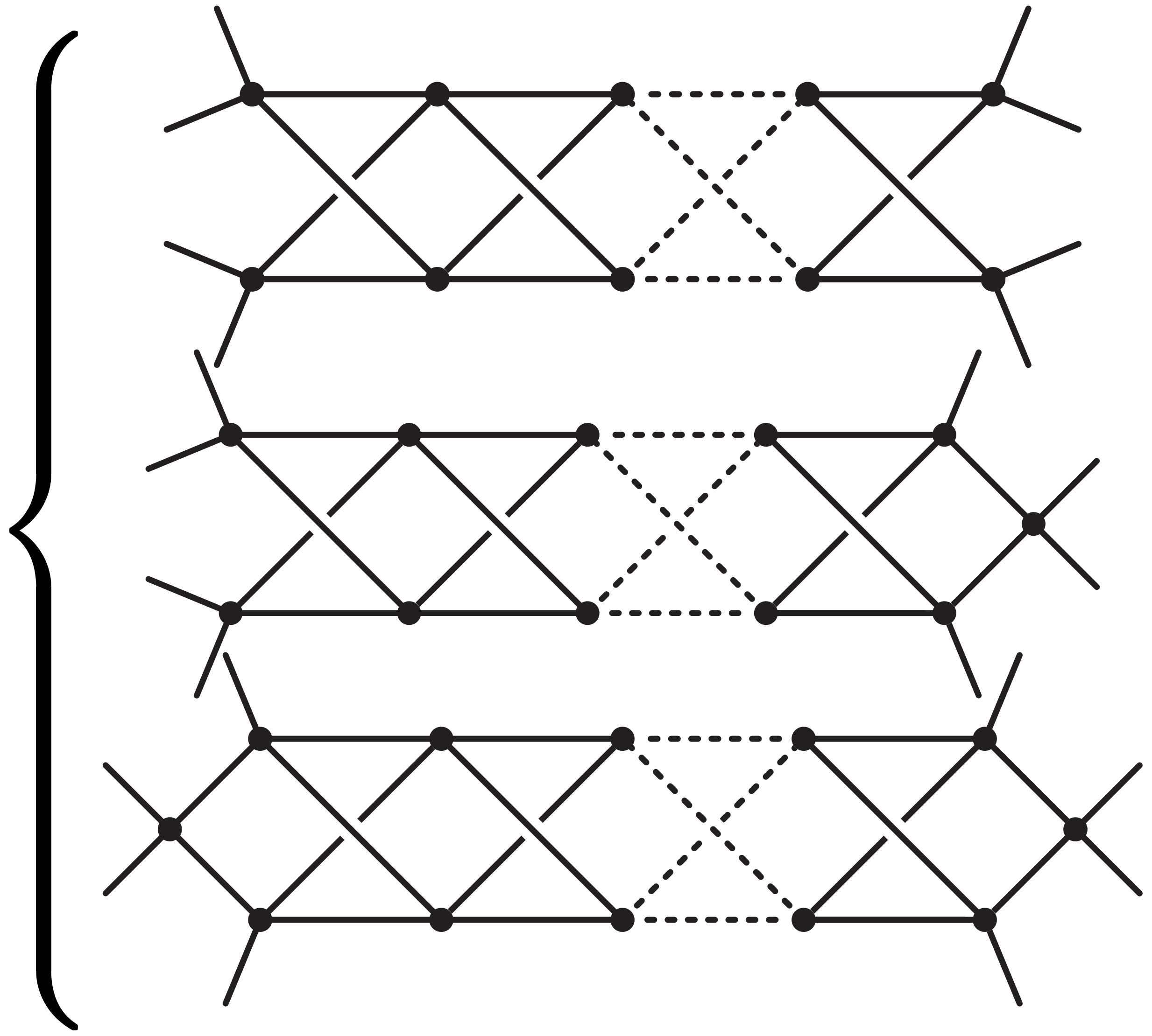


$n = 3$



Simplest Example:

$n = 2(\ell - 1)$



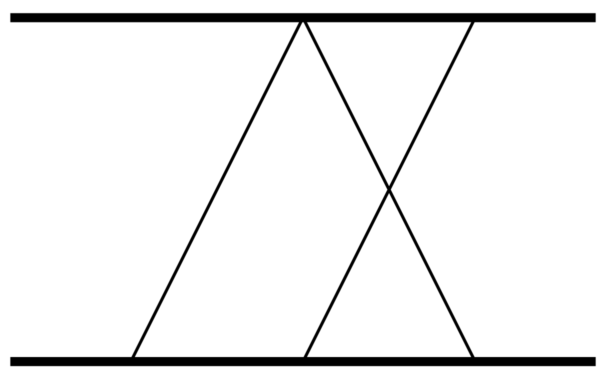
and more…

Calabi–Yaus: “A (bounded) bestiary”

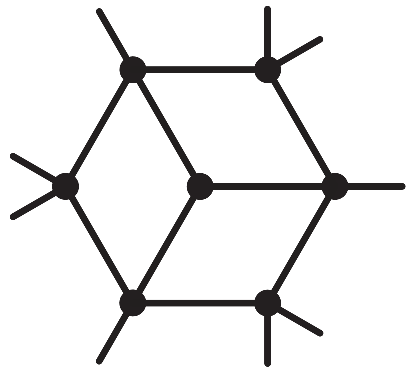
[Bourjaily, McLeod, von Hippel, Wilhelm, ’19]

$n =$ dimension of manifold

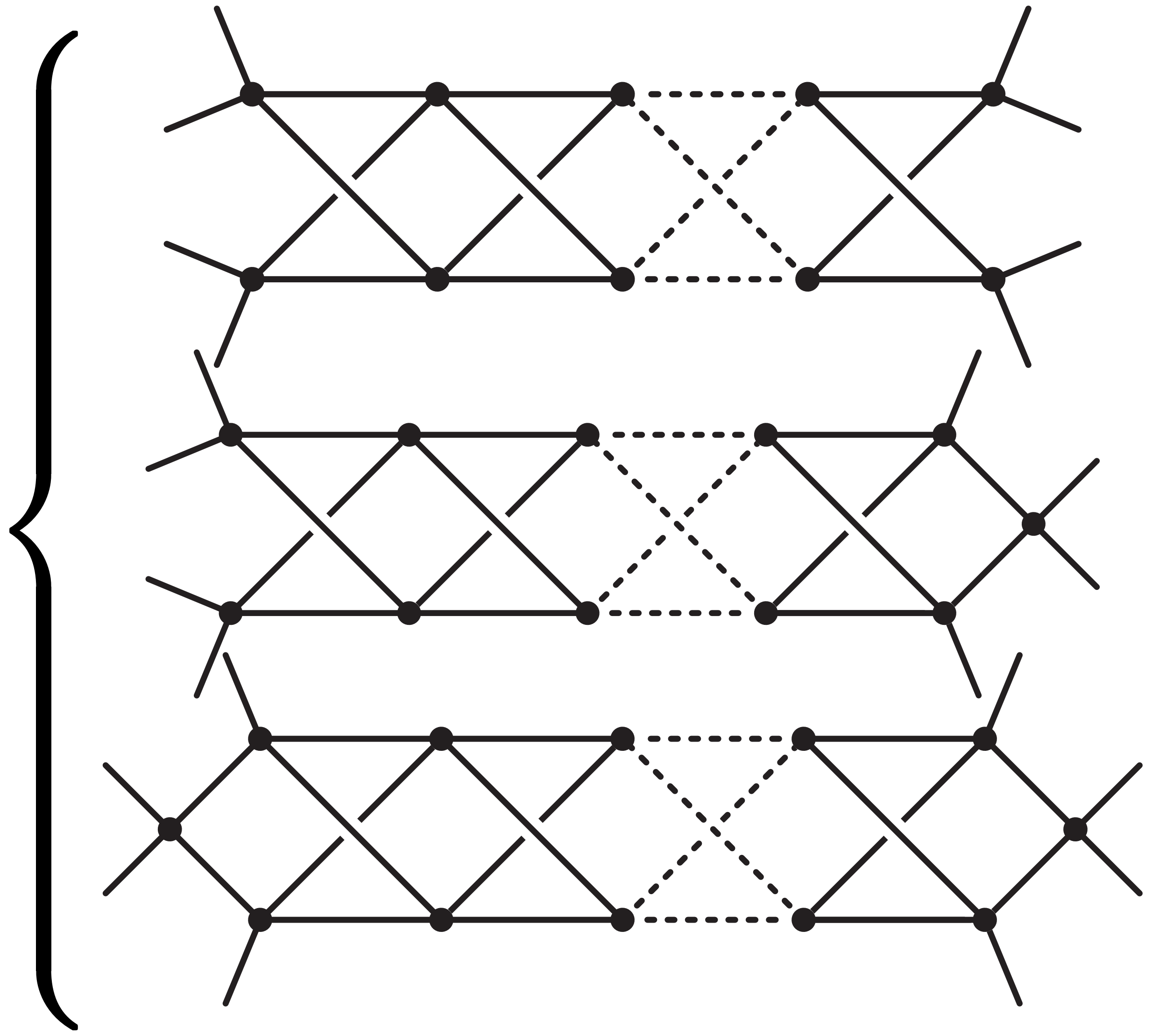
$n = 2$



$n = 3$

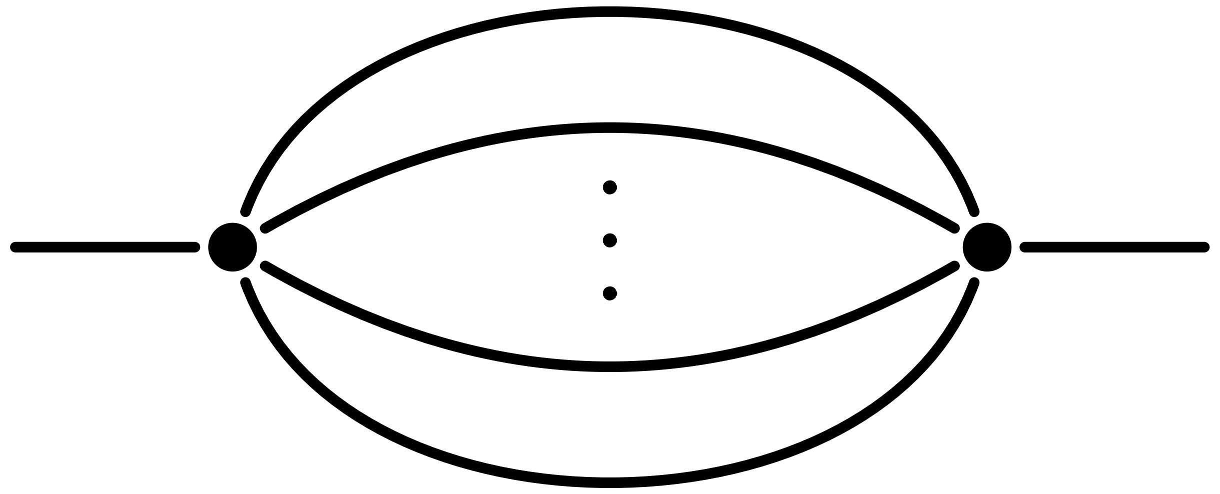


$n = 2(\ell - 1)$



and more…

Simplest Example: Banana Integrals

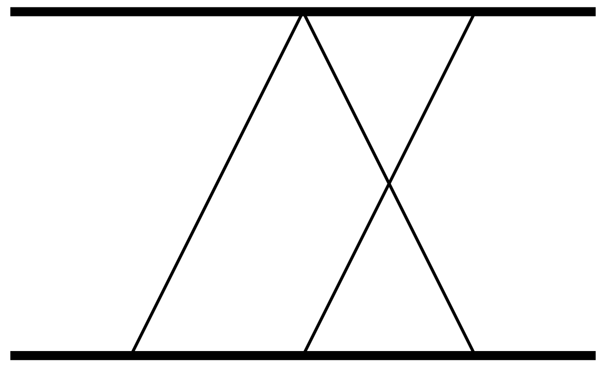


Calabi–Yaus: “A (bounded) bestiary”

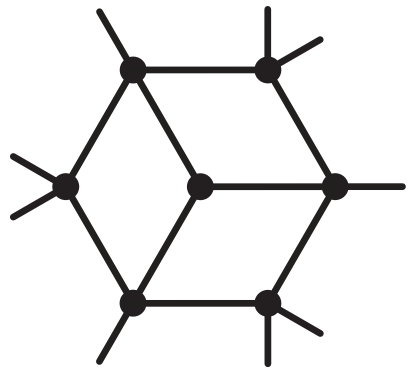
[Bourjaily, McLeod, von Hippel, Wilhelm, ’19]

