All-orders celestial OPEs from the worldsheet

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14 Dec, 2022

QCD meets gravity 2022
Based on arxiv:2111.02279, 2111.15584, 2211.17124
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Quick background on celestial holography

Codimension 2 correspondence between quantum gravity in 4d Minkowski space and some conformal field theory on 2d celestial sphere.

$$\text{Observation: } \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,1)$$

Conformal primary operators $\mathcal{O}_{h,\bar{h}}(z,\bar{z})$ in usual 2d CFTs diagonalizes dilation $(\Delta = h + \bar{h})$ and rotation $(s = h - \bar{h})$.

The analogous object in 4d diagonalizes Lorentz boost and Lorentz rotation. For massless particles, parametrize null momenta

$$p_{\alpha\dot{\alpha}} = \kappa_\alpha \tilde{\kappa}_{\dot{\alpha}} = (1,\ z)^{\mathsf{T}}\ \omega\,(1\ \ \bar{z})\,. \tag{1}$$

Single particle state labelled by $|\omega, z, \bar{z}, s\rangle$, diagonalize Lorentz boost by Mellin transform

$$\int_0^\infty \frac{\mathrm{d}\omega}{\omega}\omega^{\Delta}|\omega, z, \bar{z}, s\rangle = |\Delta, z, \bar{z}, s\rangle\,. \tag{2}$$

[Pasterski-Shao-Strominger '17] [Pasterski-Shao-Strominger '17]

Quick background on celestial holography

The *dynamical* statement is:

$$\text{Scattering amplitudes } A(p_1(z_1,\omega_1),\dots,p_n(z_n,\omega_n)) \text{ in 4d}$$
$$\iff \text{Correlation function } \tilde{A}(\mathcal{O}_1(z_1,\Delta_1),\dots,\mathcal{O}_n(z_n,\Delta_n)) \text{ in 2d}$$

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In (2,2)-signature, the OPE limit $z_i \to z_j$ in 2d implies $p_i \cdot p_j = \omega_i\omega_j z_{ij}\bar{z}_{ij} \to 0$ in 4d.

Holomorphic collinear limit of scattering amplitudes in 4d

$\iff$ Celestial OPE limit on 2d celetial sphere

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Does $\lim_{z_i \to z_j} \mathcal{O}_i(z_i)\mathcal{O}_j(z_j)$ have an OPE like structure?

Yes! Simply by Mellin transforming momentum space collinear splitting functions.

Example OPE

Celestial OPE between two positive helicity gluons:

$$\mathcal{O}^{a}_{\Delta_i}(z_i, \bar{z}_i)\, \mathcal{O}^{b}_{\Delta_j}(z_j, \bar{z}_j) \sim \frac{f^{ab}{}_{c}}{z_{ij}}\, B(\Delta_i - 1, \Delta_j - 1)\, \mathcal{O}^{c}_{\Delta_i+\Delta_j-1}(z_j, \bar{z}_j) \quad (3)$$

where $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Euler Beta function.

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where $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Euler Beta function.

Including all the $SL(2,\mathbb{R})_R$ or $\bar{z}_{ij}$ descendants:

$$\mathcal{O}^{\mathrm{a}}_{\Delta_i}(z_i,\bar{z}_i)\,\mathcal{O}^{\mathrm{b}}_{\Delta_j}(z_j,\bar{z}_j)\sim \frac{f^{\mathrm{ab}}{}_{\mathrm{c}}}{z_{ij}}\sum_{m=0}^{\infty}B(\Delta_i+m-1,\Delta_j-1)\,\frac{\bar{z}_{ij}^m}{m!}\,\bar{\partial}_j^m\mathcal{O}^{\mathrm{c}}_{\Delta_i+\Delta_j-1}(z_j,\bar{z}_j) \quad (4)$$

[Fan-Fotopoulos-Taylor '19] [Pate-Raclariu-Strominger-Yuan '19]

[Himwich-Pate-Singh '21] [Guevara-Himwich-Pate-Strominger '21] . . .

