Minimally divergent integral bases and special relations

Pavel Novichkov (IPhT CEA/Saclay)

Work with David Kosower, Giulio Gambuti, and Lorenzo Tancredi

QCD Meets Gravity 2022, December 15

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or

Finite Integrals and Where to Find Them

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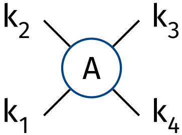
Motivation



$$= c_1 \mathrm{Master}_1 + \cdots + c_N \mathrm{Master}_N$$

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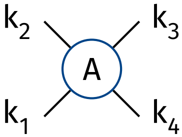

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1. Minimally divergent bases: reduce the number of divergent masters

   Q: Which integrals are finite?

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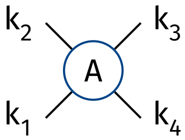


$$= c_1 \text{Master}_1 + \cdots + c_N \text{Master}_N$$

1. Minimally divergent bases: reduce the number of divergent masters

   Q: Which integrals are finite?

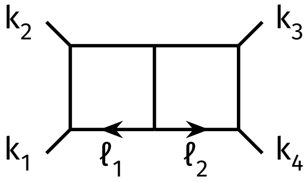
2. Special relations: remove masters which are redundant to $O(\epsilon)$

   Q: Which integrals are $O(\epsilon)$?

Problem statement

Num = $\text{Poly}\left(\ell_i \cdot \ell_j, \ell_i \cdot k_j\right)$ with coefficients being $\text{Rational}\left(k_i \cdot k_j\right)$

$$\int d\ell \, \frac{\text{Num}}{\text{Den}_1 \cdots \text{Den}_E}$$



Den = $\left(\sum \pm\ell_i \pm k_j\right)^2 - m^2 + i\varepsilon$

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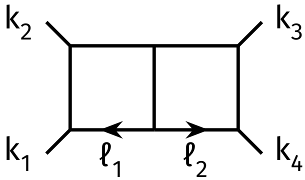
$\text{Num} = \text{Poly}\left(\ell_i \cdot \ell_j, \ell_i \cdot k_j\right)$ with coefficients being $\text{Rational}\left(k_i \cdot k_j\right)$

Find Num such that $\int d\ell \, \frac{\text{Num}}{\text{Den}_1 \cdots \text{Den}_E} = O\left(\epsilon^r\right)$



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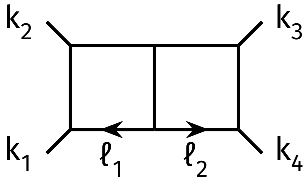

Find Num such that $\int d\ell \, \frac{\text{Num}}{\text{Den}_1 \cdots \text{Den}_E} = O\left(\epsilon^r\right)$

$\text{Den} = \left(\sum \pm \ell_i \pm k_j\right)^2 - m^2 + i\varepsilon$

This talk:
$r = 0$ (finite)
$r = 1$ (4d-vanishing)

UV divergences

Weinberg’s theorem

An integral is UV-finite, if:

- it converges superficially
- all its subintegrations converge superficially

[Weinberg 1960; Hahn, Zimmermann 1968]

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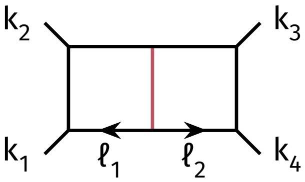
Subintegration =
hold a subset of edge momenta fixed



$\ell_1 + \ell_2$ = fixed

UV-finite numerators

- form a linear space

$$\text{coef}_1 \cdot \text{Num}_1 + \text{coef}_2 \cdot \text{Num}_2 \quad \text{is UV-finite}$$

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- overall divergence → upper bound on order in ℓ

$$\text{Num}\left(\ell_1, \ell_2\right) = c_1 + c_2\left(\ell_1 \cdot k_1\right) + \cdots + c_N\left(\ell_2^2\right)^2\left(\ell_2 \cdot k_3\right)$$