n = dimension of manifold

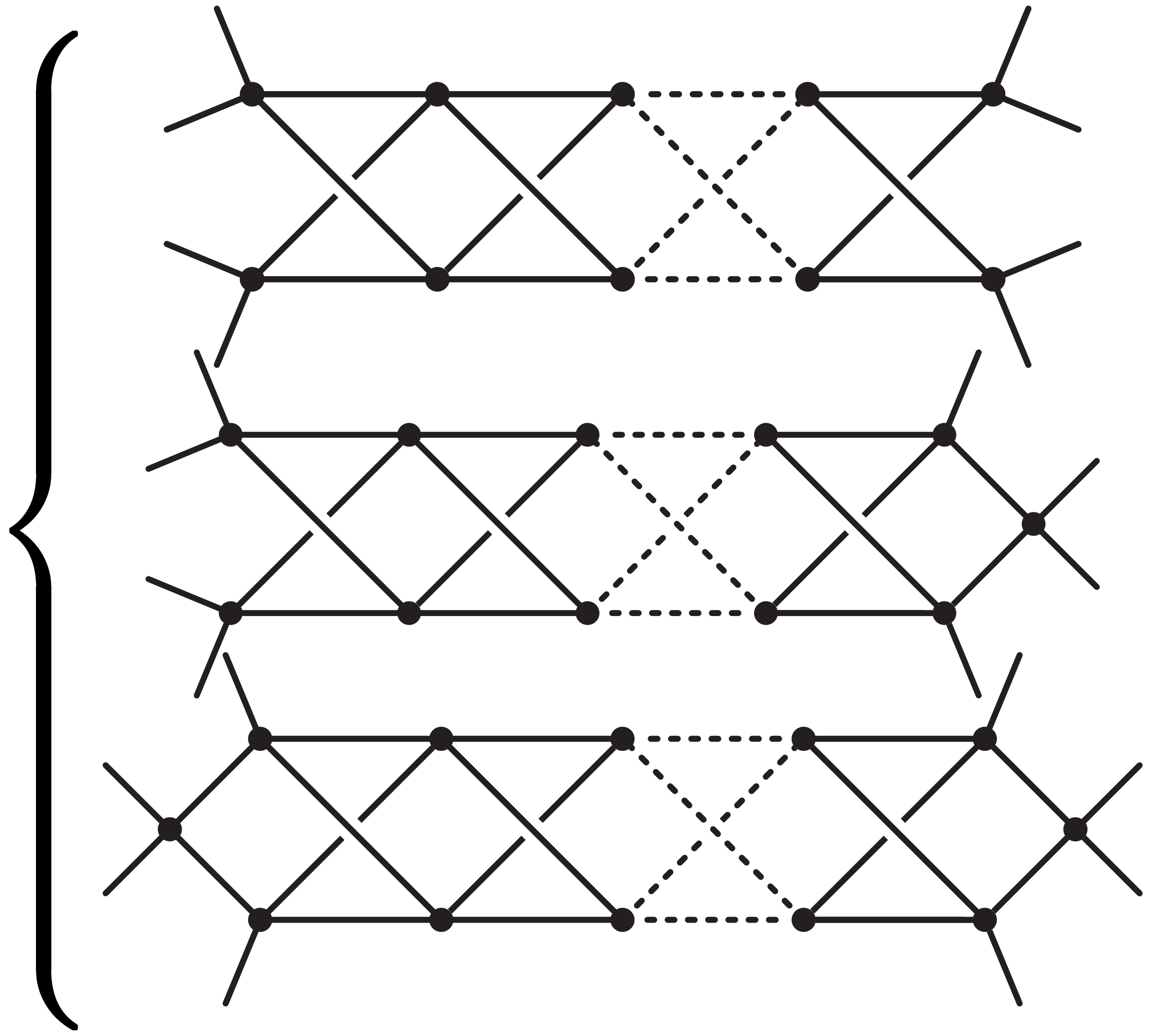
$n = 2$



$n = 3$

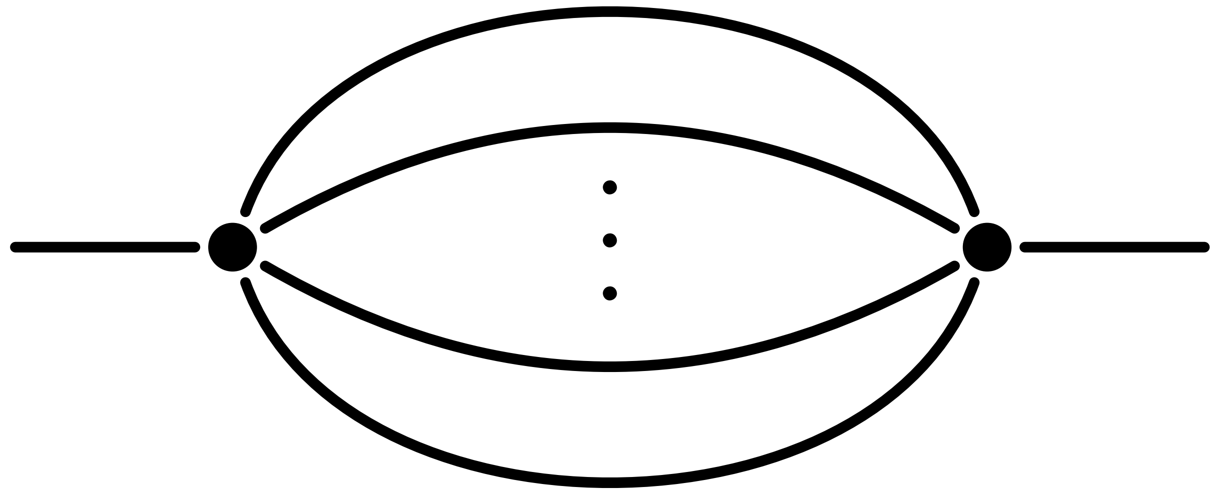


$n = 2(\ell - 1)$



and more…

Simplest Example: Banana Integrals



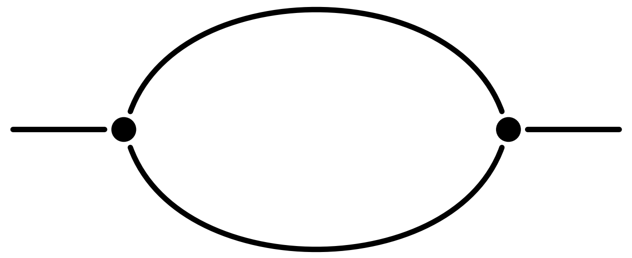
Bananas: A Calabi–Yau Prototype

Bananas: A Calabi–Yau Prototype

Calabi–Yau…

Bananas: A Calabi–Yau Prototype

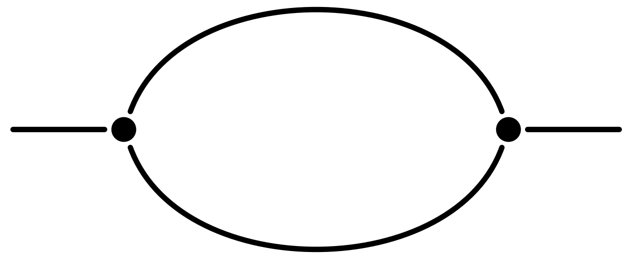
Calabi–Yau…



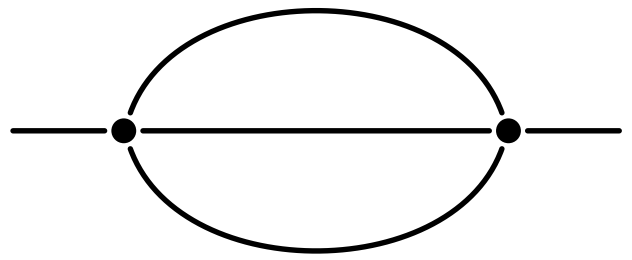
0-fold

Bananas: A Calabi–Yau Prototype

Calabi–Yau…



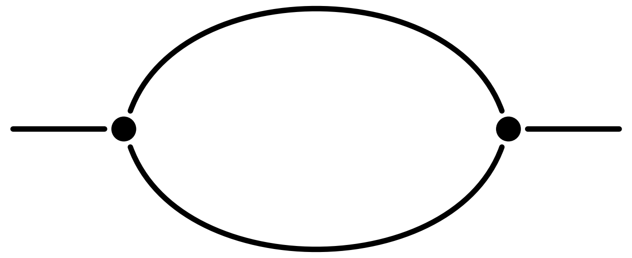
0-fold



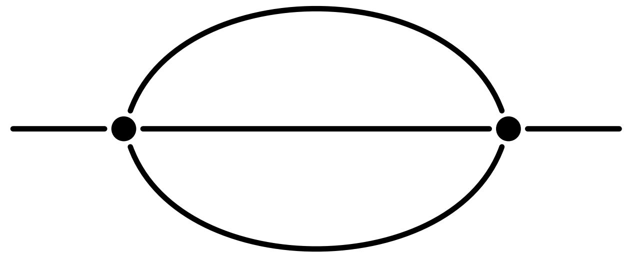
1-fold = elliptic curve

Bananas: A Calabi–Yau Prototype

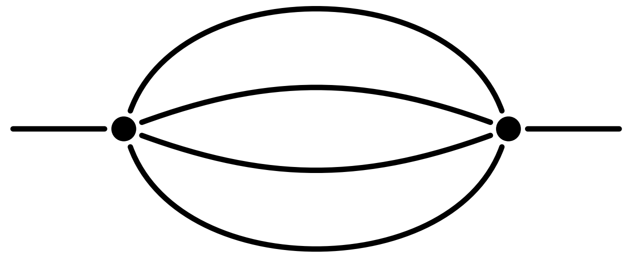
Calabi–Yau…



0-fold



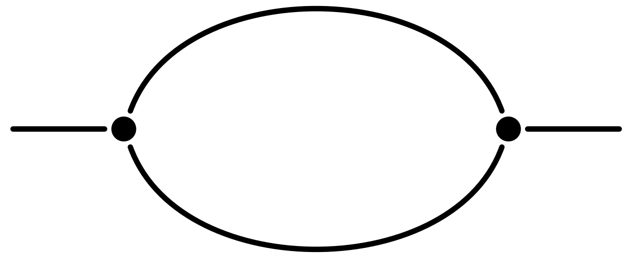
1-fold = elliptic curve



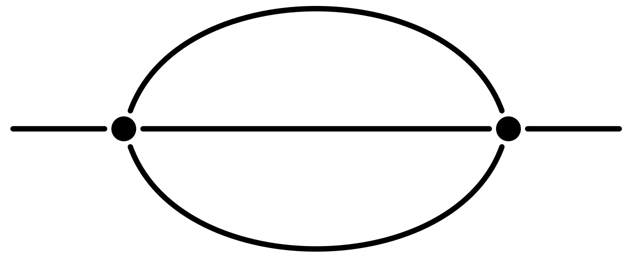
2-fold = K3 surface

Bananas: A Calabi–Yau Prototype

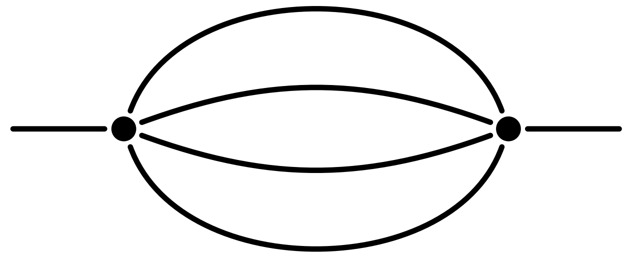
Calabi–Yau…



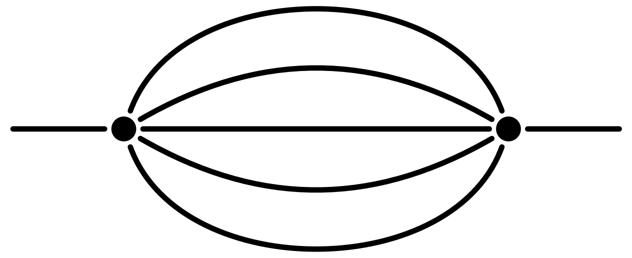
0-fold



1-fold = elliptic curve



2-fold = K3 surface

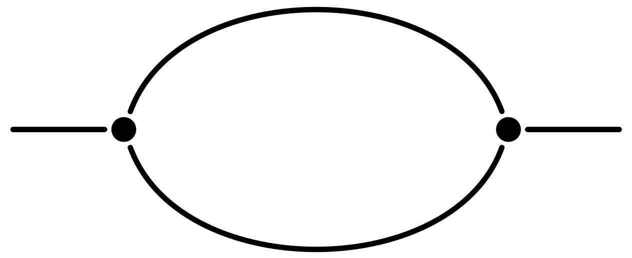


3-fold

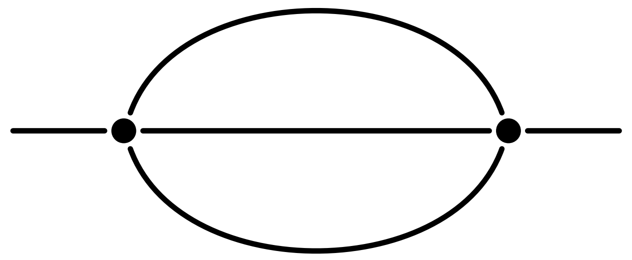
⋮

Bananas: A Calabi–Yau Prototype

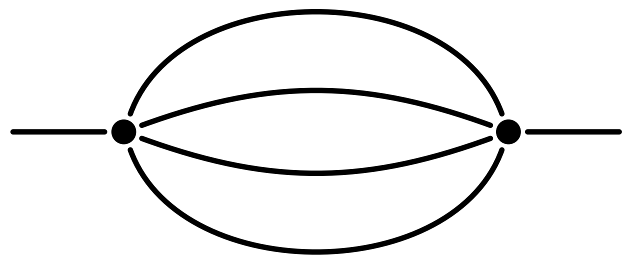
Calabi–Yau…



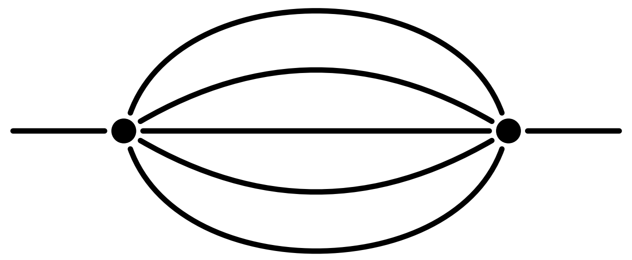
0-fold



1-fold = elliptic curve

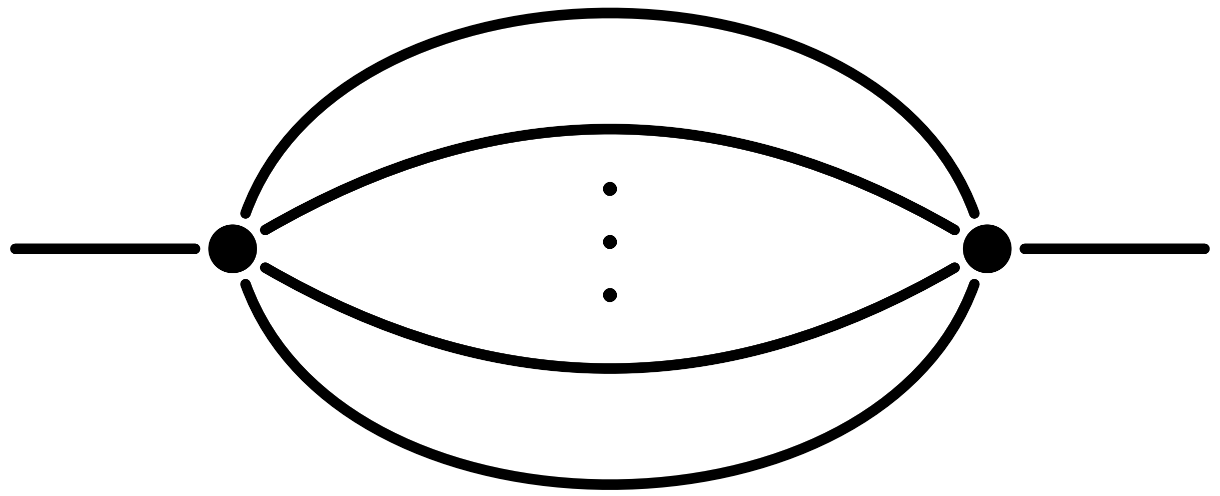


2-fold = K3 surface



3-fold

⋮



ℓ-loop Banana integral

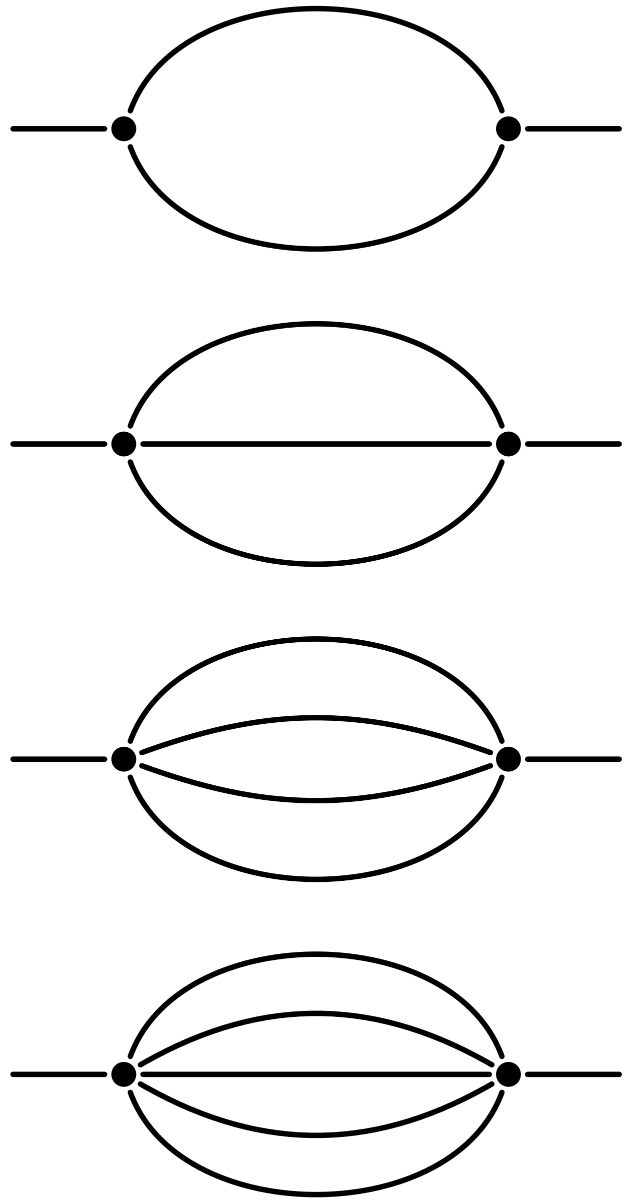
$$\hat{=}$$

$(\ell-1)$-fold Calabi–Yau manifold

ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]

Bananas: A Calabi–Yau Prototype

Calabi–Yau…



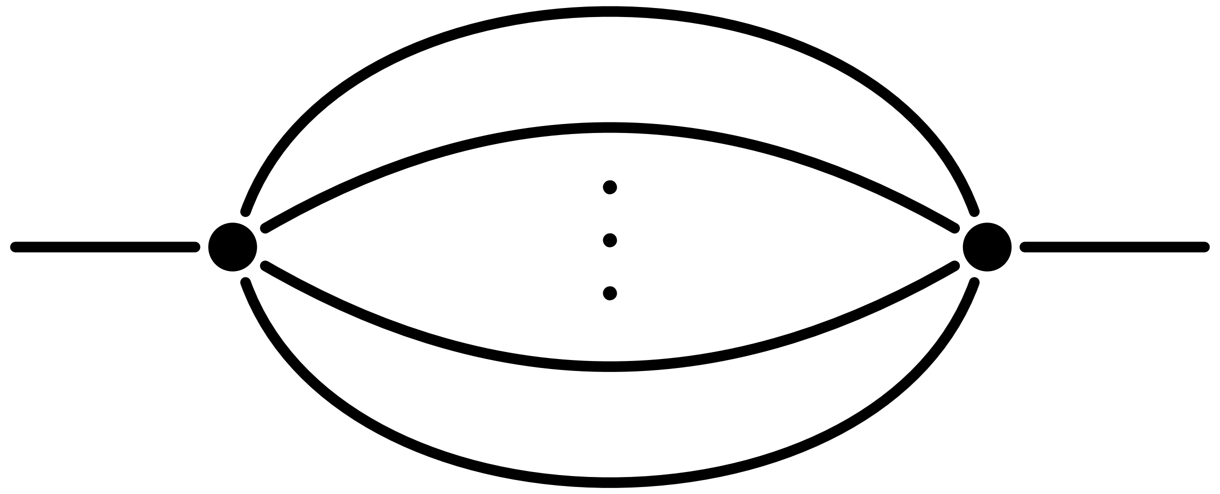
0-fold

1-fold = elliptic curve

2-fold = K3 surface

3-fold

⋮



ℓ-loop Banana integral

$\hat{=}$

$(\ell-1)$-fold Calabi–Yau manifold

ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]

Simplification: Equal-mass → single scale

Kinematic variable

$$x = \frac{p^2}{m^2} \qquad y = -\frac{m^2}{p^2}$$

Obtain $dI = \varepsilon A I$ [1304.1806]

Obtain $dI = \varepsilon A I$ [1304.1806]

Find basis and variables, such that

Obtain $dI = \varepsilon A I$ [1304.1806]

Find basis and variables, such that

- A independent of ε

Obtain $dI = \varepsilon A I$ [1304.1806]

Find basis and variables, such that

- A independent of ε
- A consisting of functions we "understand well"

Obtain $dI = \varepsilon A I$ [1304.1806]

Find basis and variables, such that

- A independent of ε
- A consisting of functions we "understand well"

Analytic understanding
and/or
fast numerical evaluation

Obtain $dI = \varepsilon A I$ [1304.1806]

Find basis and variables, such that

- A independent of ε
- A consisting of functions we "understand well"

Analytic understanding
and/or
fast numerical evaluation

Given boundary value I_0

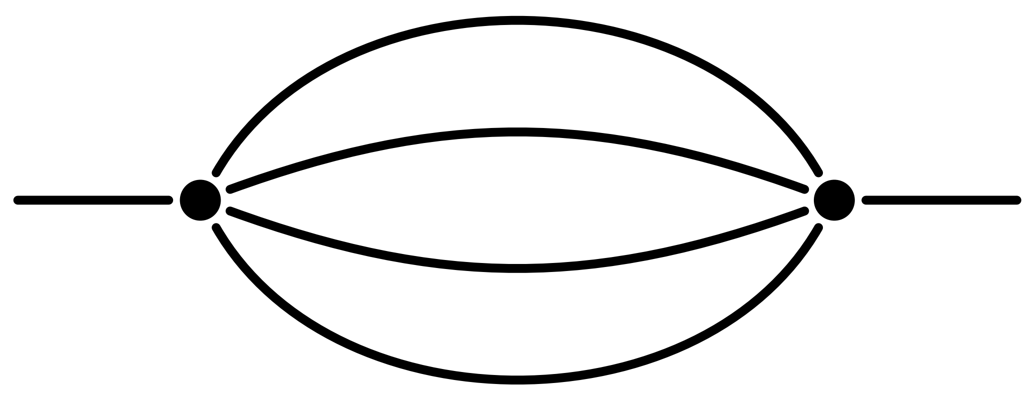
Can then trivially evaluated at any order in ε: $I = \mathbb{P}\exp\left(\varepsilon \int A\right) I_0$