Question: Is there a dynamical theory generating such OPEs without referring to the symmetry or kinematical constraints in the theory?

Twistor string worldsheet theory

The Berkovits-Witten twistor string theory is a chiral worldsheet model governing degree d holomorphic maps $Z^I(\sigma)$ from a Riemann sphere Σ to supertwistor space $\mathbb{PT} \cong \mathbb{CP}^{3|4} \setminus \mathbb{CP}^1$ with
$Z^I = (Z^A, \chi^a) = (\mu^{\dot\alpha}, \lambda_\alpha, \chi^a)\,, \quad \alpha = 0,1\,,\ \dot\alpha = \dot 0, \dot 1\,,\ a = 1, \ldots, 4.$

Vertex operator of $+$ helicity gluon:

$$\mathcal{O}^{\mathrm{a}} = \int_\Sigma \mathrm{d}\sigma\, j^{\mathrm{a}}(\sigma) \int_{\mathbb{C}^*} \frac{\mathrm{d}s}{s}\, \bar\delta^2\left(\kappa - s\,\lambda(\sigma)\right)\, e^{\mathrm{i}\, s[\mu(\sigma)\,\tilde\kappa]}\,. \qquad (5)$$

j^{a} from current algebra system on the worldsheet, Kac-Moody algebra formally with level $k \to 0$. Relevant OPE:

$$j^{\mathrm{a}}(\sigma_i)\, j^{\mathrm{b}}(\sigma_j) \sim \frac{f^{\mathrm{ab}}{}_{\mathrm{c}}\, j^{\mathrm{c}}(\sigma_j)}{\sigma_{ij}}\,. \qquad (6)$$

[Witten '03] [Berkovits '04]

Celestial OPEs from the worldsheet (leading order)

Simply taking raw OPE between two + helicity gluons gives:

$$\mathcal{O}^{a}(z_i,\bar{z}_i)\,\mathcal{O}^{b}(z_j,\bar{z}_j) \sim \int \mathrm{d}\sigma_i\,\frac{f^{ab}{}_{c}\,j^{c}(\sigma_j)}{\sigma_{ij}}\,\frac{\mathrm{d}s_i}{s_i}\,\frac{\mathrm{d}s_j}{s_j}\,\bar{\delta}^2\left(z_i - s_i\,\lambda(\sigma_i)\right)$$

$$\bar{\delta}^2\left(z_j - s_j\,\lambda(\sigma_j)\right)\,\exp\left(\mathrm{i}\,\omega_i\,s_i\,[\mu(\sigma_i)\,\bar{z}_i] + \mathrm{i}\,\omega_j\,s_j\,[\mu(\sigma_j)\,\bar{z}_j]\right). \quad (7)$$

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$$\bar{\delta}^2\left(z_j - s_j\,\lambda(\sigma_j)\right)\,\exp\Big(\mathrm{i}\,\omega_i\,s_i\,[\mu(\sigma_i)\,\bar{z}_i] + \mathrm{i}\,\omega_j\,s_j\,[\mu(\sigma_j)\,\bar{z}_j]\Big)\,. \quad (7)$$

Integrating w.r.t one of the holomorphic delta functions and integrate by parts to localize on $\frac{1}{\sigma_{ij}}$:

$$\mathcal{O}^{a}(z_i,\bar{z}_i)\,\mathcal{O}^{b}(z_j,\bar{z}_j) \sim f^{ab}{}_{c}\int \mathrm{d}\sigma_i\,j^{c}(\sigma_j)\,\bar{\delta}(\sigma_{ij})\,\frac{\mathrm{d}s_j}{s_j}\,\frac{\langle\iota\,\lambda(\sigma_i)\rangle}{\langle z_i\,\lambda(\sigma_i)\rangle}$$
$$\bar{\delta}^2\left(z_j - s_j\,\lambda(\sigma_j)\right)\,\exp\left(\mathrm{i}\,\omega_i\,\frac{[\mu(\sigma_i)\,\bar{z}_i]}{\langle\iota\,\lambda(\sigma_i)\rangle} + \mathrm{i}\omega_j\,s_j\,[\mu(\sigma_j)\,\bar{z}_j]\right)\,. \quad (8)$$

with ι_α some reference spinor.

