Bethe-Salpeter equation for classical gravitational bound states

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based on work with T.Adamo

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Motivation and introduction (I)

- The recent discovery of gravitational waves calls for new analytical techniques to study classical gravitational bound states

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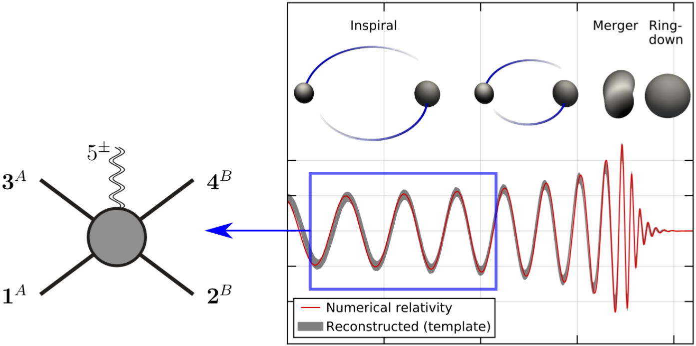
- The recent discovery of gravitational waves calls for new analytical techniques to study classical gravitational bound states
- We focus on the long-distance inspiral regime of binary compact systems, which can be studied with perturbative scattering amplitudes



See related talks by [Kavanagh,Roiban; Aoude, Alessio, Bautista, Brown, Cristofoli, Foffa, Isabella, Long, Jakobsen, Kälin, Ochirov, Pichini, Sergola]

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- Question 2: The analytic S-matrix structure encodes the bound state physics only through an infinite sum . . . How can perform the resummation?
- Question 3: How can we compute scattering and bound state observables?

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- Framework: QFT scattering amplitudes techniques for the classical gravitational interaction of two massive (spinless or spinning) point particles

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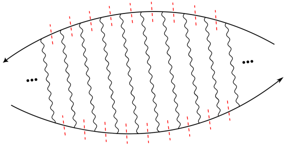

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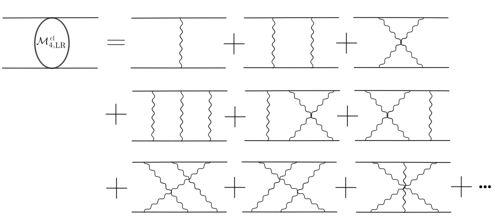

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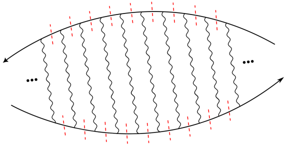

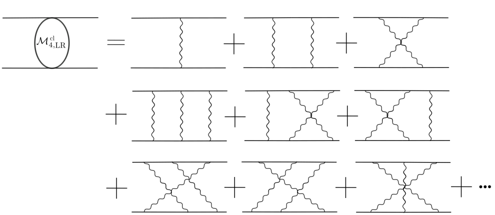

- Let's make this precise . . .

Revisiting the leading eikonal resummation (II)

- Consider the conservative 4-pt amplitude: the classical HEFT expansion [Damgaard,Aoude,Haddad,Helset;Brandhuber,Chen,Travaglini,Wen] is equivalent, at leading order, to the eikonal resummation $-t \ll s$

$$
\begin{aligned}
p_1^\mu &:= p_A^\mu + \hbar\frac{\bar{q}^\mu}{2}\,, \qquad (p_1')^\mu := p_A^\mu - \hbar\frac{\bar{q}^\mu}{2}\,, \qquad s = (p_A + p_B)^2\,,\\
p_2^\mu &:= p_B^\mu - \hbar\frac{\bar{q}^\mu}{2}\,, \qquad (p_2')^\mu := p_B^\mu + \hbar\frac{\bar{q}^\mu}{2}\,, \qquad t = -\hbar^2|\vec{\bar{q}}|^2\,,
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where p_A, p_B are the classical momenta and q is the momentum transfer.

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where p_A, p_B are the classical momenta and q is the momentum transfer.

- Given that we have

$$p_A \cdot q = p_B \cdot q = 0\,,$$

it is natural to work in impact parameter space ($x_\perp \sim$ natural length scale)

$$\widetilde{\mathcal{M}}_4^{\text{cl}}(p_A, p_B; x_\perp) := \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(2p_A \cdot q)\hat{\delta}(2p_B \cdot q) e^{i\frac{q\cdot x_\perp}{\hbar}} \mathcal{M}_4^{\text{cl}}(p_A, p_B; q)\,,$$

Revisiting the leading eikonal resummation (III)