Part 1

Part 2

Part 1

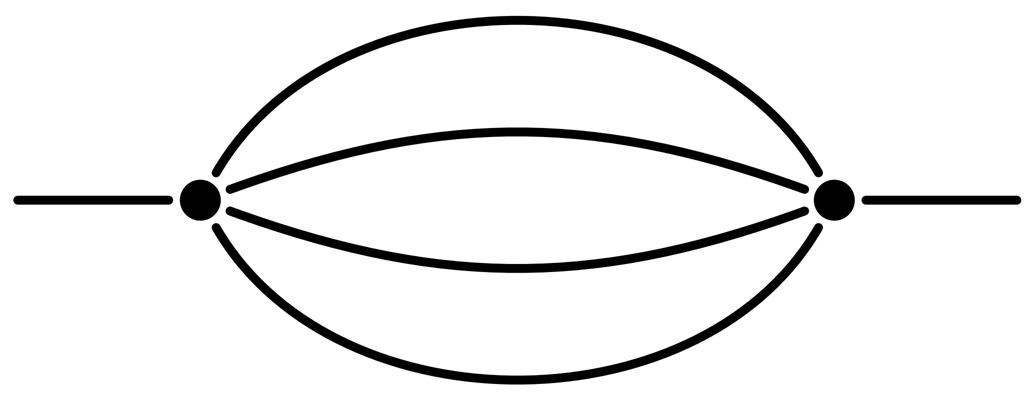
Three-loop Banana



Part 2

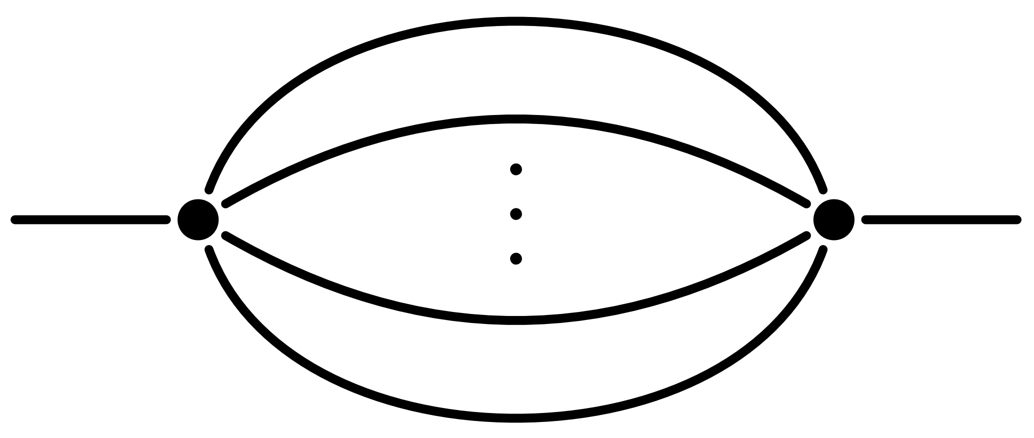
Part 1

Three-loop Banana



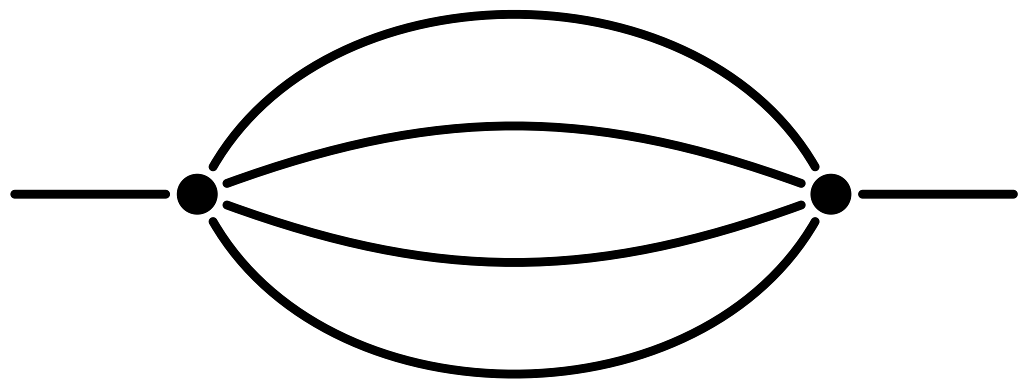
Part 2

($\geq$Four)-loop Banana



Part 1

Three-loop Banana



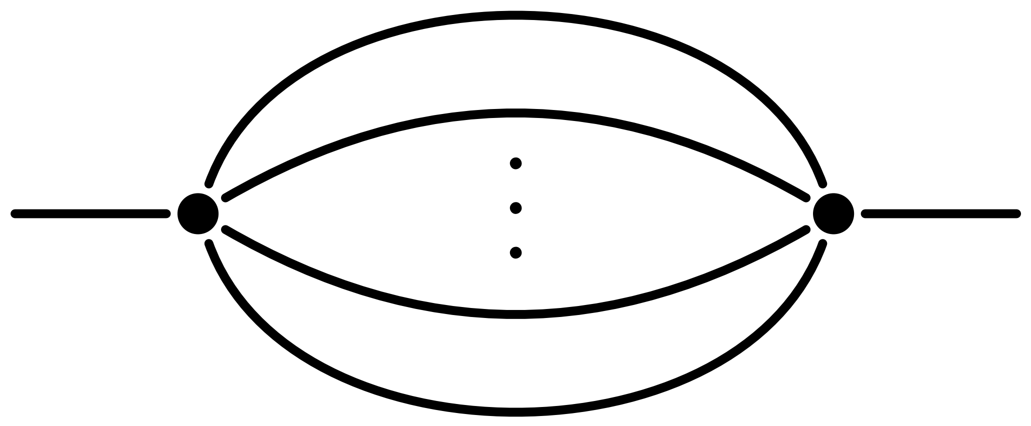
“Trivial” Calabi–Yaus

Essentially elliptic

2207.12893

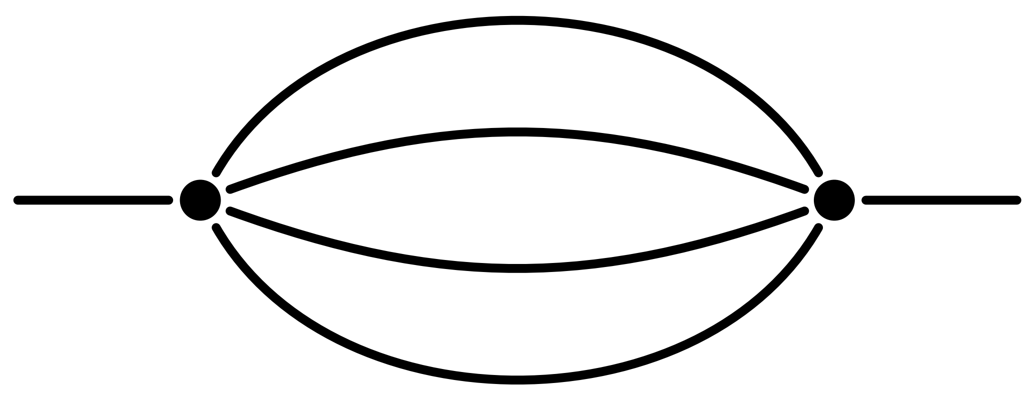
Part 2

($\geq$Four)-loop Banana



Part 1

Three-loop Banana



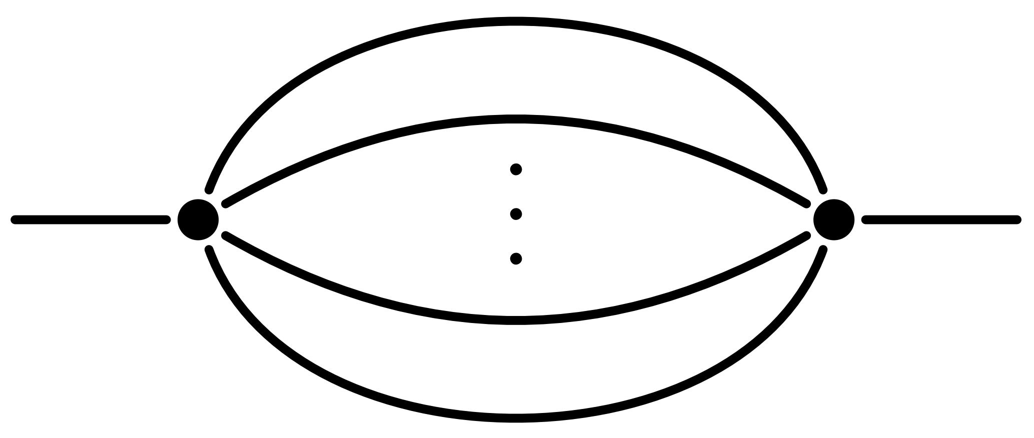
“Trivial” Calabi–Yaus

Essentially elliptic

2207.12893

Part 2

($\geq$Four)-loop Banana



“Non-trivial” Calabi–Yaus

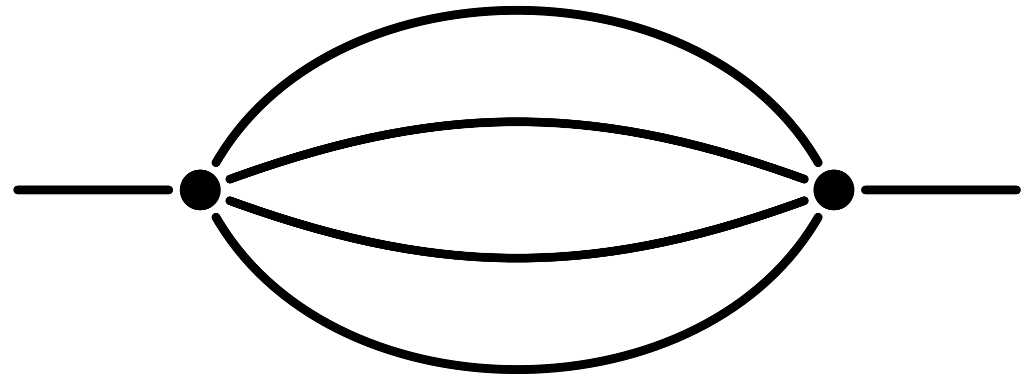
Non-elliptic

2211.04292

2212.xxxxx

Part 1

Three-loop Banana



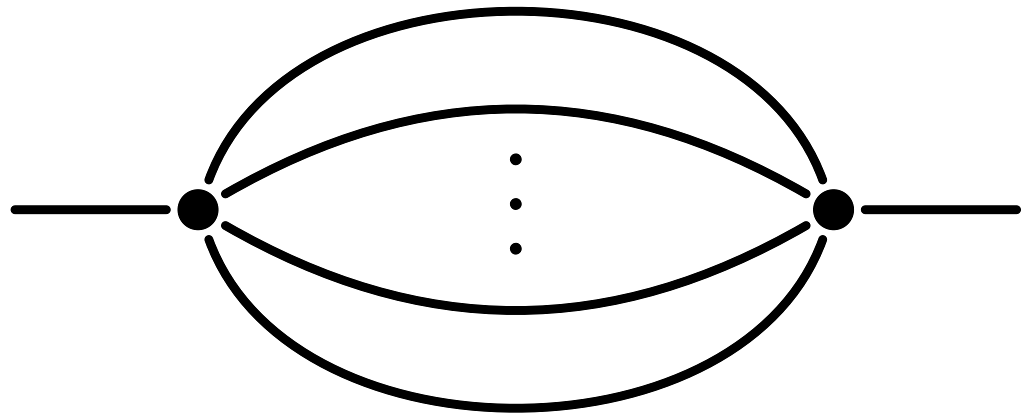
“Trivial” Calabi–Yaus

Essentially elliptic

2207.12893

Part 2

(≥Four)-loop Banana



“Non-trivial” Calabi–Yaus

Non-elliptic

2211.04292

2212.xxxxx

The Three-Loop Banana Integral

Simplest example of Feynman integral beyond elliptic:

Calabi–Yau 2-fold

Equal-mass case: closely connected to sunrise integral

Extensively studied in the past:

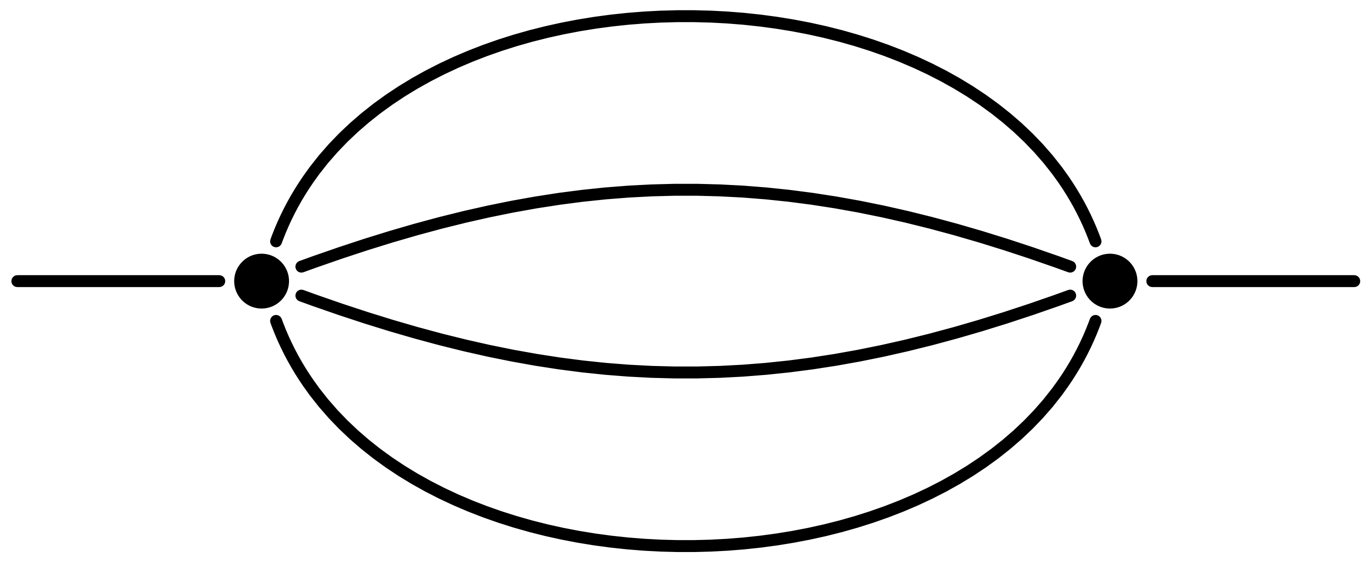
Leading term in ε [Bloch, Kerr, Vanhove, 14']

ε-factorized form [Primo, Tancredi,17']

Master integrals in $d = 2$ in terms of eMPLs $\tilde{\Gamma}$ [Broedel, Duhr, Dulat, Marzucca, Penante, 19']

DEQ with meromorphic modular forms [Broedel, Duhr, Matthes, 21']

ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]



Singularities:

$$x = \frac{p^2}{m^2} = 0,4,16,\infty$$

Picard-Fuchs Differential Operator

Annihilates $\mathrm{MaxCut}(I)$ / periods of Calabi–Yau

Picard-Fuchs Differential Operator

Annihilates $\mathrm{MaxCut}(I)$ / periods of Calabi–Yau

Defines geometry

Picard-Fuchs Differential Operator

Annihilates $\mathrm{MaxCut}(I)$ / periods of Calabi–Yau

Defines geometry

3-loop banana in $d = 2$:

$$\mathcal{L}_3^{(0)} = \frac{d^3}{dx^3} + \left[\frac{3}{x} + \frac{3}{2\,(x-4)} + \frac{3}{2\,(x-16)}\right]\frac{d^2}{dx^2} + \frac{7x^2-68x+64}{x^2\,(x-4)\,(x-16)}\frac{d}{dx} + \frac{1}{x^2\,(x-16)}.$$

with solutions $\mathcal{L}_3^{(0)}\,\omega_i = 0$ where $\omega_i = \mathrm{MaxCut}(I_{1111})|_{\gamma_i}$ on three independent contours γ_i

Picard-Fuchs Differential Operator

Annihilates $\mathrm{MaxCut}(I)$ / periods of Calabi–Yau

Defines geometry

3-loop banana in $d=2$:

$$\mathcal{L}_3^{(0)} = \frac{d^3}{dx^3} + \left[\frac{3}{x} + \frac{3}{2\,(x-4)} + \frac{3}{2\,(x-16)}\right]\frac{d^2}{dx^2} + \frac{7x^2-68x+64}{x^2\,(x-4)\,(x-16)}\frac{d}{dx} + \frac{1}{x^2\,(x-16)}.$$

with solutions $\mathcal{L}_3^{(0)}\,\omega_i = 0$ where $\omega_i = \mathrm{MaxCut}(I_{1111})|_{\gamma_i}$ on three independent contours γ_i

$\mathcal{L}_3^{(0)}$ is a symmetric square

[Verrill, 96’; Joyce, 72’]

Picard-Fuchs Differential Operator

Annihilates $\mathrm{MaxCut}(I)$ / periods of Calabi–Yau

→ Defines geometry

3-loop banana in $d = 2$:

$$\mathcal{L}_3^{(0)} = \frac{d^3}{dx^3} + \left[\frac{3}{x} + \frac{3}{2(x-4)} + \frac{3}{2(x-16)}\right]\frac{d^2}{dx^2} + \frac{7x^2 - 68x + 64}{x^2(x-4)(x-16)}\frac{d}{dx} + \frac{1}{x^2(x-16)}.$$

with solutions $\mathcal{L}_3^{(0)}\,\omega_i = 0$ where $\omega_i = \mathrm{MaxCut}(I_{1111})|_{\gamma_i}$ on three independent contours γ_i

$\mathcal{L}_3^{(0)}$ is a symmetric square

[Verrill, 96'; Joyce, 72']

There exists an operator

$$\mathcal{L}_2^{(0)} = \frac{d^2}{dx^2} + \left[\frac{1}{x} + \frac{1}{2(x-4)} + \frac{1}{2(x-16)}\right]\frac{d}{dx} + \frac{(x-8)}{4x(x-4)(x-16)}$$

with solutions ψ_1, ψ_2, $\mathcal{L}_2^{(0)}\,\psi_i = 0$ such that

Picard-Fuchs Differential Operator

Annihilates $\mathrm{MaxCut}(I)$ / periods of Calabi–Yau

Defines geometry

3-loop banana in $d=2$:

$$\mathcal{L}_3^{(0)} = \frac{d^3}{dx^3} + \left[\frac{3}{x} + \frac{3}{2\,(x-4)} + \frac{3}{2\,(x-16)}\right]\frac{d^2}{dx^2} + \frac{7x^2-68x+64}{x^2\,(x-4)\,(x-16)}\frac{d}{dx} + \frac{1}{x^2\,(x-16)}.$$

with solutions $\mathcal{L}_3^{(0)}\,\omega_i = 0$ where $\omega_i = \mathrm{MaxCut}(I_{1111})|_{\gamma_i}$ on three independent contours γ_i

$\mathcal{L}_3^{(0)}$ is a symmetric square

[Verrill, 96'; Joyce, 72']

There exists an operator

$$\mathcal{L}_2^{(0)} = \frac{d^2}{dx^2} + \left[\frac{1}{x} + \frac{1}{2\,(x-4)} + \frac{1}{2\,(x-16)}\right]\frac{d}{dx} + \frac{(x-8)}{4x\,(x-4)\,(x-16)}$$

with solutions $\psi_1,\ \psi_2,\ \mathcal{L}_2^{(0)}\,\psi_i = 0$ such that

$$\omega_i \in \langle \psi_1^2,\ \psi_1\psi_2,\ \psi_2^2 \rangle$$

Picard-Fuchs Differential Operator

Annihilates $\mathrm{MaxCut}(I)$ / periods of Calabi–Yau

Defines geometry

3-loop banana in $d = 2$:

$$\mathcal{L}_3^{(0)} = \frac{d^3}{dx^3} + \left[\frac{3}{x} + \frac{3}{2\,(x-4)} + \frac{3}{2\,(x-16)}\right]\frac{d^2}{dx^2} + \frac{7x^2-68x+64}{x^2\,(x-4)\,(x-16)}\frac{d}{dx} + \frac{1}{x^2\,(x-16)}.$$

with solutions $\mathcal{L}_3^{(0)}\,\omega_i = 0$ where $\omega_i = \mathrm{MaxCut}(I_{1111})|_{\gamma_i}$ on three independent contours γ_i

$\mathcal{L}_3^{(0)}$ is a symmetric square

[Verrill, 96'; Joyce, 72']

There exists an operator

$$\mathcal{L}_2^{(0)} = \frac{d^2}{dx^2} + \left[\frac{1}{x} + \frac{1}{2\,(x-4)} + \frac{1}{2\,(x-16)}\right]\frac{d}{dx} + \frac{(x-8)}{4x\,(x-4)\,(x-16)}$$

Sunrise in disguise

with solutions $\psi_1,\,\psi_2,\ \mathcal{L}_2^{(0)}\,\psi_i = 0$ such that

$$\omega_i \in \langle \psi_1^2,\, \psi_1\psi_2,\, \psi_2^2 \rangle$$

ε-Factorization: Sunrise

ε-Factorization: Sunrise

Make the ansatz

$$\begin{aligned} I_1 &= \varepsilon^2 I_{110}, \\ I_2 &= \varepsilon^2 \frac{\pi}{\psi_1} I_{111}, \\ I_3 &= \frac{1}{\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \end{aligned}$$