Celestial OPEs from the worldsheet (leading order)

Delta functions automatically localise on worldsheet OPE limit $\sigma_i \to \sigma_j$, which correspond to OPE limit on celestial sphere $z_i \to z_j$.
At leading order in z_{ij}:

Worldsheet OPE limit on twistor string worldsheet
$\iff$ Celestial OPE limit on 2d celestial sphere

After some manipulations
$\implies$ The celestial OPE at leading order emerges!

Provides a dynamical way to generate celestial OPEs at leading order.

All orders in z_{ij}?

The putative CCFT is not unitary nor local

$\implies$ Need full OPE data

- Previous work uses Virasoro and Kac-Moody descendants to organize order by order in z_{ij} in the MHV sector
- Here instead use soft gluon descendants

[Banerjee-Ghosh-Gonzo '20] [Banerjee-Ghosh-Paul '20] [Ebert-Sharma-Wang '21]

[Banerjee-Ghosh '20] [Banerjee-Ghosh-Samal '21] [Banerjee-Ghosh-Paul '21]

Worldsheet at MHV level

At MHV level, degree 1 holomorphic maps $Z^A(\sigma) = U^A_\alpha\, \sigma^\alpha$, fix $GL(2, \mathbb{C})$ with $\lambda_\alpha(\sigma) = \sigma_\alpha$ and remaining moduli $\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha$.

Doing the integral w.r.t the delta function in positive helicity gluon operator gives operator on celestial sphere in an effective 2d CFT:

$$\mathcal{O}^{\mathsf{a}}(z, \tilde\kappa) = \int_\Sigma \mathrm{d}\lambda\, j^{\mathsf{a}}\,(\lambda)\bar\delta(\lambda_\alpha - \kappa_\alpha)\, e^{\mathrm{i}[\mu(\lambda)\tilde\kappa]} = j^{\mathsf{a}}(z)\, e^{\mathrm{i}[\mu(z)\tilde\kappa]}\,, \qquad (9)$$

where we have only parametrized $\kappa_\alpha = (1\ \ z)$ using 2d coordinates on the celestial sphere.

$$\text{worldsheet } \Sigma \xleftrightarrow{\text{holomorphically}} \text{celestial sphere}$$

Worldsheet at MHV level

The effective 2d CFT on the celestial sphere:

$$S^{\mathrm{MHV}} = \frac{1}{2\pi}\int_{S^2} \tilde{\lambda}_{\dot{\alpha}}(z)\,\bar{\partial}\mu^{\dot{\alpha}}(z) + S_{\mathrm{current}}\Big|_{S^2}\,. \tag{10}$$

To capture all orders OPE between vertex operators, we also look at all orders OPE in the worldsheet CFT:

$$j^{\mathsf{a}}(z_i)\,j^{\mathsf{b}}(z_j) = \frac{f^{\mathsf{ab}}{}_{\mathsf{c}}\,j^{\mathsf{c}}(z_j)}{z_{ij}} + \sum_{n=0}^{\infty}\frac{z_{ij}^n}{n!}\underbrace{:\partial^n j^{\mathsf{a}}\,j^{\mathsf{b}}:(z_j)}_{\frac{1}{2\pi \mathrm{i}}\oint_{C(z_j)}\mathrm{d}z_i\,\frac{\partial^n j^{\mathsf{a}}(z_i)\,j^{\mathsf{b}}(z_j)}{z_{ij}}}\,. \tag{11}$$