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- subdivergences → linear constraints on c_i

$$\text{Num}\left(\ell_1, \lambda\ell_2\right) = \underbrace{\left(c_N \ldots\right)}_{=0}\lambda^5 + (\ldots)\,\lambda^4 + \ldots$$

IR divergences

UV vs IR

UV

$$\left\{ \begin{array}{l} \ell_1 = \infty, \\ \ell_1 + \ell_2 = C \end{array} \right\}$$

divergent surface

$$\left\{ \begin{array}{l} \ell_1 \to \lambda \ell_1, \\ \ell_2 \to C - \lambda \ell_1 \end{array} \right\}$$

power-counting rule

UV vs IR

UV

$$\begin{Bmatrix} \ell_1 = \infty, \\ \ell_1 + \ell_2 = C \end{Bmatrix} \qquad \begin{Bmatrix} \ell_1 \to \lambda \ell_1, \\ \ell_2 \to C - \lambda \ell_1 \end{Bmatrix}$$

divergent surface — power-counting rule

IR

$$\begin{Bmatrix} \ell_1 = k_1, \\ \ell_2 = x k_4 \end{Bmatrix} \qquad \begin{Bmatrix} \ell_1 \to k_1 + \lambda^2 \ell_s, \\ \ell_2 \to x k_4 + \lambda^2 \eta_4 + \lambda \ell_\perp \end{Bmatrix}$$

[Agarwal, Magnea et al. 2021; Collins 2011; see also Anastasiou, Sterman 2018]

IR-divergent surfaces

$$\int_{-1}^{1} \frac{dx}{x}$$

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$$\lim_{\varepsilon \to +0} \int_{-1}^{1} \frac{dx}{x + i\varepsilon}$$

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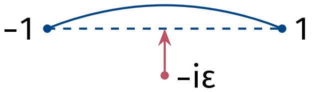
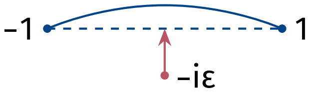


IR-divergent surfaces

no divergence

$$\lim_{\varepsilon \to +0} \int_{-1}^{1} \frac{dx}{x + i\varepsilon}$$



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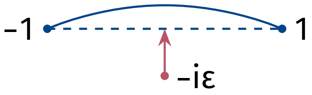
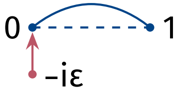


end-point divergence

$$\lim_{\varepsilon \to +0} \int_{0}^{1} \frac{dx}{x + i\varepsilon}$$



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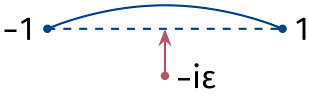

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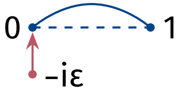
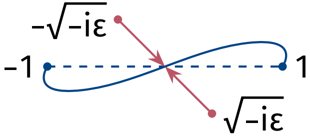


pinch divergence

$$\lim_{\varepsilon \to +0} \int_{-1}^{1} \frac{dx}{x^2 + i\varepsilon}$$



Landau equations

mixed representation

$$\int d\ell \int_0^\infty d\alpha \, \frac{\cdots}{\underbrace{(\alpha_1 \mathrm{Den}_1 + \cdots + \alpha_E \mathrm{Den}_E)}_{Q(\alpha,\ell)}^{\cdots}}$$

$$\Downarrow$$

$$\alpha_e = 0 \quad \text{or} \quad \frac{\partial}{\partial \alpha_e} Q(\alpha, \ell) = 0,$$

$$\frac{\partial}{\partial \ell_i} Q(\alpha, \ell) = 0$$

[Bjorken 1959; Landau 1959; Nakanishi 1959; see also Collins 2020]

Landau equations

mixed representation

Feynman parameter representation

$$\int d\ell \int_0^\infty d\alpha \, \frac{\cdots}{\underbrace{(\alpha_1 \mathrm{Den}_1 + \cdots + \alpha_E \mathrm{Den}_E)}_{Q(\alpha,\ell)}^{\cdots}} \quad \longrightarrow \quad \int_0^\infty d\alpha \, \frac{\cdots}{F(\alpha)^{\cdots}}$$

$$\alpha_e = 0 \quad \text{or} \quad \frac{\partial}{\partial \alpha_e} Q(\alpha, \ell) = 0, \qquad \frac{\partial}{\partial \ell_i} Q(\alpha, \ell) = 0$$

$$\longleftarrow \quad \alpha_e = 0 \quad \text{or} \quad \frac{\partial}{\partial \alpha_e} F(\alpha) = 0$$