ε-Factorization: Sunrise

Make the ansatz

$$I_1 = \varepsilon^2 I_{110},$$
$$I_2 = \varepsilon^2 \frac{\pi}{\psi_1} I_{111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$

$$\tau = \frac{\psi_2}{\psi_1}$$

Periods of elliptic curve

ε-Factorization: Sunrise

Make the ansatz

$$I_1 = \varepsilon^2 I_{110},$$
$$I_2 = \varepsilon^2 \frac{\pi}{\psi_1} I_{111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$

$$\tau = \frac{\psi_2}{\psi_1}$$

Periods of elliptic curve

$$\mathrm{d}I = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & 1 \\ \eta_3 & \eta_4 & \eta_2 \end{pmatrix} I$$

ε-Factorization: Sunrise

Make the ansatz

$$I_1 = \varepsilon^2 I_{110},$$
$$I_2 = \varepsilon^2 \frac{\pi}{\psi_1} I_{111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$

$$\tau = \frac{\psi_2}{\psi_1}$$

Periods of elliptic curve

$$\mathrm{d}I = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & 1 \\ \eta_3 & \eta_4 & \eta_2 \end{pmatrix} I$$

A organised by modular weight

ε-Factorization: Sunrise

Make the ansatz

$$I_1 = \varepsilon^2 I_{110},$$
$$I_2 = \varepsilon^2 \frac{\pi}{\psi_1} I_{111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$

$$\tau = \frac{\psi_2}{\psi_1}$$

Periods of elliptic curve

$$\mathrm{d}I = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & 1 \\ \eta_3 & \eta_4 & \eta_2 \end{pmatrix} I$$

A organised by modular weight

η_k : Modular forms of $\Gamma_1(6)$ of weight k, independent of ε

ε-Factorization: Sunrise

Make the ansatz

$$\begin{aligned} I_1 &= \varepsilon^2 I_{110}, \\ I_2 &= \varepsilon^2 \frac{\pi}{\psi_1} I_{111}, \\ I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \end{aligned}$$

$$\tau = \frac{\psi_2}{\psi_1}$$

Periods of elliptic curve

$$\mathrm{d}I = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & 1 \\ \eta_3 & \eta_4 & \eta_2 \end{pmatrix} I$$

A organised by modular weight

η_k : Modular forms of $\Gamma_1(6)$ of weight k, independent of ε

✓ A independent of ε

✓ A consists of modular forms

ε-Factorization: Sunrise

Make the ansatz

$$I_1 = \varepsilon^2 I_{110},$$
$$I_2 = \varepsilon^2 \frac{\pi}{\psi_1} I_{111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$

$$\tau = \frac{\psi_2}{\psi_1}$$

Periods of elliptic curve

$$\mathrm{d}I = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & 1 \\ \eta_3 & \eta_4 & \eta_2 \end{pmatrix} I$$

A organised by modular weight

η_k : Modular forms of $\Gamma_1(6)$ of weight k, independent of ε

✓ A independent of ε

✓ A consists of modular forms ("well understood")

ε-Factorization: Three-loop Ansatz

ε-Factorization: Three-loop Ansatz

Make the ansatz

$$\begin{aligned} I_1 &= \varepsilon^3 I_{1110}, \\ I_2 &= \varepsilon^3 \frac{1}{\omega} I_{1111}, \\ I_3 &= \frac{1}{\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\ I_4 &= \frac{1}{\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3. \end{aligned}$$

ε-Factorization: Three-loop Ansatz

Make the ansatz

$$
\begin{aligned}
I_1 &= \varepsilon^3 I_{1110}, \\
I_2 &= \varepsilon^3 \frac{1}{\omega} I_{1111}, \\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3.
\end{aligned}
$$

No assumptions for ω and τ required

ε-Factorization: Three-loop Ansatz

Make the ansatz

$$I_1 = \varepsilon^3 I_{1110},$$

$$I_2 = \varepsilon^3 \frac{1}{\omega} I_{1111},$$

$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$

$$I_4 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3.$$

No assumptions for ω and τ required

$$\downarrow$$

$$dI = \tilde{A} I$$

ε-Factorization: Three-loop Ansatz

Make the ansatz

$$I_1 = \varepsilon^3 I_{1110},$$
$$I_2 = \varepsilon^3 \frac{1}{\omega} I_{1111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$
$$I_4 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3.$$

No assumptions for ω and τ required

$$dI = \tilde{A} I$$

Requiring $\tilde{A} = \varepsilon A$ → constraints on ω, J, F_{32}, F_{42}, F_{43}

ε-Factorization: Three-loop Ansatz

Make the ansatz

$$I_1 = \varepsilon^3 I_{1110},$$
$$I_2 = \varepsilon^3 \frac{1}{\omega} I_{1111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$
$$I_4 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3.$$

No assumptions for ω and τ required

$$dI = \tilde{A} I$$

Requiring $\tilde{A} = \varepsilon A$ $\rightarrow$ constraints on ω, J, F_{32}, F_{42}, F_{43}

J: $\frac{\mathrm{d}x}{\mathrm{d}\tau}$

Eliminate Non-ε-Factorized Pieces

$$dI = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{2,2} & 1 & 0 \\ 0 & \tilde{A}_{3,2} & \tilde{A}_{3,3} & 1 \\ \tilde{A}_{4,1} & \tilde{A}_{4,2} & \tilde{A}_{4,3} & \tilde{A}_{4,4} \end{pmatrix} I$$

Eliminate Non-ε-Factorized Pieces

Already ε-factorized

$$dI = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{2,2} & 1 & 0 \\ 0 & \tilde{A}_{3,2} & \tilde{A}_{3,3} & 1 \\ \tilde{A}_{4,1} & \tilde{A}_{4,2} & \tilde{A}_{4,3} & \tilde{A}_{4,4} \end{pmatrix} I$$

Eliminate Non-ε-Factorized Pieces

Already ε-factorized

$$dI = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{2,2} & 1 & 0 \\ 0 & \tilde{A}_{3,2} & \tilde{A}_{3,3} & 1 \\ \tilde{A}_{4,1} & \tilde{A}_{4,2} & \tilde{A}_{4,3} & \tilde{A}_{4,4} \end{pmatrix} I$$

$\tilde{A}_{4,k}$ contains term ε^{-4+k} through ε

Eliminate Non-ε-Factorized Pieces

Already ε-factorized

$$dI = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{2,2} & 1 & 0 \\ 0 & \tilde{A}_{3,2} & \tilde{A}_{3,3} & 1 \\ \tilde{A}_{4,1} & \tilde{A}_{4,2} & \tilde{A}_{4,3} & \tilde{A}_{4,4} \end{pmatrix} I$$

$\tilde{A}_{4,k}$ contains term ε^{-4+k} through ε

Five variables, six constraints

$\omega, J, F_{32}, F_{42}, F_{43}$

Eliminate Non-ε-Factorized Pieces

Already ε-factorized

$$dI = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{2,2} & 1 & 0 \\ 0 & \tilde{A}_{3,2} & \tilde{A}_{3,3} & 1 \\ \tilde{A}_{4,1} & \tilde{A}_{4,2} & \tilde{A}_{4,3} & \tilde{A}_{4,4} \end{pmatrix} I$$

$\tilde{A}_{4,k}$ contains term ε^{-4+k} through ε

Five variables, six constraints

$\omega, J, F_{32}, F_{42}, F_{43}$

$\rightarrow$ One non-trivial constraint!

Eliminate Non-ε-Factorized Pieces

Already ε-factorized

$$dI = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{2,2} & 1 & 0 \\ 0 & \tilde{A}_{3,2} & \tilde{A}_{3,3} & 1 \\ \tilde{A}_{4,1} & \tilde{A}_{4,2} & \tilde{A}_{4,3} & \tilde{A}_{4,4} \end{pmatrix} I$$

$\tilde{A}_{4,k}$ contains term ε^{-4+k} through ε

Five variables, six constraints

$\omega, J, F_{32}, F_{42}, F_{43}$

$\rightarrow$ One non-trivial constraint!

Satisfied for $\omega = (x\psi_1^{\text{sun}})^2 \quad \tau = \dfrac{\psi_2^{\text{sun}}}{\psi_1^{\text{sun}}}$

Eliminate Non-ε-Factorized Pieces

Already ε-factorized

$$dI = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{2,2} & 1 & 0 \\ 0 & \tilde{A}_{3,2} & \tilde{A}_{3,3} & 1 \\ \tilde{A}_{4,1} & \tilde{A}_{4,2} & \tilde{A}_{4,3} & \tilde{A}_{4,4} \end{pmatrix} I$$

$\tilde{A}_{4,k}$ contains term ε^{-4+k} through ε

Five variables, six constraints

$\omega, J, F_{32}, F_{42}, F_{43}$

$\rightarrow$ **One non-trivial constraint!**

Satisfied for $\omega = (x\psi_1^{\text{sun}})^2 \quad \tau = \dfrac{\psi_2^{\text{sun}}}{\psi_1^{\text{sun}}}$

Periods of elliptic curve

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = (2\pi i)\varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -f_{2,a} - f_{2,b} & 1 & 0 \\ 0 & f_{4,b} & -f_{2,a} + 2f_{2,b} & 1 \\ f_{4,a} & f_6 & f_{4,b} & -f_{2,a} - f_{2,b} \end{pmatrix} I$$

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = (2\pi i)\varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -f_{2,a} - f_{2,b} & 1 & 0 \\ 0 & f_{4,b} & -f_{2,a} + 2f_{2,b} & 1 \\ f_{4,a} & f_6 & f_{4,b} & -f_{2,a} - f_{2,b} \end{pmatrix} I$$

Constraints allow symmetry

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = (2\pi i)\varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -f_{2,a} - f_{2,b} & 1 & 0 \\ 0 & f_{4,b} & -f_{2,a} + 2f_{2,b} & 1 \\ f_{4,a} & f_6 & f_{4,b} & -f_{2,a} - f_{2,b} \end{pmatrix} I$$

Alphabet: $\mathcal{A} = \{1, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_6\}$.

Constraints allow symmetry

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = (2\pi i)\varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -f_{2,a} - f_{2,b} & 1 & 0 \\ 0 & f_{4,b} & -f_{2,a} + 2f_{2,b} & 1 \\ f_{4,a} & f_6 & f_{4,b} & -f_{2,a} - f_{2,b} \end{pmatrix} I$$

Constraints allow symmetry

Alphabet: $\mathcal{A} = \{1, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_6\}$.

Function space of Alphabet

Meromorphic modular forms + Special function F_2

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = (2\pi i)\varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -f_{2,a} - f_{2,b} & 1 & 0 \\ 0 & f_{4,b} & -f_{2,a} + 2f_{2,b} & 1 \\ f_{4,a} & f_6 & f_{4,b} & -f_{2,a} - f_{2,b} \end{pmatrix} I$$

Alphabet: $\mathcal{A} = \{1, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_6\}\,.$

Constraints allow symmetry

Function space of Alphabet

Meromorphic modular forms + Special function F_2

Iterated integral of meromorphic modular form of weight 6

$F_2 = I(1, g_6; \tau) \quad g_6 = \frac{x\,(x-8)\,(x+8)^3}{864\,(4-x)^{\frac{3}{2}}\,(16-x)^{\frac{3}{2}}} \left(\frac{\psi_1}{\pi}\right)^6$

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = (2\pi i)\varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -f_{2,a} - f_{2,b} & 1 & 0 \\ 0 & f_{4,b} & -f_{2,a} + 2f_{2,b} & 1 \\ f_{4,a} & f_6 & f_{4,b} & -f_{2,a} - f_{2,b} \end{pmatrix} I$$

Alphabet: $\mathcal{A} = \{1, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_6\}$.

Constraints allow symmetry

Function space of Alphabet

Meromorphic modular forms + Special function F_2

Iterated integral of meromorphic modular form of weight 6

$F_2 = I(1, g_6; \tau)$ $g_6 = \frac{x\,(x-8)\,(x+8)^3}{864\,(4-x)^{\frac{3}{2}}\,(16-x)^{\frac{3}{2}}} \left(\frac{\psi_1}{\pi}\right)^6$

$$I_2 = \varepsilon^3 \left(\frac{4}{3}\zeta_3 + I(1,1,f_{4,a};\tau)\right) + \mathcal{O}(\epsilon^4)$$

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = (2\pi i)\varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -f_{2,a} - f_{2,b} & 1 & 0 \\ 0 & f_{4,b} & -f_{2,a} + 2f_{2,b} & 1 \\ f_{4,a} & f_6 & f_{4,b} & -f_{2,a} - f_{2,b} \end{pmatrix} I$$

Alphabet: $\mathcal{A} = \{1, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_6\}$.

Constraints allow symmetry

Function space of Alphabet

Meromorphic modular forms + Special function F_2

Iterated integral of meromorphic modular form of weight 6

$F_2 = I(1, g_6; \tau)$ $g_6 = \frac{x\,(x-8)\,(x+8)^3}{864\,(4-x)^{\frac{3}{2}}\,(16-x)^{\frac{3}{2}}} \left(\frac{\psi_1}{\pi}\right)^6$

$$I_2 = \varepsilon^3 \left(\frac{4}{3}\zeta_3 + I(1,1,f_{4,a};\tau) \right) + \mathcal{O}(\epsilon^4)$$

Obtained expressions for all masters up to ε^6

Numerics via q-expansion

“Trivial” Calabi–Yau Summary

“Trivial” Calabi–Yau Summary

ε**-factorized form:** Ansatz, then solve constraints algorithmically

“Trivial” Calabi–Yau Summary

ε-factorized form: Ansatz, then solve constraints algorithmically

Symmetric square: Three-loop banana integral related to elliptic curve

“Trivial” Calabi–Yau Summary

ε**-factorized form:** Ansatz, then solve constraints algorithmically

Symmetric square: Three-loop banana integral related to elliptic curve

Function space: Meromorphic modular forms, plus iterated integrals thereof (F_2)

“Trivial” Calabi–Yau Summary

ε**-factorized form:** Ansatz, then solve constraints algorithmically

Symmetric square: Three-loop banana integral related to elliptic curve

Function space: Meromorphic modular forms, plus iterated integrals thereof (F_2)

Expectation: This generalizes beyond the banana!

“Trivial” Calabi–Yau Summary

ε**-factorized form:** Ansatz, then solve constraints algorithmically

Symmetric square: Three-loop banana integral related to elliptic curve

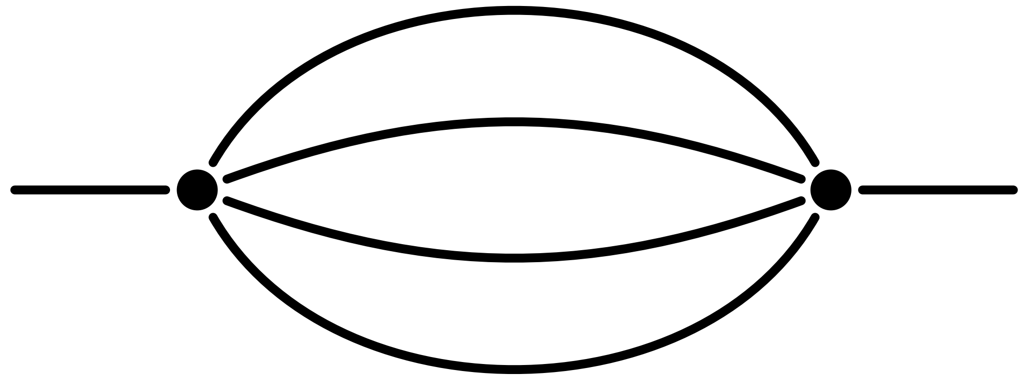
Function space: Meromorphic modular forms, plus iterated integrals thereof (F_2)

Expectation: This generalizes beyond the banana!

Single scale + 2-fold Calabi–Yau integral + degree 3 Picard–Fuchs operator	⟶	“trivial” Calabi–Yau *i.e.* symmetric power of elliptic curve	⟶	ε-factorised DEQ from Ansatz Function space *essentially elliptic*

Part 1

Three-loop Banana



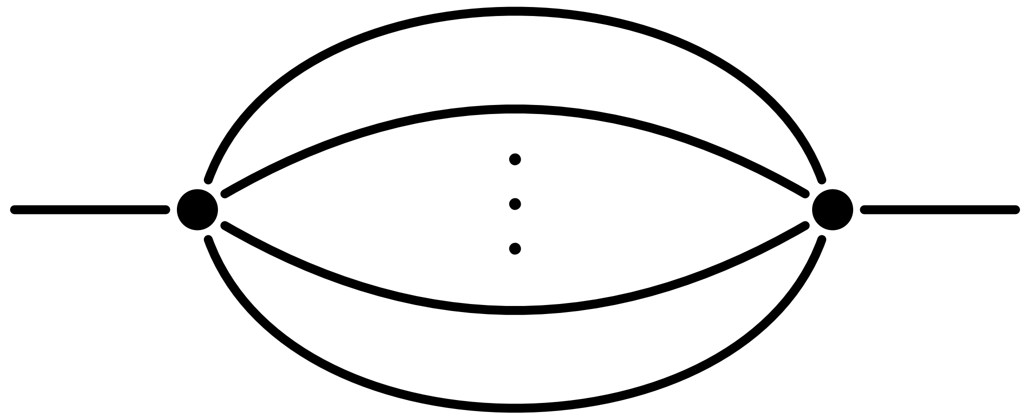
“Trivial” Calabi–Yaus

Essentially elliptic

2207.12893

Part 2

(≥Four)-loop Banana



“Non-trivial” Calabi–Yaus

Non-elliptic

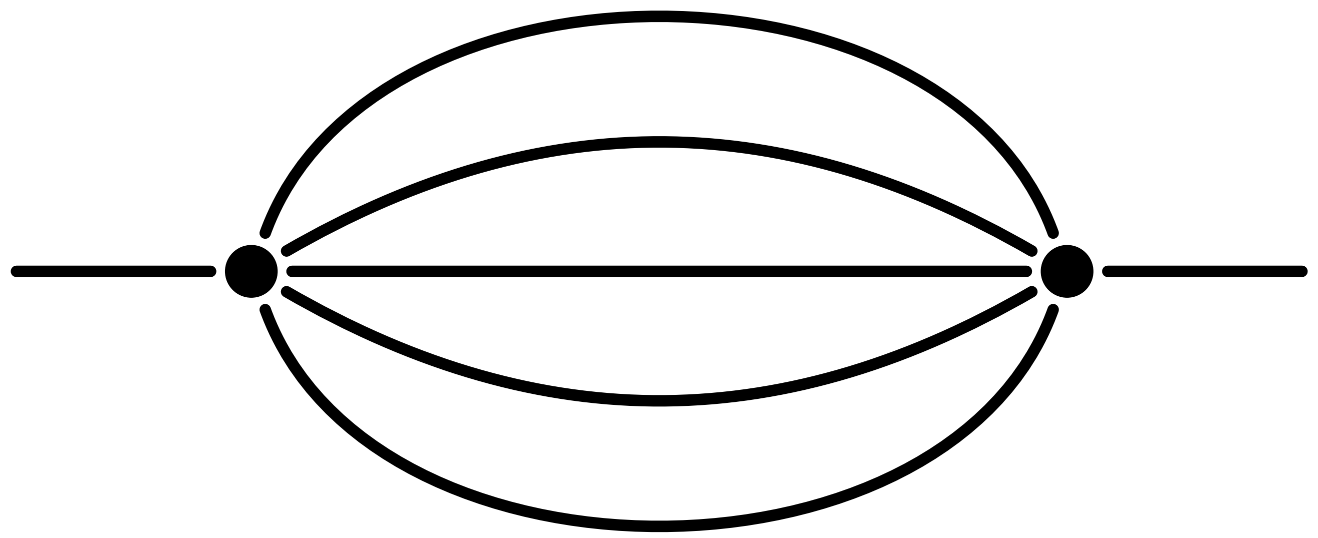
2211.04292

2212.xxxxx

The Four-Loop Banana Integral

First banana integral with "non-trivial" Calabi–Yau:

Not related to elliptic curves



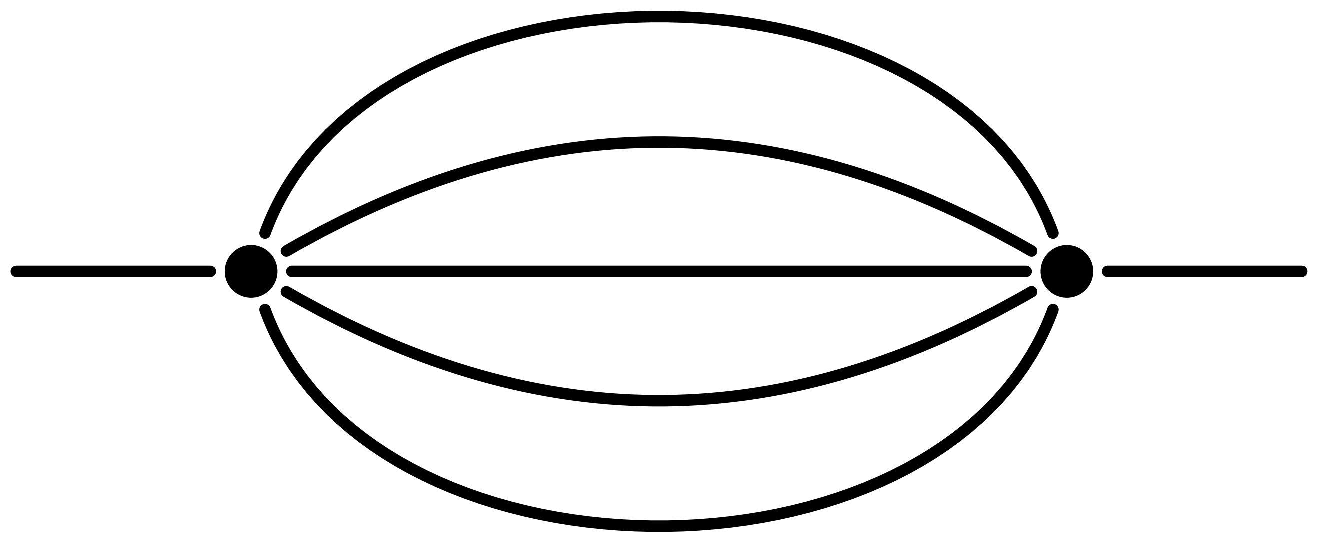
Singularities:

$y = -\frac{m^2}{p^2} = 0, -1, -\frac{1}{9}, -\frac{1}{25}, \infty$

The Four-Loop Banana Integral

First banana integral with "non-trivial" Calabi–Yau:

Not related to elliptic curves



Integral already studied in the past

ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]

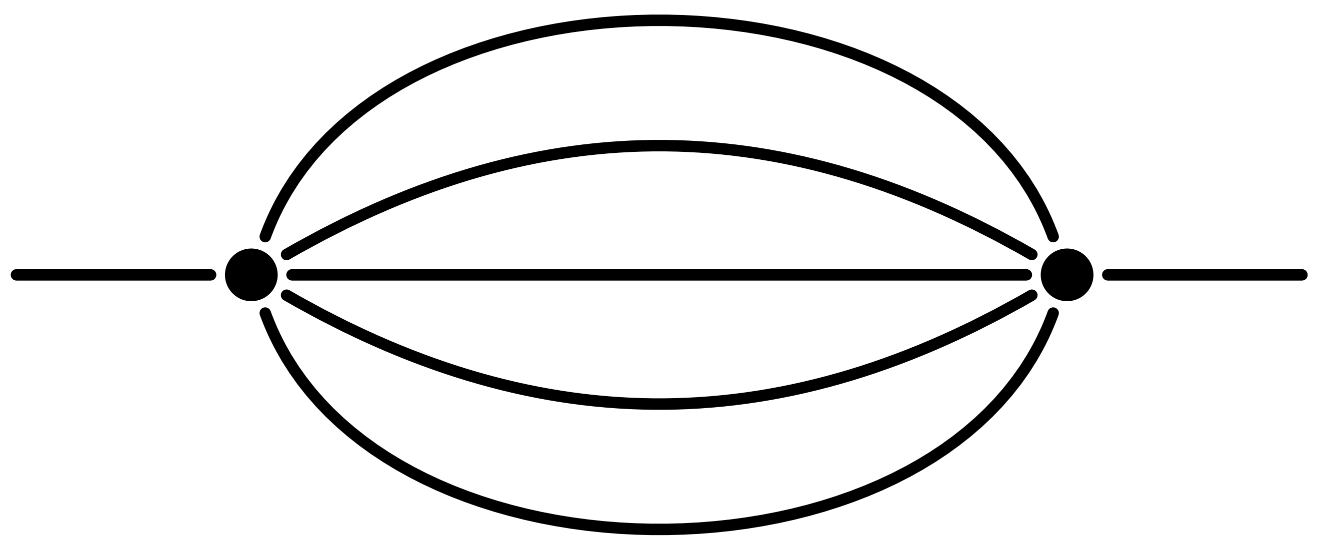
Singularities:

$$y = -\frac{m^2}{p^2} = 0, -1, -\frac{1}{9}, -\frac{1}{25}, \infty$$

The Four-Loop Banana Integral

First banana integral with "non-trivial" Calabi–Yau:

Not related to elliptic curves



Integral already studied in the past

ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]

Algebraic Variety from graph polynomial

Hypersurface in $\mathbb{CP}^4$ with

Singularities:

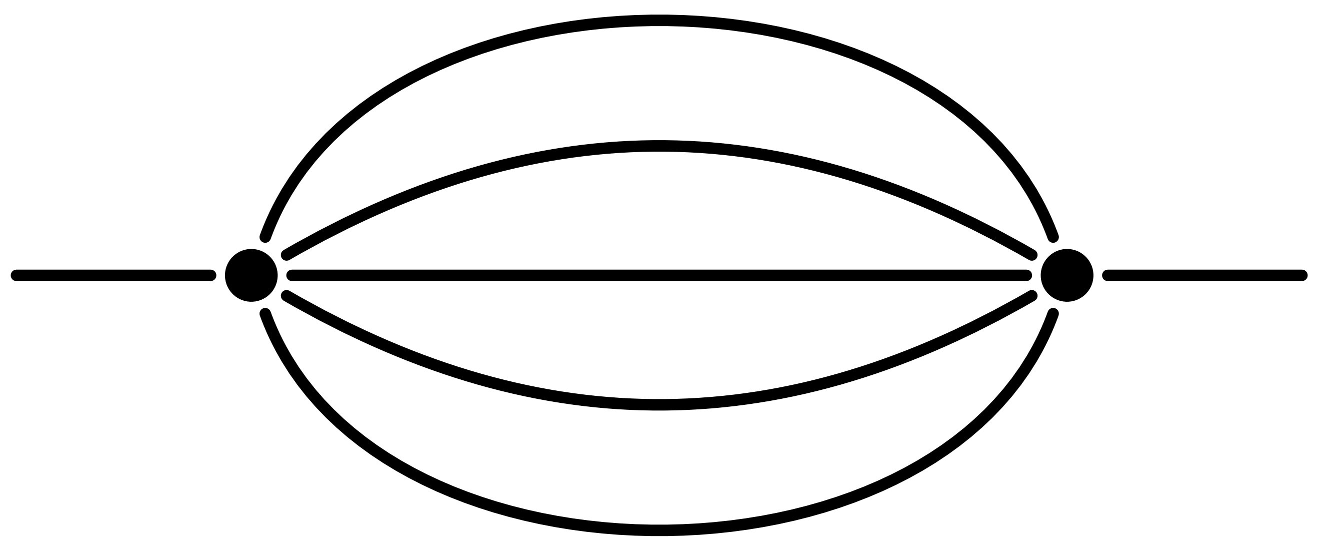
$y = -\frac{m^2}{p^2} = 0, -1, -\frac{1}{9}, -\frac{1}{25}, \infty$

$$1/y = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} + \frac{1}{\alpha_5}\right)$$

The Four-Loop Banana Integral

First banana integral with "non-trivial" Calabi–Yau:

Not related to elliptic curves



Singularities:

$y = -\frac{m^2}{p^2} = 0, -1, -\frac{1}{9}, -\frac{1}{25}, \infty$

Integral already studied in the past

ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]

Algebraic Variety from graph polynomial

Hypersurface in $\mathbb{CP}^4$ with

$$1/y = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} + \frac{1}{\alpha_5}\right)$$

Calabi–Yau very well known

Studied in [Hulek, Verrill, 05'; …]

Known as `AESZ34` [Almkvist, van Enckevort, van Straten, Zudilin]

ε-Factorization: Four-loop Ansatz

ε-Factorization: Four-loop Ansatz

Guess the pattern?

ε-Factorization: Four-loop Ansatz

Guess the pattern?

$$
\begin{aligned}
I_1 &= \varepsilon^4 I_{11110}, \\
I_2 &= \varepsilon^4 \frac{1}{\omega} I_{11111}, \\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3 \\
I_5 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.
\end{aligned}
$$

?

ε-Factorization: Four-loop Ansatz

Guess the pattern?

$$
\begin{aligned}
I_1 &= \varepsilon^4 I_{11110}, \\
I_2 &= \varepsilon^4 \frac{1}{\omega} I_{11111}, \\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3 \\
I_5 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.
\end{aligned}
$$

?

$dI = \varepsilon A I$ leads to inconsistent constraints!
$\rightarrow$ No solution!

ε-Factorization: Four-loop Ansatz (fixed)

ε-Factorization: Four-loop Ansatz (fixed)

Modify ansatz!

$$
\begin{aligned}
I_1 &= \varepsilon^4 I_{11110}, \\
I_2 &= \varepsilon^4 \frac{1}{\omega} I_{11111}, \\
I_3 &= \frac{1}{\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon} \frac{1}{K_1} \frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3 \\
I_5 &= \frac{1}{\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.
\end{aligned}
$$

ε-Factorization: Four-loop Ansatz (fixed)

Modify ansatz!

$$
\begin{aligned}
I_1 &= \varepsilon^4 I_{11110}, \\
I_2 &= \varepsilon^4 \frac{1}{\omega} I_{11111}, \\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3 \\
I_5 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.
\end{aligned}
$$

ℓ-loop Banana Integrals define special Calabi–Yau manifolds

Picard–Fuchs operators are **Calabi–Yau operators**

[Almkvist, van Enckevort, van Straten, Zudilin, 05']
[M. Bogner, 13']

ε-Factorization: Four-loop Ansatz (fixed)

Modify ansatz!

$$
\begin{aligned}
I_1 &= \varepsilon^4 I_{11110}, \\
I_2 &= \varepsilon^4 \frac{1}{\omega} I_{11111}, \\
I_3 &= \frac{1}{\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon} \frac{1}{K_1} \frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3 \\
I_5 &= \frac{1}{\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.
\end{aligned}
$$

ℓ-loop Banana Integrals define special Calabi–Yau manifolds

Picard–Fuchs operators are **Calabi–Yau operators**

[Almkvist, van Enckevort, van Straten, Zudilin, 05']
[M. Bogner, 13']

K_1 is a Y-invariant of Calabi–Yau operator

ε-Factorization: Four-loop Ansatz (fixed)

Modify ansatz!

$$I_1 = \varepsilon^4 I_{11110},$$
$$I_2 = \varepsilon^4 \frac{1}{\omega} I_{11111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$
$$I_4 = \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3$$
$$I_5 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.$$

ℓ-loop Banana Integrals define special Calabi–Yau manifolds

Picard–Fuchs operators are **Calabi–Yau operators**

[Almkvist, van Enckevort, van Straten, Zudilin, 05']
[M. Bogner, 13']

K_1 is a Y-invariant of Calabi–Yau operator

Start appearing at 3-fold

ε-Factorization: Four-loop Ansatz (fixed)

Modify ansatz!

$$I_1 = \varepsilon^4 I_{11110},$$
$$I_2 = \varepsilon^4 \frac{1}{\omega} I_{11111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$
$$I_4 = \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3$$
$$I_5 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.$$

ℓ-loop Banana Integrals define special Calabi–Yau manifolds

Picard–Fuchs operators are **Calabi–Yau operators**

[Almkvist, van Enckevort, van Straten, Zudilin, 05']
[M. Bogner, 13']

K_1 is a Y-invariant of Calabi–Yau operator

Start appearing at 3-fold

$dI = \varepsilon A I$ leads to consistent constraints!

No prior knowledge of K_1 required!
Fixed by constraints (up to rescaling)

ε-Factorization: Four-loop Ansatz (fixed)

Modify ansatz!

$$I_1 = \varepsilon^4 I_{11110},$$
$$I_2 = \varepsilon^4 \frac{1}{\omega} I_{11111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$
$$I_4 = \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3$$
$$I_5 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.$$

ℓ-loop Banana Integrals define special Calabi–Yau manifolds

Picard–Fuchs operators are **Calabi–Yau operators**

[Almkvist, van Enckevort, van Straten, Zudilin, 05']
[M. Bogner, 13']

K_1 is a Y-invariant of Calabi–Yau operator

Start appearing at 3-fold

$dI = \varepsilon A I$ leads to consistent constraints!

No prior knowledge of K_1 required!

Fixed by constraints (up to rescaling)

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -\, m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i \tau)$, $\tau = \omega_2/\omega_1$

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i \tau)$, $\tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Four-Loop solutions

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i\tau)$, $\tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Four-Loop solutions

$$q(y) = \exp\left(2\pi i \omega_2/\omega_1\right)$$
$$\omega = \omega_1$$
$$K_1 = \mathrm{d}^2/\mathrm{d}\tau^2(\omega_3/\omega_1)$$
$$J$$

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i\tau), \tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i\tau)$, $\tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

Four-Loop solutions

$q(y) = \exp(2\pi i\omega_2/\omega_1)$	$y - 8y^2 + 92y^3 - 1288y^4 + 20398y^5 + \mathcal{O}(y^6)$
$\omega = \omega_1$	$q + 3q^2 + q^3 + 23q^4 - 101q^5 + \mathcal{O}(q^6)$
$K_1 = \mathrm{d}^2/\mathrm{d}\tau^2(\omega_3/\omega_1)$	$1 - q + 17q^2 - 253q^3 + 3345q^4 - 43751q^5 + \mathcal{O}(q^6)$
J	$q + 16q^2 + 108q^3 + 672q^4 + 2570q^5 + \mathcal{O}(q^6)$

What is the function space of "non-trivial" Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i \tau)$, $\tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

Four-Loop solutions

$q(y) = \exp\left(2\pi i \omega_2/\omega_1\right)$	$y - 8y^2 + 92y^3 - 1288y^4 + 20398y^5 + \mathcal{O}(y^6)$
$\omega = \omega_1$	$q + 3q^2 + q^3 + 23q^4 - 101q^5 + \mathcal{O}(q^6)$
$K_1 = \mathrm{d}^2/\mathrm{d}\tau^2(\omega_3/\omega_1)$	$1 - q + 17q^2 - 253q^3 + 3345q^4 - 43751q^5 + \mathcal{O}(q^6)$
J	$q + 16q^2 + 108q^3 + 672q^4 + 2570q^5 + \mathcal{O}(q^6)$
F_{32}	
F_{42}	
$\vdots$	

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i \tau), \tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

Four-Loop solutions

$q(y) = \exp(2\pi i \omega_2/\omega_1)$	$y - 8y^2 + 92y^3 - 1288y^4 + 20398y^5 + \mathcal{O}(y^6)$
$\omega = \omega_1$	$q + 3q^2 + q^3 + 23q^4 - 101q^5 + \mathcal{O}(q^6)$
$K_1 = \mathrm{d}^2/\mathrm{d}\tau^2(\omega_3/\omega_1)$	$1 - q + 17q^2 - 253q^3 + 3345q^4 - 43751q^5 + \mathcal{O}(q^6)$
J	$q + 16q^2 + 108q^3 + 672q^4 + 2570q^5 + \mathcal{O}(q^6)$
F_{32}	$c_{32} + 8q - 32q^2 + 512q^3 - 5872q^4 + 70008q^5 + \mathcal{O}(q^6)$
F_{42}	$c_{42} + 8q - 240q^2 + 4816q^3 - 90448q^4 + 1444008q^5 + c_{32}(-9q + 176q^2 - 2956q^3 + 44568q^4 - 611106q^5) + c_{32}^2(q - 16q^2 + 220q^3 - 2600q^4 + 30018q^5) + \mathcal{O}(q^6)$
$\vdots$	

What is the function space of "non-trivial" Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i \tau)$, $\tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

Four-Loop solutions

$q(y) = \exp(2\pi i \omega_2/\omega_1)$	$y - 8y^2 + 92y^3 - 1288y^4 + 20398y^5 + \mathcal{O}(y^6)$
$\omega = \omega_1$	$q + 3q^2 + q^3 + 23q^4 - 101q^5 + \mathcal{O}(q^6)$
$K_1 = \mathrm{d}^2/\mathrm{d}\tau^2(\omega_3/\omega_1)$	$1 - q + 17q^2 - 253q^3 + 3345q^4 - 43751q^5 + \mathcal{O}(q^6)$
J	$q + 16q^2 + 108q^3 + 672q^4 + 2570q^5 + \mathcal{O}(q^6)$
F_{32}	$c_{32} + 8q - 32q^2 + 512q^3 - 5872q^4 + 70008q^5 + \mathcal{O}(q^6)$
F_{42}	$c_{42} + 8q - 240q^2 + 4816q^3 - 90448q^4 + 1444008q^5$ $+ c_{32}(-9q + 176q^2 - 2956q^3 + 44568q^4 - 611106q^5)$ $+ c_{32}^2(q - 16q^2 + 220q^3 - 2600q^4 + 30018q^5)$ $+ \mathcal{O}(q^6)$
$\vdots$	

(first four rows:) Predictable from just Picard–Fuchs operator

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp\left(2\pi i \tau\right), \tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

Four-Loop solutions

$q(y) = \exp\left(2\pi i \omega_2/\omega_1\right)$	$y - 8y^2 + 92y^3 - 1288y^4 + 20398y^5 + \mathcal{O}(y^6)$	Predictable from just Picard–Fuchs operator
$\omega = \omega_1$	$q + 3q^2 + q^3 + 23q^4 - 101q^5 + \mathcal{O}(q^6)$	
$K_1 = \mathrm{d}^2/\mathrm{d}\tau^2(\omega_3/\omega_1)$	$1 - q + 17q^2 - 253q^3 + 3345q^4 - 43751q^5 + \mathcal{O}(q^6)$	
J	$q + 16q^2 + 108q^3 + 672q^4 + 2570q^5 + \mathcal{O}(q^6)$	
F_{32}	$c_{32} + 8q - 32q^2 + 512q^3 - 5872q^4 + 70008q^5 + \mathcal{O}(q^6)$	Need to solve constraints
F_{42}	$c_{42} + 8q - 240q^2 + 4816q^3 - 90448q^4 + 1444008q^5$ $+c_{32}(-9q + 176q^2 - 2956q^3 + 44568q^4 - 611106q^5)$ $+c_{32}^2(q - 16q^2 + 220q^3 - 2600q^4 + 30018q^5)$ $+\mathcal{O}(q^6)$	
$\vdots$		