Soft gluon vertex operator

For the positive helicity gluon operator, we can expand in powers of the energy scale ω

$$\mathcal{O}^{\mathsf{a}}(z,\omega(1\ \bar{z}))\ =\ \sum_k \omega^k\,\frac{\mathrm{i}^k}{k!} j^{\mathsf{a}}(z)[\mu(z)\bar{z}]^k\ =\ \sum_k \frac{\mathrm{i}^k}{k!} j^{\mathsf{a}}(z)[\mu(z)\tilde{\kappa}]^k\,. \quad (12)$$

And define soft gluon vertex operators as

$$J^{\mathsf{a}}[k](z,\tilde{\kappa}) := \frac{\mathrm{i}^k}{k!}\, j^{\mathsf{a}}(z)\,[\mu(z)\,\tilde{\kappa}]^k = \sum_{n+l=k} \frac{\tilde{\kappa}_{\dot{0}}{}^n\,\tilde{\kappa}_{\dot{1}}{}^l}{n!\,l!}\, J^{\mathsf{a}}[n,l]\,, \quad k\in\mathbb{Z}_{\geq 0}\,, \quad (13)$$

with $J^{\mathsf{a}}[n,l] = \mathrm{i}^{n+l}\, j^{\mathsf{a}}(z)\left(\mu^{\dot{0}}\right)^n(z)\left(\mu^{\dot{1}}\right)^l(z)$.

All orders collinear expansion in momentum space

Computing the $++$ gluon OPE gives

$$\mathcal{O}_i^{\mathrm{a}}(z_i, \tilde{\kappa}_i)\, \mathcal{O}_j^{\mathrm{b}}(z_j, \tilde{\kappa}_j) = j^{\mathrm{a}}(z_i)\, j^{\mathrm{b}}(z_j)\, e^{\mathrm{i}\,[\mu(z_i)\,\tilde{\kappa}_i]}\, e^{\mathrm{i}\,[\mu(z_j)\,\tilde{\kappa}_j]}$$
$$= \left(\sum_{m=0}^{\infty} z_{ij}^{m-1}\, j_{-m}^{\mathrm{a}}\, j^{\mathrm{b}}\right) \left(\sum_{n=0}^{\infty} \frac{z_{ij}^n}{n!}\, \partial_{z_j}^n \left(e^{\mathrm{i}\,[\mu\,\tilde{\kappa}_i]}\right)\, e^{\mathrm{i}\,[\mu\,\tilde{\kappa}_j]}\right), \tag{14}$$

normal ordered and evaluated at z_j where

$$j_{-m}^{\mathrm{a}}\, j^{\mathrm{b}} := \begin{cases} f^{\mathrm{ab}}{}_{\mathrm{c}}\, j^{\mathrm{c}}(z_j) & \text{when } m = 0\,; \\ \dfrac{\partial_{z_j}^{m-1} j^{\mathrm{a}}\, j^{\mathrm{b}}(z_j)}{(m-1)!} & \text{when } m \geq 1\,. \end{cases} \tag{15}$$

All orders collinear expansion in momentum space

We choose to organize the OPEs using soft gluon operators, their action on $\mathcal{O}^{\mathsf{a}}(z, \tilde{\kappa})$:

$$
\begin{aligned}
&J^{\mathsf{a}}[k](z_i, \tilde{\kappa}_i)\, \mathcal{O}_j^{\mathsf{b}}(z_j, \tilde{\kappa}_j) \\
&= \frac{[i\,\partial_j]^k}{k!}\, \frac{f^{\mathsf{ab}}{}_{\mathsf{c}}\, \mathcal{O}_j^{\mathsf{c}}(z_j, \tilde{\kappa}_j)}{z_{ij}} + \sum_{p=1}^{\infty} z_{ij}^{p-1} \underbrace{\frac{\mathrm{i}^k}{k!} \sum_{n=0}^{p} \frac{1}{n!}\, j^{\mathsf{a}}_{n-p}\, j^{\mathsf{b}}\, \partial_{z_j}^n \Big([\mu\,\tilde{\kappa}_i]^k \Big)\ e^{\mathrm{i}\,[\mu\,\tilde{\kappa}_j]}}_{J^{\mathsf{a}}_{-p}[k](\tilde{\kappa}_i)\, \mathcal{O}_j^{\mathsf{b}}(z_j, \tilde{\kappa}_j)} .
\end{aligned}
$$