[Bjorken 1959; Landau 1959; Nakanishi 1959; see also Collins 2020]

Two types of solutions

1. kinematics-independent → divergences

Two types of solutions

1. kinematics-independent → divergences

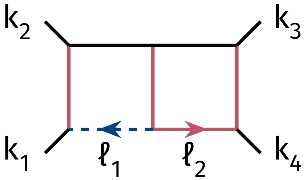
2. kinematics-dependent → Landau singularities [see William's talk]

$$\text{Integral} \supset \log\left(\frac{m^2 - s}{m^2}\right) \Rightarrow \text{singularity at } s = m^2$$

General structure of solutions

Edge momenta

soft	collinear to k_i	hard
$\ell = 0$	$\ell = xk_i$	
$\ell \to \lambda^2 \ell_s$	$\ell \to xk_i + \lambda^2 \eta_i + \lambda \ell_\perp$	



$$\ell_1 = 0, \quad \ell_2 = xk_4$$

[Coleman, Norton 1965; Sterman 1978; Libby, Sterman 1978]

IR-finite numerators

- linear constraints on c_i as in the UV case

$$\text{Num}(\lambda^2 \ell_s, x k_4 + \lambda^2 \eta_4 + \lambda \ell_\perp) = \underbrace{\left(c_i \ldots\right)}_{=0} + (\ldots)\,\lambda + \ldots$$

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- form a polynomial ideal

$$\text{poly}_1 \cdot \text{Num}_1 + \text{poly}_2 \cdot \text{Num}_2 \quad \text{is IR-finite}$$

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- form a polynomial ideal

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Conjecture: IR-finite ideal
can be built using Gram determinants

$$G\begin{pmatrix} p_1 \cdots p_n \\ q_1 \cdots q_n \end{pmatrix} = \det\left(2p_i \cdot q_j\right)$$

$O(\epsilon)$ numerators

1. Start with the most general finite numerator

$$\text{Num}(\ell) = c_1 \text{Num}_1 + \cdots + c_N \text{Num}_N$$

$\mathcal{O}(\epsilon)$ numerators

1. Start with the most general finite numerator

$$\text{Num}(\ell) = c_1 \text{Num}_1 + \cdots + c_N \text{Num}_N$$

2. Restrict it to 4d loop momenta

$$\ell = b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_\perp \qquad k_\perp \cdot k_i = 0$$

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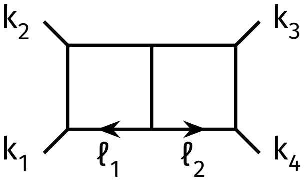
Results

Result: planar double box

max. order in ℓ	1	2	3	4	5
# finite integrals	0	2	18	89	247
# $O(\epsilon)$ integrals	0	0	0	1	7



31 IR-div. surfaces

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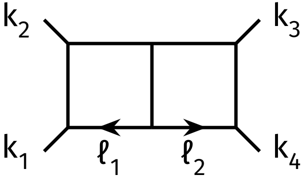

31 IR-div. surfaces

$$
\begin{aligned}
\mathrm{Num} = {} & \frac{1}{2}(s_{12}+s_{23})(\ell_1\cdot\ell_2) + (\ell_1\cdot k_3)(\ell_2\cdot k_1) + (\ell_1\cdot k_3)(\ell_2\cdot k_3) \\
& - \frac{s_{23}}{s_{12}}(\ell_1\cdot k_1)(\ell_2\cdot k_1) - \frac{s_{23}}{s_{12}}(\ell_1\cdot k_1)(\ell_2\cdot k_3) \\
& - \left(1+\frac{s_{23}}{s_{12}}\right)(\ell_1\cdot k_1)(\ell_2\cdot k_2) + \left(1+\frac{s_{23}}{s_{12}}\right)(\ell_1\cdot k_2)(\ell_2\cdot k_3)
\end{aligned}
$$