What is the function space of “non-trivial” Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i \tau)$, $\tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

Four-Loop solutions

$q(y) = \exp(2\pi i \omega_2/\omega_1)$	$y - 8y^2 + 92y^3 - 1288y^4 + 20398y^5 + \mathcal{O}(y^6)$
$\omega = \omega_1$	$q + 3q^2 + q^3 + 23q^4 - 101q^5 + \mathcal{O}(q^6)$
$K_1 = \mathrm{d}^2/\mathrm{d}\tau^2(\omega_3/\omega_1)$	$1 - q + 17q^2 - 253q^3 + 3345q^4 - 43751q^5 + \mathcal{O}(q^6)$
J	$q + 16q^2 + 108q^3 + 672q^4 + 2570q^5 + \mathcal{O}(q^6)$
F_{32}	$c_{32} + 8q - 32q^2 + 512q^3 - 5872q^4 + 70008q^5 + \mathcal{O}(q^6)$
F_{42}	$c_{42} + 8q - 240q^2 + 4816q^3 - 90448q^4 + 1444008q^5$ $+ c_{32}(-9q + 176q^2 - 2956q^3 + 44568q^4 - 611106q^5)$ $+ c_{32}^2(q - 16q^2 + 220q^3 - 2600q^4 + 30018q^5)$ $+ \mathcal{O}(q^6)$
$\vdots$	

Predictable from just Picard–Fuchs operator (for $q(y)$, ω, K_1, J)

Need to solve constraints (for F_{32}, F_{42}, …)

Remaining freedom c_{32}, c_{42}, etc.
→ can impose symmetry on A

What is the function space of "non-trivial" Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i \tau)$, $\tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

Four-Loop solutions

$q(y) = \exp(2\pi i \omega_2/\omega_1)$	$y - 8y^2 + 92y^3 - 1288y^4 + 20398y^5 + \mathcal{O}(y^6)$
$\omega = \omega_1$	$q + 3q^2 + q^3 + 23q^4 - 101q^5 + \mathcal{O}(q^6)$
$K_1 = \mathrm{d}^2/\mathrm{d}\tau^2(\omega_3/\omega_1)$	$1 - q + 17q^2 - 253q^3 + 3345q^4 - 43751q^5 + \mathcal{O}(q^6)$
J	$q + 16q^2 + 108q^3 + 672q^4 + 2570q^5 + \mathcal{O}(q^6)$
F_{32}	$c_{32} + 8q - 32q^2 + 512q^3 - 5872q^4 + 70008q^5 + \mathcal{O}(q^6)$
F_{42}	$c_{42} + 8q - 240q^2 + 4816q^3 - 90448q^4 + 1444008q^5$ $+ c_{32}(-9q + 176q^2 - 2956q^3 + 44568q^4 - 611106q^5)$ $+ c_{32}^2(q - 16q^2 + 220q^3 - 2600q^4 + 30018q^5)$ $+ \mathcal{O}(q^6)$
$\vdots$	

Predictable from just Picard–Fuchs operator (the first four rows, above the dotted line)

Need to solve constraints (the rows below the dotted line)

Remaining freedom c_{32}, c_{42}, etc.
$\rightarrow$ can impose symmetry on A

Fast numerical evaluation
(Within convergence radius)

Five-, Six-, All-Loop Ansatz

Five-, Six-, All-Loop Ansatz

$$
\begin{aligned}
I_1 &= \varepsilon^{\ell} I_{1\ldots10}, \\
I_2 &= \varepsilon^{\ell} \frac{1}{\omega} I_{1\ldots1}, \\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3 \\
I_5 &= \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4 \\
&\vdots \\
I_{\ell-1} &= \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-2} + \sum_{i=2}^{\ell-2} F_{\ell-1,i} I_i \\
I_{\ell} &= \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-1} + \sum_{i=2}^{\ell-1} F_{\ell,i} I_i \\
I_{\ell+1} &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell} + \sum_{i=2}^{\ell} F_{\ell+1,i} I_i
\end{aligned}
$$

Five-, Six-, All-Loop Ansatz

$$
\begin{aligned}
I_1 &= \varepsilon^\ell I_{1\ldots10}, \\
I_2 &= \varepsilon^\ell \frac{1}{\omega} I_{1\ldots1}, \\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3 \\
I_5 &= \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4 \\
&\vdots \\
I_{\ell-1} &= \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-2} + \sum_{i=2}^{\ell-2} F_{\ell-1,i} I_i \\
I_\ell &= \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-1} + \sum_{i=2}^{\ell-1} F_{\ell,i} I_i \\
I_{\ell+1} &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_\ell + \sum_{i=2}^{\ell} F_{\ell+1,i} I_i
\end{aligned}
$$

Five-, Six-, All-Loop Ansatz

$$
\begin{aligned}
I_1 &= \varepsilon^\ell I_{1\ldots10},\\
I_2 &= \varepsilon^\ell \frac{1}{\omega} I_{1\ldots1},\\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,\\
I_4 &= \frac{1}{\varepsilon}\boxed{\frac{1}{K_1}}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3\\
I_5 &= \frac{1}{\varepsilon}\boxed{\frac{1}{K_2}}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4\\
&\vdots\\
I_{\ell-1} &= \frac{1}{\varepsilon}\boxed{\frac{1}{K_2}}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-2} + \sum_{i=2}^{\ell-2} F_{\ell-1,i} I_i\\
I_\ell &= \frac{1}{\varepsilon}\boxed{\frac{1}{K_1}}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-1} + \sum_{i=2}^{\ell-1} F_{\ell,i} I_i\\
I_{\ell+1} &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_\ell + \sum_{i=2}^{\ell} F_{\ell+1,i} I_i
\end{aligned}
$$

Checked up to seven loops

Ansatz with K_i being Y-invariants leads to consistent constraints

Five-, Six-, All-Loop Ansatz

$$
\begin{aligned}
I_1 &= \varepsilon^\ell I_{1\ldots10}, \\
I_2 &= \varepsilon^\ell \frac{1}{\omega} I_{1\ldots1}, \\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3 \\
I_5 &= \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4 \\
&\vdots \\
I_{\ell-1} &= \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-2} + \sum_{i=2}^{\ell-2} F_{\ell-1,i} I_i \\
I_\ell &= \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-1} + \sum_{i=2}^{\ell-1} F_{\ell,i} I_i \\
I_{\ell+1} &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_\ell + \sum_{i=2}^{\ell} F_{\ell+1,i} I_i
\end{aligned}
$$

Checked up to seven loops

Ansatz with K_i being Y-invariants leads to consistent constraints

Checked up to six loops

Analytic expressions for Masters in terms of iterated integrals

Five-, Six-, All-Loop Ansatz

$$
\begin{aligned}
I_1 &= \varepsilon^{\ell} I_{1\ldots10},\\
I_2 &= \varepsilon^{\ell} \frac{1}{\omega} I_{1\ldots1},\\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,\\
I_4 &= \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3\\
I_5 &= \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4\\
&\vdots\\
I_{\ell-1} &= \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-2} + \sum_{i=2}^{\ell-2} F_{\ell-1,i} I_i\\
I_{\ell} &= \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-1} + \sum_{i=2}^{\ell-1} F_{\ell,i} I_i\\
I_{\ell+1} &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell} + \sum_{i=2}^{\ell} F_{\ell+1,i} I_i
\end{aligned}
$$

Checked up to seven loops

Ansatz with K_i being Y-invariants leads to consistent constraints

Checked up to six loops

Analytic expressions for Masters in terms of iterated integrals

$$I_2 = \left[I(1,K_1,K_2,K_1,1,A_{71};\tau) + \text{boundary}\right]\varepsilon^7 + \mathcal{O}(\varepsilon^8) \quad \text{etc.}$$

Five-, Six-, All-Loop Ansatz

$$I_1 = \varepsilon^{\ell} I_{1\ldots10},$$

$$I_2 = \varepsilon^{\ell} \frac{1}{\omega} I_{1\ldots1},$$

$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$

$$I_4 = \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3$$

$$I_5 = \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4$$

$$\vdots$$

$$I_{\ell-1} = \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-2} + \sum_{i=2}^{\ell-2} F_{\ell-1,i} I_i$$

$$I_{\ell} = \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-1} + \sum_{i=2}^{\ell-1} F_{\ell,i} I_i$$

$$I_{\ell+1} = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell} + \sum_{i=2}^{\ell} F_{\ell+1,i} I_i$$

Checked up to seven loops

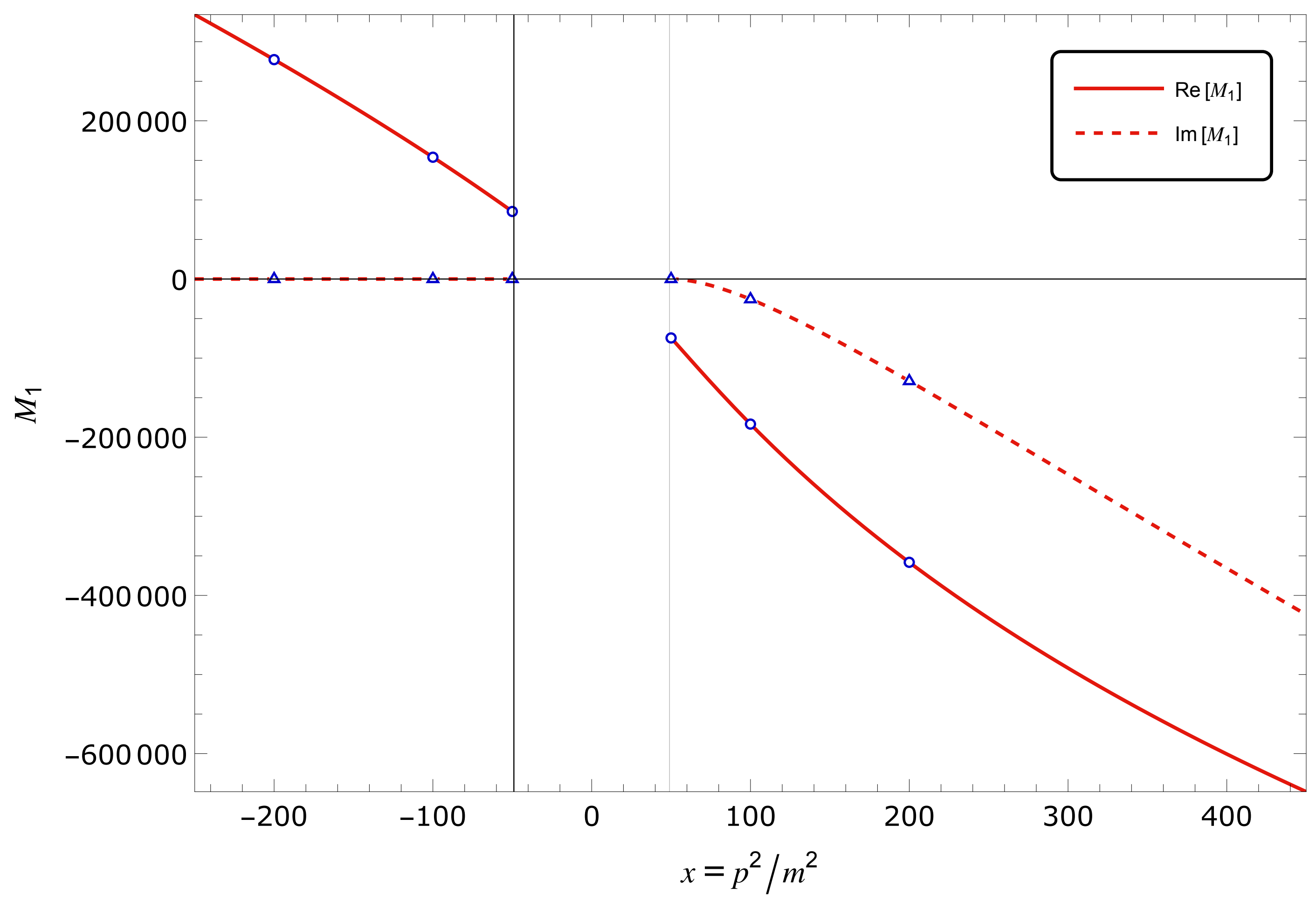
Ansatz with K_i being Y-invariants leads to consistent constraints

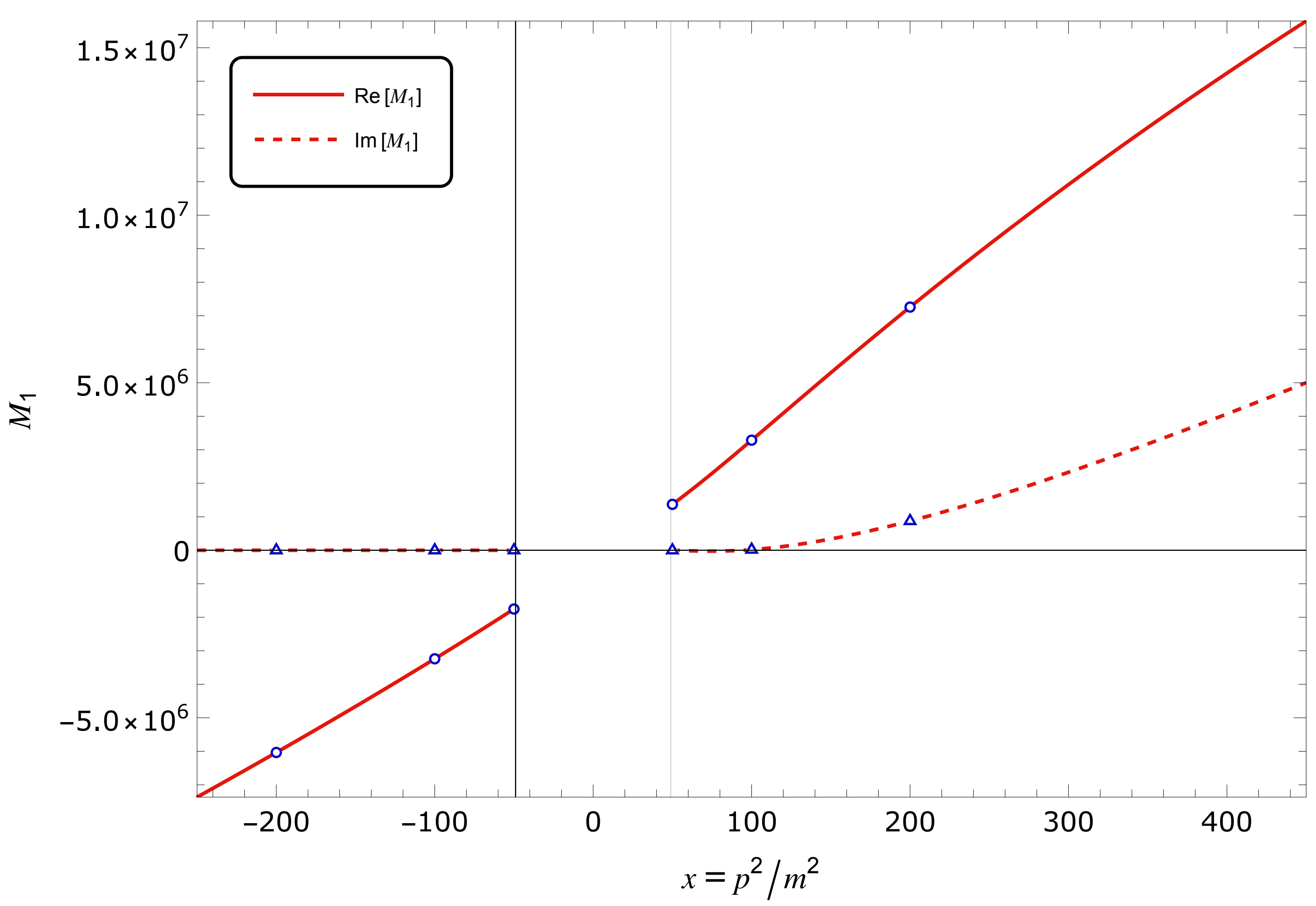
Checked up to six loops

Analytic expressions for Masters in terms of iterated integrals

$$I_2 = \left[I(1, K_1, K_2, K_1, 1, A_{71}; \tau) + \text{boundary} \right] \varepsilon^7 + \mathcal{O}(\varepsilon^8) \quad \text{etc.}$$

Numeric evaluation using q-expansion: agrees with `SecDec`





“Non-Trivial” Calabi–Yau Summary

“Non-Trivial” Calabi–Yau Summary

ε-factorized form:

“Non-Trivial” Calabi–Yau Summary

ε-factorized form: Ansatz with information from Calabi–Yau operators
$\rightarrow$ Solve constraints algorithmically

“Non-Trivial” Calabi–Yau Summary

ε**-factorized form:** Ansatz with information from Calabi–Yau operators
→ Solve constraints algorithmically

Function space: currently unknown

“Non-Trivial” Calabi–Yau Summary

ε**-factorized form:** Ansatz with information from Calabi–Yau operators
→ Solve constraints algorithmically

Function space: currently unknown

Numerics: can obtain fast converging q-expansion

“Non-Trivial” Calabi–Yau Summary

ε**-factorized form:** Ansatz with information from Calabi–Yau operators
→ Solve constraints algorithmically

Function space: currently unknown

Numerics: can obtain fast converging q-expansion

Expectation: Generalizes to other Calabi–Yau integrals

“Non-Trivial” Calabi–Yau Summary

ε-factorized form: Ansatz with information from Calabi–Yau operators
$\rightarrow$ Solve constraints algorithmically

Function space: currently unknown

Numerics: can obtain fast converging q-expansion

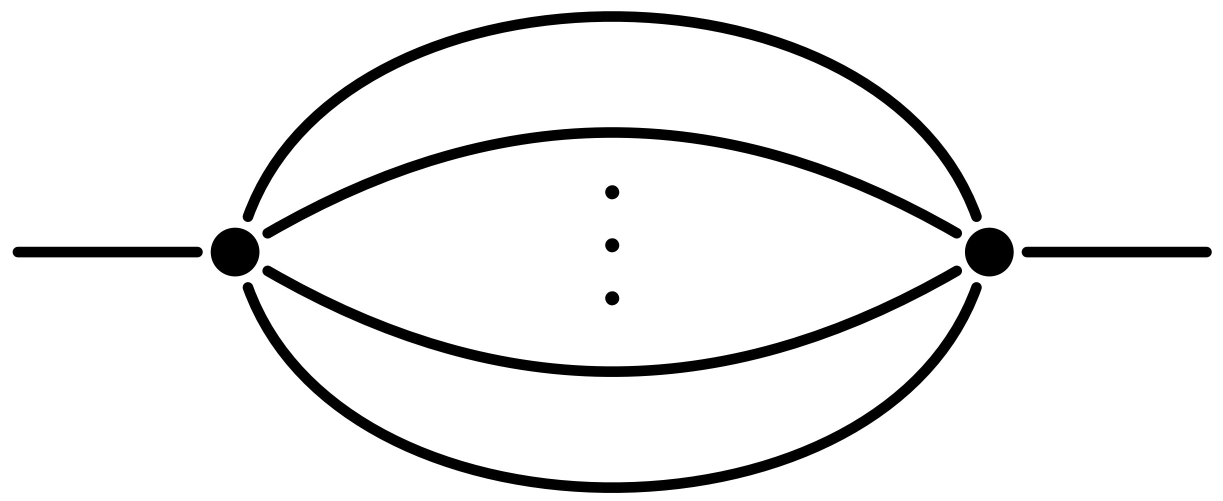
Expectation: Generalizes to other Calabi–Yau integrals