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\begin{aligned}
&J^{\mathsf{a}}[k](z_i,\tilde{\kappa}_i)\,\mathcal{O}_j^{\mathsf{b}}(z_j,\tilde{\kappa}_j)\\
&=\frac{[i\,\partial_j]^k}{k!}\,\frac{f^{\mathsf{ab}}{}_{\mathsf{c}}\,\mathcal{O}_j^{\mathsf{c}}(z_j,\tilde{\kappa}_j)}{z_{ij}}+\sum_{p=1}^{\infty} z_{ij}^{p-1}\,\underbrace{\frac{\mathrm{i}^k}{k!}\sum_{n=0}^{p}\frac{1}{n!}\,j^{\mathsf{a}}_{n-p}\,j^{\mathsf{b}}\,\partial_{z_j}^n\Big([\mu\,\tilde{\kappa}_i]^k\Big)\;e^{\mathrm{i}\,[\mu\,\tilde{\kappa}_j]}}_{J^{\mathsf{a}}_{-p}[k](\tilde{\kappa}_i)\,\mathcal{O}_j^{\mathsf{b}}(z_j,\tilde{\kappa}_j)}\,.
\end{aligned}
$$

Reorganizing the OPE (14) using the soft descendants $J^{\mathsf{a}}_{-p}[k]$:

$$
\mathcal{O}_i^{\mathsf{a}}\,\mathcal{O}_j^{\mathsf{b}}=\sum_{p=0}^{\infty} z_{ij}^{p-1}\sum_{k=0}^{p}\sum_{\ell=0}^{p-k}\frac{\left(-[\tilde{\kappa}_i\,\partial_{\tilde{\kappa}_j}]\right)^{\ell}}{\ell!}\,J^{\mathsf{a}}_{-p}[k](\tilde{\kappa}_i)\,\mathcal{O}^{\mathsf{b}}(z_j,\tilde{\kappa}_i+\tilde{\kappa}_j)\,. \quad (16)
$$

This is the all orders collinear expansion in the MHV sector.

All orders $SL(2,\mathbb{R})_L$ descendants in conformal primary basis

We can also parametrize $\tilde{\kappa}_{\dot{\alpha}} = \omega \binom{1}{\bar{z}}$ and Mellin transform in ω to obtain the all-order celestial OPE in conformal primary basis:

$$\mathcal{O}^{a}_{+,\Delta_i}(z_i,\bar{z}_i)\,\mathcal{O}^{b}_{+,\Delta_j}(z_j,\bar{z}_j) = \sum_{p=0}^{\infty} z_{ij}^{p-1} \sum_{k=0}^{p}\sum_{\ell=0}^{p-k}\sum_{m=0}^{\infty} \frac{\bar{z}_{ij}^m}{m!}\,\frac{\bar{D}_k^{\ell}}{\ell!}\,J^{a}_{-p}[k](\bar{z}_i)$$
$$\times\; B(\Delta_i+k+\ell+m-1,\,\Delta_j-1)\,\bar{\partial}_j^m\,\mathcal{O}^{b}_{+,\Delta_i+\Delta_j+k-1}(z_j,\bar{z}_j)\,. \quad (17)$$

with $\bar{D}_j = -\bar{z}_{ij}\,\bar{\partial}_j + \Delta_i + \Delta_j + k - 3$.

Discussion

- This gives a genuinely dynamical CFT on 2d celestial sphere to generate observables to match those from gauge theory in 4d, which provides a good starting point for a bottom up construction of celestial holography
- The momentum space formulae gives all-orders collinear expansion and the one in conformal primary basis gives all-orders celestial OPE
- Can obtain all-order expressions for other helicity and orientation configurations
- Matching with known results in low orders in z_{ij} using null state
- Relation to infinite dimensional chiral algebra living on the celestial sphere
- Beyond MHV sector?