Result: planar double box

order 2

$$G\begin{pmatrix}\ell_1 & k_1 & k_2\\ \ell_2 & k_3 & k_4\end{pmatrix} \qquad G\begin{pmatrix}\ell_1 & k_1 & k_2\\ k_1 & k_2 & k_4\end{pmatrix} G\begin{pmatrix}\ell_2 & k_3 & k_4\\ k_1 & k_2 & k_4\end{pmatrix}$$

order 3

$$(\ell_1 - k_1)^2\, G\begin{pmatrix}\ell_2 & k_3 & k_4\\ k_1 & k_2 & k_4\end{pmatrix} \qquad G\begin{pmatrix}\ell_1 & k_1 & k_2\\ k_1 & k_2 & k_4\end{pmatrix} G\begin{pmatrix}\ell_2 & k_1 & k_2 & k_4\end{pmatrix}$$

$$(\ell_2 - k_4)^2\, G\begin{pmatrix}\ell_1 & k_1 & k_2\\ k_1 & k_2 & k_4\end{pmatrix} \qquad G\begin{pmatrix}\ell_2 & k_3 & k_4\\ k_1 & k_2 & k_4\end{pmatrix} G\begin{pmatrix}\ell_1 & k_1 & k_2 & k_4\end{pmatrix}$$

order 4

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$$(\ell_2 - k_4)^2\, G\begin{pmatrix} \ell_1 & k_1 & k_2 \\ k_1 & k_2 & k_4 \end{pmatrix} \qquad G\begin{pmatrix} \ell_2 & k_3 & k_4 \\ k_1 & k_2 & k_4 \end{pmatrix} G\left(\ell_1 \quad k_1 \quad k_2 \quad k_4\right)$$

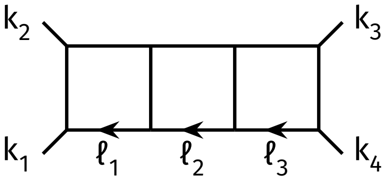
order 4

$$(\ell_1 - k_1)^2\, G\left(\ell_2 \quad k_3 \quad k_4\right) \qquad (\ell_1 - k_1)^2\,(\ell_2 - k_4)^2$$

$$(\ell_2 - k_4)^2\, G\left(\ell_1 \quad k_1 \quad k_2\right) \qquad \boxed{G\left(\ell_1 \quad \ell_2 \quad k_1 \quad k_2 \quad k_4\right)} \quad O(\epsilon)$$

Result: 3-loop ladder

max. order in ℓ	1	2	3	4	5	6	7
# finite integrals	0	2	26	184	850	2807	6044
# Gram generators	0	2	6	9	–	–	–
# $O(\epsilon)$ integrals	0	0	0	4	42	?	?



71 IR-div. surfaces

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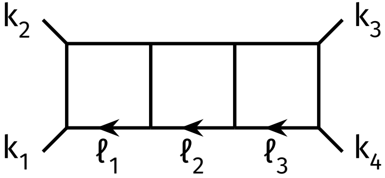

71 IR-div. surfaces

$$G\begin{pmatrix}\ell_1 & \ell_2 & k_1 & k_2 & k_4\\ \ell_1 & \ell_3 & k_1 & k_2 & k_4\end{pmatrix} \quad G\begin{pmatrix}\ell_1 & \ell_2 & k_1 & k_2 & k_4\\ \ell_2 & \ell_3 & k_1 & k_2 & k_4\end{pmatrix}$$

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Summary

1. We have developed an algorithmic procedure for finding sets of finite and $O(\epsilon)$ integrals for a given diagram

2. IR-finite integrals can be compactly described in terms of generating numerators

3. These integrals can be used to construct minimally divergent bases and to find special relations on the masters