Single scale +
n-fold Calabi–Yau integral +
degree (n+1) Picard–Fuchs operator
$\rightarrow$
Calabi–Yau operator
“non-trivial” Calabi–Yau
$\rightarrow$
ε-factorised DEQ from Ansatz
Solve via series expansion

Conclusions

Conclusions

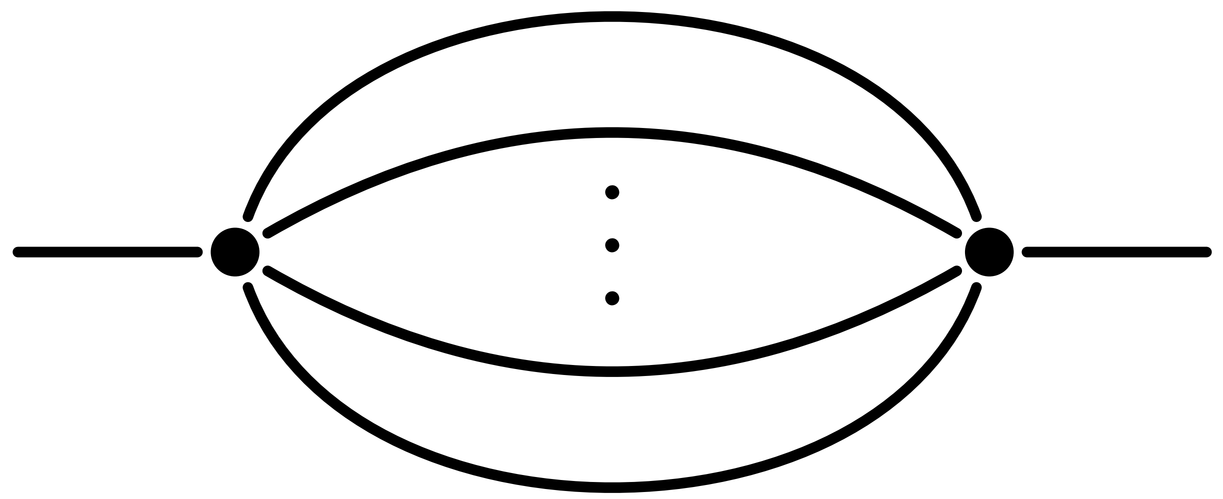
Banana integrals: Simplest example of Calabi–Yau integrals



Conclusions

Banana integrals: Simplest example of Calabi–Yau integrals

Simplification: Equal-mass = single scale

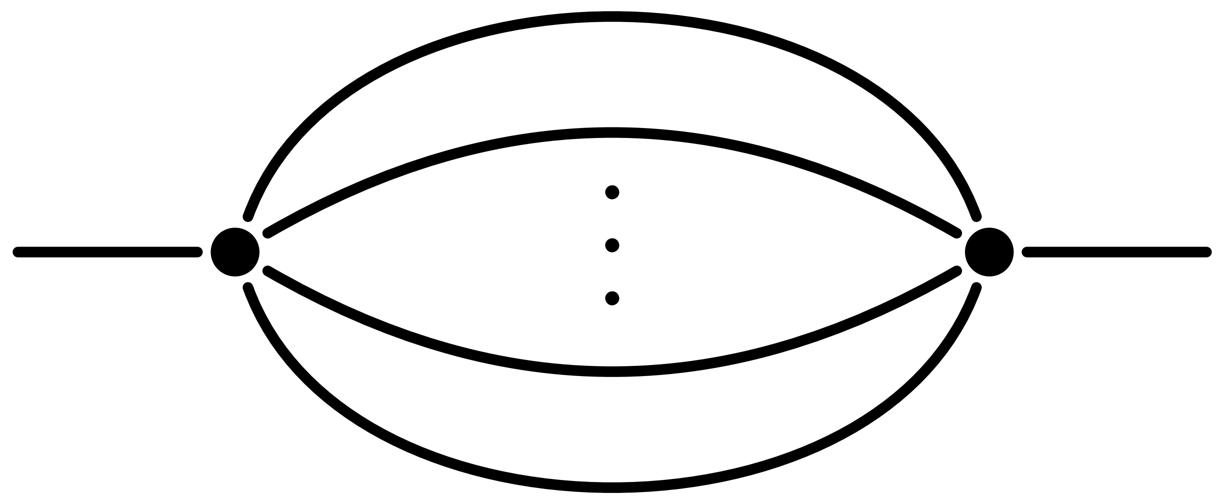


Conclusions

Banana integrals: Simplest example of Calabi–Yau integrals

Simplification: Equal-mass = single scale

Single scale integral
n-fold Calabi–Yau,
degree (n+1) Picard–Fuchs operator

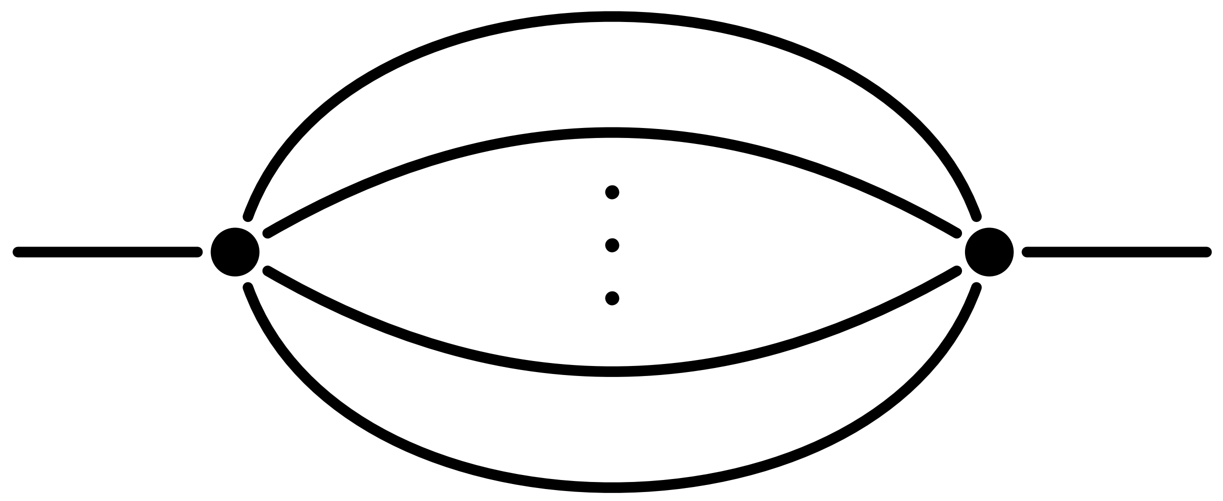


Conclusions

Banana integrals: Simplest example of Calabi–Yau integrals

Simplification: Equal-mass = single scale

Single scale integral
n-fold Calabi–Yau,
degree (n+1) Picard–Fuchs operator



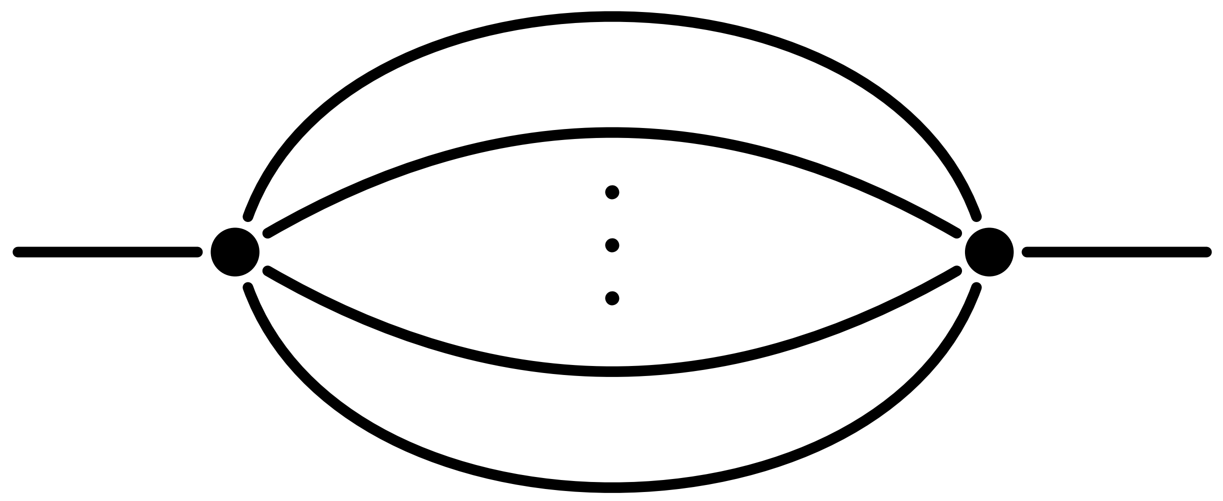
Ansatzing allows to find ε-factorised form algorithmically

Conclusions

Banana integrals: Simplest example of Calabi–Yau integrals

Simplification: Equal-mass = single scale

Single scale integral
n-fold Calabi–Yau,
degree (n+1) Picard–Fuchs operator



Ansatzing allows to find ε-factorised form algorithmically

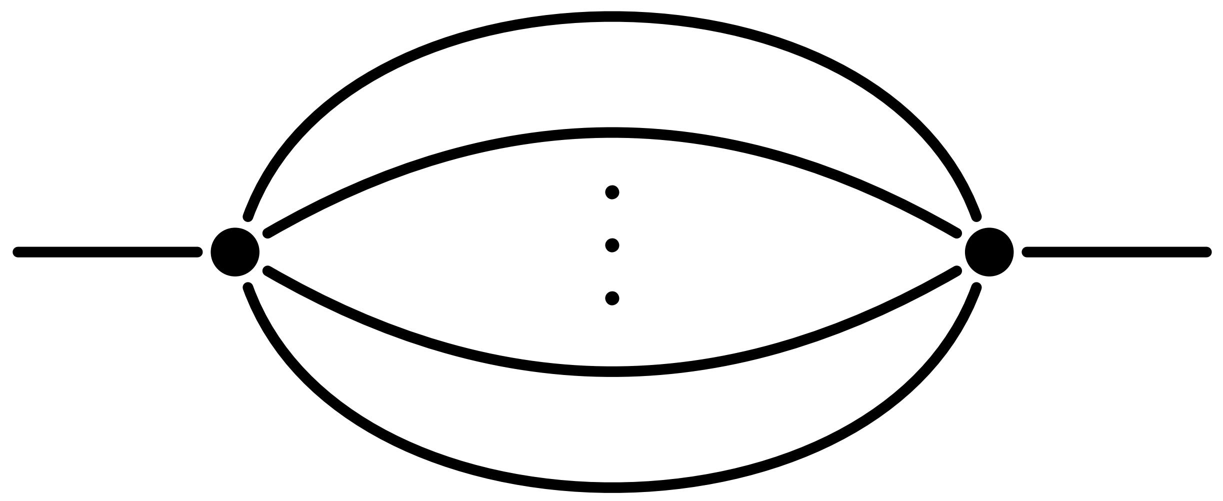
Use information from theory of Calabi–Yau operators

Conclusions

Banana integrals: Simplest example of Calabi–Yau integrals

Simplification: Equal-mass = single scale

Single scale integral
n-fold Calabi–Yau,
degree (n+1) Picard–Fuchs operator



Ansatzing allows to find ε-factorised form algorithmically

Use information from theory of Calabi–Yau operators

Calabi-Yau 2-fold

Picard–Fuchs is symmetric square of elliptic curve

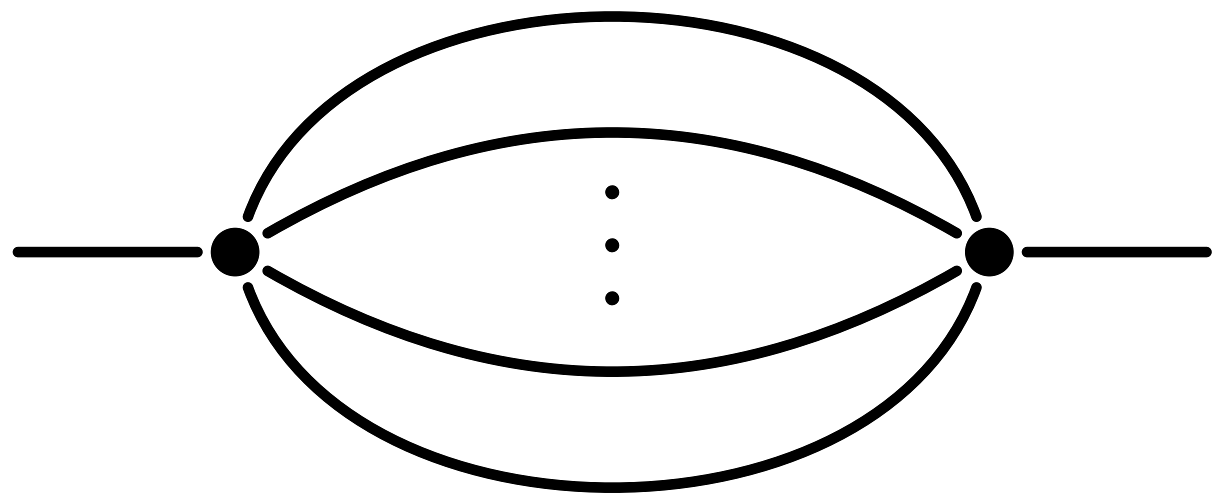
Modular forms

Conclusions

Banana integrals: Simplest example of Calabi–Yau integrals

Simplification: Equal-mass = single scale

Single scale integral
n-fold Calabi–Yau,
degree (n+1) Picard–Fuchs operator



Ansatzing allows to find ε-factorised form algorithmically

Use information from theory of Calabi–Yau operators

Calabi-Yau 2-fold

Picard–Fuchs is symmetric square of elliptic curve

Modular forms

Calabi-Yau (≥3)-fold

Not relatable to elliptics
Function space unknown

q-expansion

Backup

Calabi–Yau Operators

ℓ-loop Banana Integrals define special Calabi–Yau manifolds
Picard–Fuchs operator are called Calabi–Yau operators

Canonical coordinate:
q-coordinate or mirror-map

$$q = \exp(2\pi i \tau) \quad \tau = \frac{\omega_2}{\omega_1}$$

For $\ell = 2$
(*i.e.* sunrise/elliptic curve)
τ = ratio of periods
q = nome,

Picard–Fuchs operator in q-coordinate
Special Local Normal Form:
[M. Bogner, 13']

$$\mathcal{L}^{(2)} = \Theta_q^2$$

$$\mathcal{L}^{(3)} = \Theta_q^3$$

$$\mathcal{L}^{(4)} = \Theta_q^2 \frac{1}{Y_1} \Theta_q^2$$

$$\mathcal{L}^{(\ell)} = \Theta_q^2 \frac{1}{Y_1} \Theta_q \frac{1}{Y_2} \Theta_q \dots \Theta_q \frac{1}{Y_2} \Theta_q \frac{1}{Y_1} \Theta_q^2$$

Logarithmic derivative
$$\Theta_q = q\frac{\mathrm{d}}{\mathrm{d}q} = \frac{\mathrm{d}}{\mathrm{d}\log q} = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau}$$

For $\ell = 4$:
Y_1 known as
Yukawa coupling
in string theory

Y_i: Y-invariants of operator

Calabi-Yau 3-fold from graph polynomial

$$F^{(4)}_{11111} = e^{4\varepsilon\gamma_E} \cdot \Gamma(1+4\varepsilon) \cdot \int_{\alpha_i \geq 0} d^5\alpha\, \delta\left(1 - \sum_{i=1}^{5} \alpha_i\right) \frac{\mathcal{U}(\alpha)^{5\varepsilon}}{\mathcal{F}(\alpha)^{1+4\varepsilon}}$$

$$\mathcal{U}(\alpha) = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} + \frac{1}{\alpha_5}\right)$$

$$\mathcal{F}(\alpha) = x\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\mathcal{U}(\alpha)$$

$$\mathrm{CY}_3 = \left\{ [\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_5] \in \mathbb{CP}^4 \,\middle|\, x\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\mathcal{U}(\alpha) = 0 \right\}$$

Frobenius Basis

$$
\begin{aligned}
\omega_1 &= \Sigma_1 \\
\omega_2 &= \log y\, \Sigma_1 + \Sigma_2 \\
\omega_3 &= \tfrac{1}{2} \log y^2\, \Sigma_1 + \log y\, \Sigma_2 + \Sigma_3 \\
\omega_4 &= \tfrac{1}{3!} \log y^3\, \Sigma_1 + \tfrac{1}{2} \log y\, \Sigma_2 + \log y \Sigma_3 + \Sigma_4
\end{aligned}
$$

$$\Sigma_i \in \mathbb{Q}[[y]]$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^2 from $A_{4,2}$:

$$L_3\, \omega = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^0 from $A_{4,4}$:

$$\frac{d \ln J}{dx} = \frac{d \ln \omega}{dx} + \frac{2\left(x^2 - 15x + 32\right)}{x\left(x-4\right)\left(x-16\right)}$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^{-1} from $A_{4,3}$ (plus previous):

$$\frac{1}{\omega}\frac{d^2\omega}{dx^2} - \frac{1}{2}\left(\frac{1}{\omega}\frac{d\omega}{dx}\right)^2 + \frac{2\left(x^2 - 15x + 32\right)}{x\left(x-4\right)\left(x-16\right)}\frac{1}{\omega}\frac{d\omega}{dx} + \frac{\left(x-8\right)}{2x\left(x-4\right)\left(x-16\right)} = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & \color{red}{A_{4,2}} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^{-1} from $\color{red}{A_{4,2}}$:

$$\frac{d^2 F_{32}}{dx^2} + \left[\frac{d\ln\omega}{dx} + \frac{2\left(x^2 - 15x + 32\right)}{x\,(x-4)\,(x-16)}\right]\frac{dF_{32}}{dx} + \frac{3J}{2\pi i}\left[-\frac{(x-10)}{(x-4)\,(x-16)}\left(\frac{d\ln\omega}{dx}\right)^2 - \frac{2\left(x^3 - 30x^2 + 228x - 640\right)}{x\,(x-4)^2\,(x-16)^2}\frac{d\ln\omega}{dx} - \frac{\left(x^3 - 28x^2 + 168x - 384\right)}{x^2\,(x-4)^2\,(x-16)^2}\right] = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^0 from $A_{4,2}$:

$$\frac{dF_{42}}{dx} - 3F_{32}\frac{dF_{32}}{dx} + \frac{3J}{2\pi i}\frac{2(x-10)}{(x-4)(x-16)}\frac{dF_{32}}{dx}$$
$$+\frac{3J}{2\pi i}\left[\frac{2(x-10)}{(x-4)(x-16)}\frac{d\ln\omega}{dx} + \frac{2\left(x^3-30x^2+228x-640\right)}{x(x-4)^2(x-16)^2}\right]F_{32}$$
$$+\frac{J^2}{(2\pi i)^2}\left[-\frac{(11x+16)}{x^2(x-16)}\frac{d\ln\omega}{dx} - \frac{(11x-14)}{x^2(x-4)(x-16)}\right] = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^0 from $A_{4,3}$:

$$\frac{dF_{43}}{dx} + 2\frac{dF_{32}}{dx} + \frac{3J}{2\pi i}\left[-\frac{2\,(x-10)}{(x-4)\,(x-16)}\frac{d\ln\omega}{dx} - \frac{2\left(x^3-30x^2+228x-640\right)}{x\,(x-4)^2\,(x-16)^2}\right] = 0$$

Solution for Normalisation ω

First constraint is just Picard-Fuchs operator

$$L_3\,\omega = 0$$

Symmetric square

$$\omega_i \in \langle \psi_1^2,\, \psi_1\psi_2,\, \psi_2^2 \rangle$$

Second constraint is non-linear

$$\frac{1}{\omega}\frac{d^2\omega}{dx^2} - \frac{1}{2}\left(\frac{1}{\omega}\frac{d\omega}{dx}\right)^2 + \frac{2\left(x^2-15x+32\right)}{x\left(x-4\right)\left(x-16\right)}\frac{1}{\omega}\frac{d\omega}{dx} + \frac{\left(x-8\right)}{2x\left(x-4\right)\left(x-16\right)} = 0$$

$$\omega_i \in \langle \psi_1^2,\, \cancel{\psi_1\psi_2},\, \psi_2^2 \rangle$$

We choose: $\omega = \psi_1^2$

Next, Fix Kinematic Variable τ

With $\omega = \psi_1^2$ the constraint for τ is

$$\frac{d\ln J}{dx} = \frac{d\ln\omega}{dx} + \frac{2\left(x^2 - 15x + 32\right)}{x\,(x-4)\,(x-16)}$$

As hoped, satisfied by $\tau = \frac{\psi_2}{\psi_1} = \frac{\psi_2^{\text{sun}}}{\psi_1^{\text{sun}}}$ $\quad J = \frac{\psi_1^2}{W}$

Wronskian $W = \psi_1 \frac{d}{dx}\psi_2 - \psi_2 \frac{d}{dx}\psi_1$

Constraints for F_{32}, F_{42}, F_{43}

Remaining differential equations are fulfilled for

$$F_{32} = F_2 - \frac{\pi i \left(x-10\right)}{\left(x-4\right)\left(x-16\right)W}\left(\frac{\psi_1}{\pi}\right)^2$$

$$F_{42} = \frac{3}{2}F_2^2 + \frac{\pi^2\left(x+8\right)^2\left(x^2-8x+64\right)}{8x^2\left(x-4\right)^2\left(x-16\right)^2W^2}\left(\frac{\psi_1}{\pi}\right)^4$$

$$F_{43} = -2F_2 - \frac{\pi i \left(x-10\right)}{\left(x-4\right)\left(x-16\right)W}\left(\frac{\psi_1}{\pi}\right)^2$$

All depend on one additional function F_2

F_2 has to satisfy

$$\frac{d^2F_2}{dx^2}+\left[\frac{2\left(x^2-15x+32\right)}{x\left(x-4\right)\left(x-16\right)}+2\left(\frac{d\ln\psi_1}{dx}\right)\right]\frac{dF_2}{dx}=\frac{\pi i\left(x-8\right)\left(x+8\right)^3}{x^2\left(x-4\right)^3\left(x-16\right)^3W}\left(\frac{\psi_1}{\pi}\right)^2$$

Solution: Iterated integral of meromorphic modular form of weight 6!

$$F_2=\left(2\pi i\right)^2\int_{i\infty}^{\tau}d\tau_1\int_{i\infty}^{\tau_1}d\tau_2\,\underbrace{\frac{x\left(x-8\right)\left(x+8\right)^3}{864\left(4-x\right)^{\frac{3}{2}}\left(16-x\right)^{\frac{3}{2}}}\left(\frac{\psi_1}{\pi}\right)^6}_{g_6}$$

Properties:

- $\overline{q}$ expansion of g_6 has only integer coefficients
- $\overline{q}^n$ coefficient of g_6 divisible by n^2
- Carrying out integration, F_2 has simple poles at $x=4,\ 16$

Basis of Modular Forms

Two classes

Holomorphic: $b_0 = \frac{\psi_1^{\text{sun}}}{\pi}$ $b_1 = y\frac{\psi_1^{\text{sun}}}{\pi}$

Meromorphic: $b_3 = \frac{1}{(y-3)}\frac{\psi_1^{\text{sun}}}{\pi}$ $b_{-3} = \frac{1}{(y+3)}\frac{\psi_1^{\text{sun}}}{\pi}$

Poles corresponding to (pseudo-)threshold

Can use these to express all modular forms appearing

Example:
$$
\begin{aligned}
f_{2,a} &= \left(\frac{1}{x-4}+\frac{1}{x-16}\right)\frac{\psi_1^2}{2\pi i W} \\
&= \left[\frac{1}{6}y^2-\frac{5}{3}y+\frac{9}{2}-\frac{6}{y-3}-\frac{24}{y+3}\right]\left(\frac{\psi_1^{\text{sun}}}{\pi}\right)^2 \\
&= \frac{1}{6}b_1^2-\frac{5}{3}b_0b_1+\frac{9}{2}b_0^2-6b_0b_3-24b_0b_{-3}.
\end{aligned}
$$

Letter $f_{2,b}$ is not a modular form, but iterated integral of one: non-trivial transformation under $\Gamma_1(6)$

Path decomposition gives us

$$(f_{2,b}|_2\gamma)(\tau) = f_{2,b}(\tau)$$
$$-6\frac{c}{c\tau+d}\frac{1}{2\pi i}I(1,1,g_6;\tau) + 18\left(\frac{c}{c\tau+d}\right)^2\frac{1}{(2\pi i)^2}I(1,1,1,g_6;\tau)$$
$$-24\left(\frac{c}{c\tau+d}\right)^3\frac{1}{(2\pi i)^3}I(1,1,1,1,g_6;\tau)$$
$$+\frac{C_{1,6}}{(c\tau+d)^2} - \frac{2\pi i C_6}{c(c\tau+d)^3}$$

Constants:
$$C_{1,6} = I\left(1,g_6;i\infty,\frac{a}{c}\right)$$
$$C_6 = I\left(g_6;i\infty,\frac{a}{c}\right)$$

Singularities obstruct simple evaluation

E.g. $a/c = 1/6$: $C_{1,6} = 5$, $C_6 = \frac{1620\zeta_3}{\pi^4} - i\frac{42}{\pi}$

Defining "Quasi-Eichler" of weight k, depth p:

$$(f|_k\gamma)(\tau) = f(\tau) + \sum_{j=1}^{p}\left(\frac{c}{c\tau+d}\right)^j f_j(\tau) + \frac{P_\gamma(\tau)}{(c\tau+d)^p}$$

$f_{2,b}$ transforms "Quasi-Eichler" of modular weight 2 and depth 3

Solution for Master Integrals

Initial condition of I_{1111} in limit $1/x \to 0$ from Mellin-Barnes representation

Master integrals to arbitrary power in ε as iterated integrals over $\{1, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_6\}$

$$I\left(f_1, \ldots, f_n; \tau\right) = (2\pi i)^n \int_{i\infty}^{\tau} d\tau_1 \ldots \int_{i\infty}^{\tau_{n-1}} d\tau_n \; f_1\left(\tau_1\right) \ldots f_n\left(\tau_n\right)$$

e.g., with $I_2 = \varepsilon^3 \frac{\pi^2}{\psi_1^2} I_{1111} = \varepsilon^3 I_2^{(3)} + \varepsilon^4 I_2^{(4)} + \mathcal{O}(\varepsilon^5)$

$$I_2^{(3)} = \frac{4}{3}\zeta_3 + I\left(1, 1, f_{4,a}; \tau\right)$$

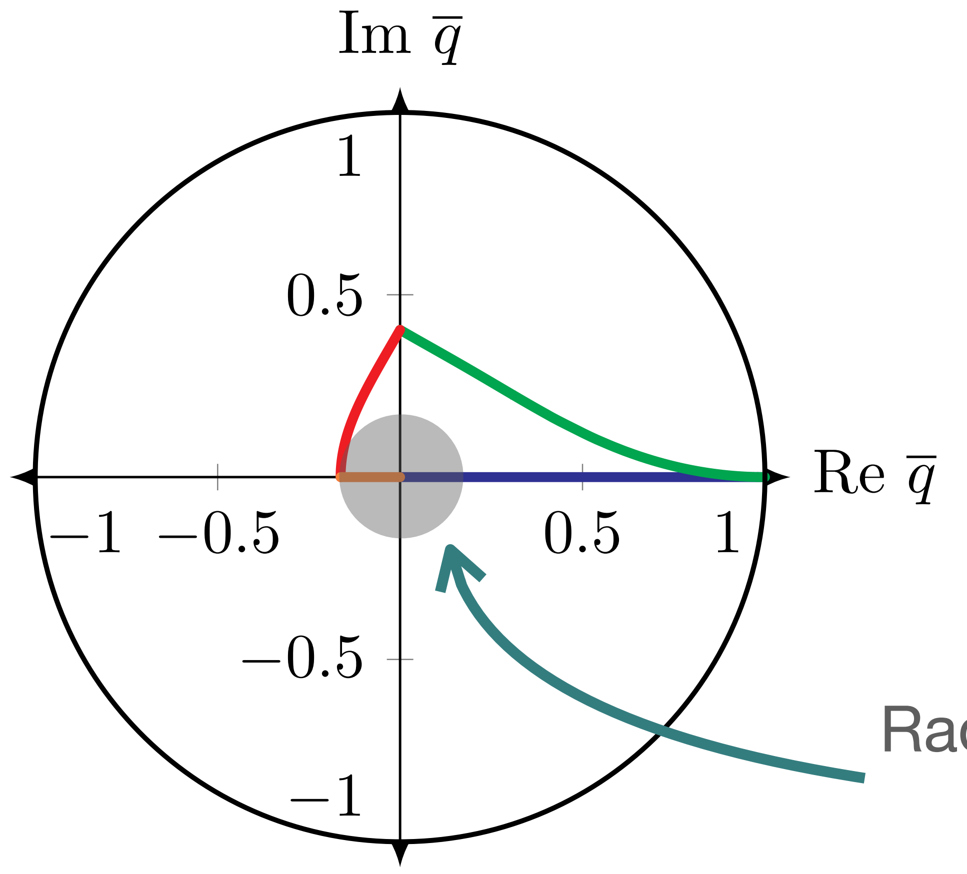
Holomorphic, agrees with [Bloch, Kerr, Vanhove]

$$\begin{aligned} I_2^{(4)} = {} & 2\zeta_4 + \frac{4}{3}\zeta_3 \left[\frac{11}{2}\ln\left(\bar{q}\right) - I\left(f_{2,a}; \tau\right) - I\left(f_{2,b}; \tau\right)\right] + \zeta_2 \ln^2\left(\bar{q}\right) - I\left(1, 1, f_{2,a}, f_{4,a}; \tau\right) \\ & - I\left(1, f_{2,a}, 1, f_{4,a}; \tau\right) - I\left(f_{2,a}, 1, 1, f_{4,a}; \tau\right) - I\left(1, 1, f_{2,b}, f_{4,a}; \tau\right) \\ & + 2I\left(1, f_{2,b}, 1, f_{4,a}; \tau\right) - I\left(f_{2,b}, 1, 1, f_{4,a}; \tau\right) \end{aligned}$$

Obtained explicit expressions for all master integrals up to ε^6

All integrals have uniform length

Numeric Verification

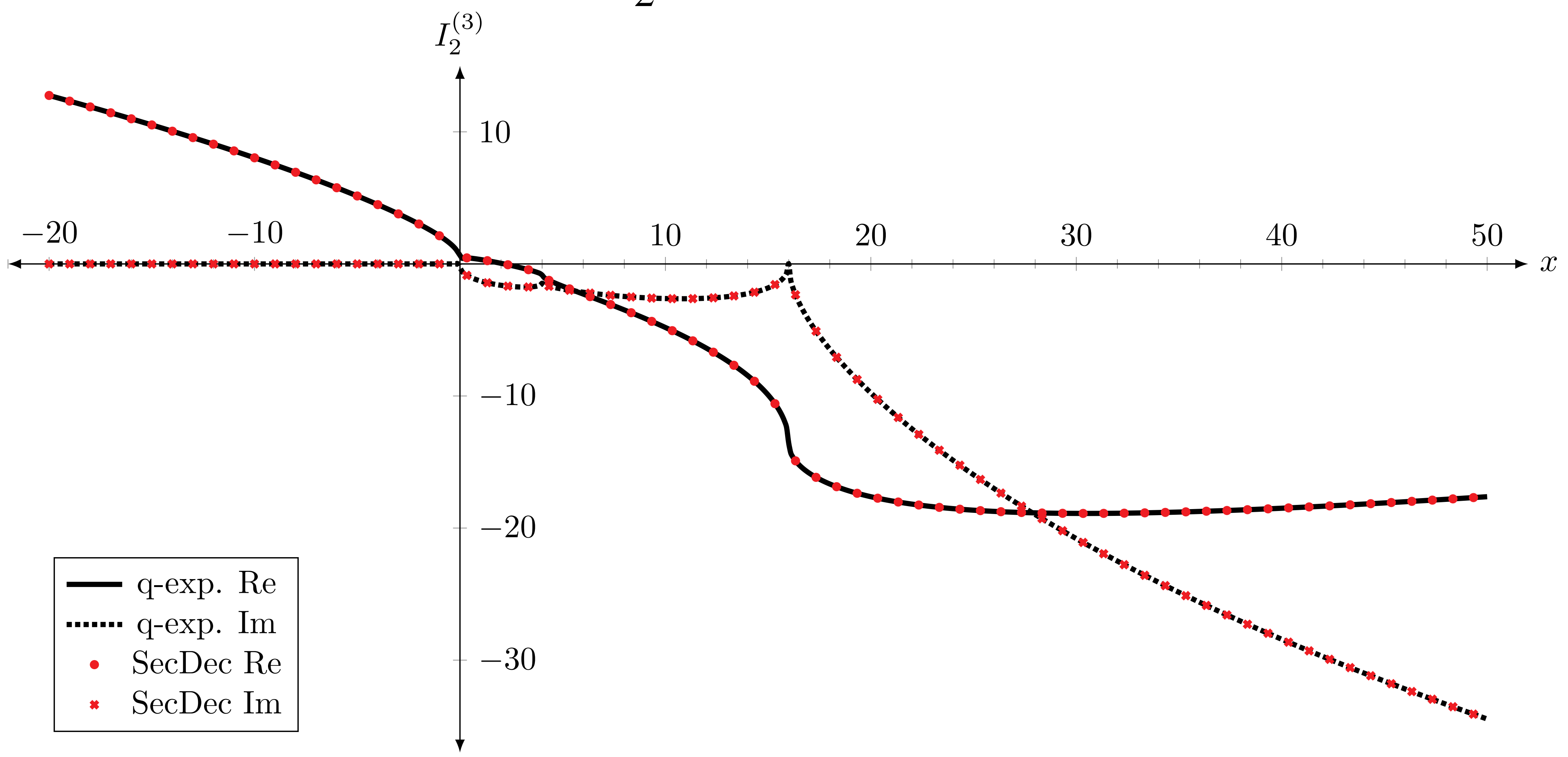


Numeric evaluation via $\overline{q}$-expansion

Singularities limit radius of convergence

Comparison against `SecDec`

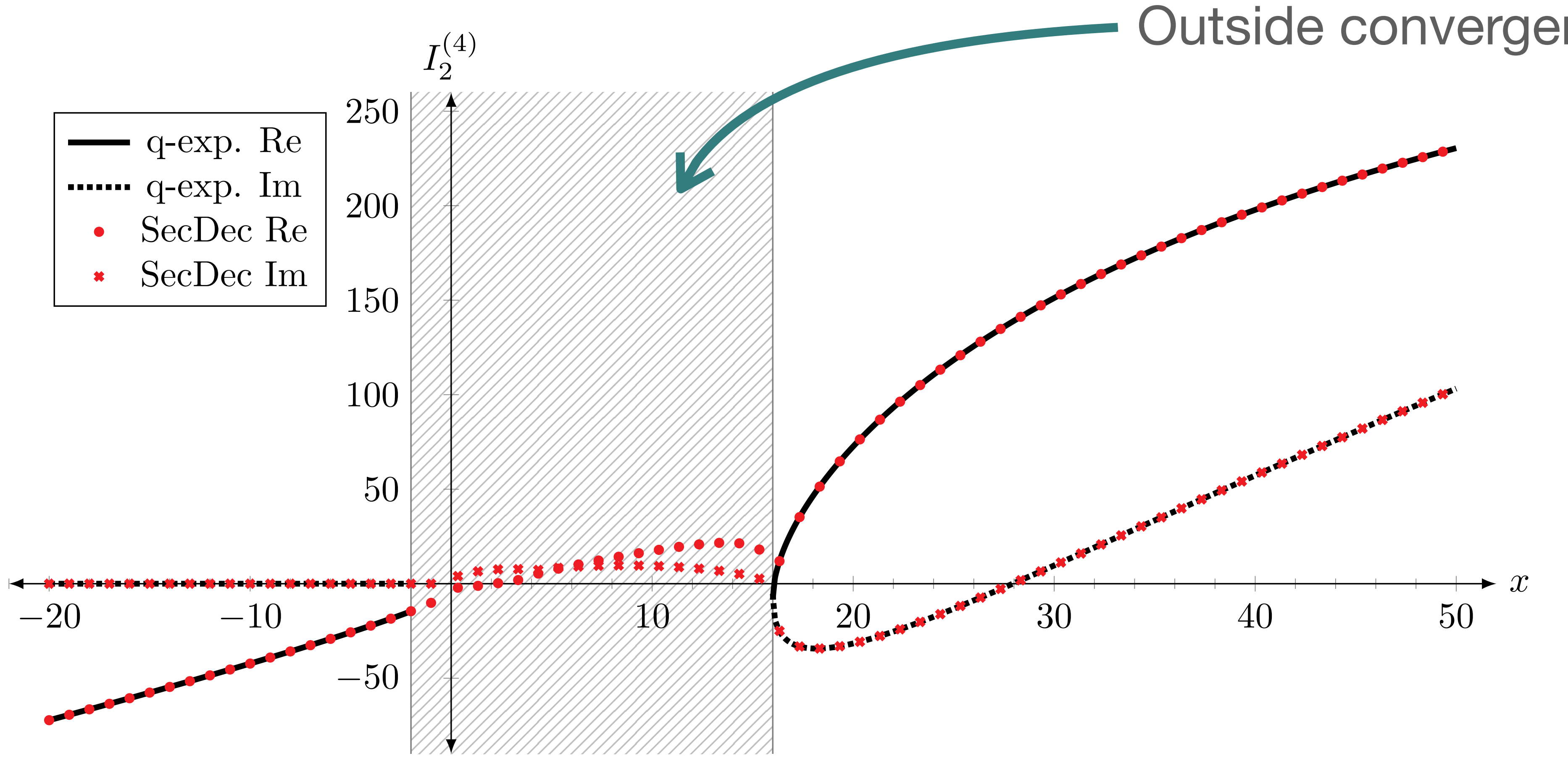
I_2, order ε^3



Only $f_{4,a}$, therefore holomorphic

I_2, order ε^4

Outside convergence radius



Also meromorphic $f_{2,a}, f_{2,b}$