- The tree-level contribution to the leading resummation (LR) is,

$$i\mathcal{M}^{(0),\text{cl}}_{4,\text{LR}}(q) = \frac{i}{q^2 + i\epsilon}\, V_{\mu\nu}(p_A)\, P^{\mu\nu\alpha\beta}\, V_{\alpha\beta}(p_B)\,, \quad V_{\mu\nu}(p) := i\,\kappa\, p_\mu\, p_\nu\,.$$

which can be rewritten in impact parameter space as

$$i\widetilde{\mathcal{M}}^{(0),\text{cl}}_{4,\text{LR}}(x_\perp) = \int \hat{d}^4 q\, e^{i\frac{q\cdot x_\perp}{\hbar}}\, \widetilde{R}^{\alpha_1\beta_1}(p_A, q) \left(\frac{i}{q^2 + i\epsilon}\, V_{\alpha_1\beta_1}(p_B)\, \hat{\delta}(2p_B \cdot q) \right),$$

$$\widetilde{R}^{\alpha\beta}(p_A, q) := P^{\mu\nu\alpha\beta}\, V_{\mu\nu}(p_A)\, \hat{\delta}(2p_A \cdot q)\,.$$

i.e. we separate artificially the contributions from particle A and B.

Revisiting the leading eikonal resummation (IV)

- The one-loop contribution is the sum of box and crossed box,

$$i\mathcal{M}_{4,\mathrm{LR}}^{(1),\mathrm{cl}}(q) = \frac{1}{2!}\int \hat{\mathrm{d}}^4 l_1 \int \hat{\mathrm{d}}^4 l_2 \hat{\delta}^4(l_1 + l_2 - q) \prod_{i=1}^{2} \left[V_{\mu_i\nu_i}(p_A) P^{\mu_i\nu_i\alpha_i\beta_i} V_{\alpha_i\beta_i}(p_B) \frac{i}{l_i^2 + i\epsilon} \right]$$

$$\times \underbrace{\left[\frac{i}{-2l_1\cdot p_A + i\epsilon} + \frac{i}{2l_1\cdot p_A + i\epsilon} \right] \left[\frac{i}{-2l_1\cdot p_B + i\epsilon} + \frac{i}{2l_1\cdot p_B + i\epsilon} \right]}_{\hat{\delta}(2l_1\cdot p_A)\hat{\delta}(2l_1\cdot p_B)} .$$

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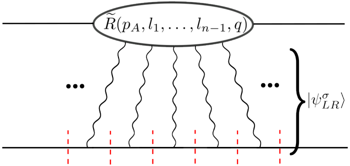
- In impact parameter space we recognize an iterative structure ...

$$i\widetilde{\mathcal{M}}^{(1),\mathrm{cl}}_{4,\mathrm{LR}}(x_\perp) = \int \hat{\mathrm{d}}^4 q\, e^{i\frac{q\cdot x_\perp}{\hbar}}\int \hat{\mathrm{d}}^4 l_1 \int \hat{\mathrm{d}}^4 l_2 \hat{\delta}^4(l_1+l_2-q)\widetilde{R}^{\alpha_1\alpha_2\beta_1\beta_2}(p_A,l_1,q)$$
$$\times \frac{1}{2!}\left(\frac{i}{l_1^2+i\epsilon}V_{\alpha_1\beta_1}(p_B)\hat{\delta}(2l_1\cdot p_B)\right)\left(\frac{i}{l_2^2+i\epsilon}V_{\alpha_2\beta_2}(p_B)\hat{\delta}(2l_2\cdot p_B)\right).$$

where does it come from?

New classical structure: coherent state of virtual gravitons

- The iterative structure suggests the introduction of a new classical object



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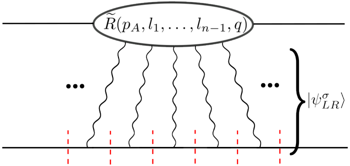

which creates the gravitational field $h_{\mu\nu}$ responsible for the interaction.

- We find that this object is a coherent state of virtual gravitons

$$|\psi^\sigma_{\mathrm{LR}}\rangle = \frac{1}{\mathcal{N}} \int \mathrm{d}\Phi(p)\phi(p) \exp\left[\int \frac{\hat{\mathrm{d}}^4 l}{l^2 + i\epsilon}\, \hat{\delta}(2p_A \cdot l) i\mathcal{M}_3^{(0),\mathrm{cl}}(p_A, l^\sigma) A^\dagger_\sigma(l)\right] |p\rangle\,,$$

which can be derived also by the in-in expectation value in $(1,3)$ signature (see [Monteiro, O'Connell, Peinador Veiga, Sergola] for a $(2,2)$ derivation).

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- 1) We can view the leading order interaction as one particle moving in the background of the other [t'Hooft;Kabat,Ortiz;Adamo,Cristofoli,Tourkine]
- 2) Having a classical field imposes on you to average over the internal graviton legs: the $1/n!$ factor comes from expanding the coherent state, as well as all the eikonal diagrams.

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- 2) Having a classical field imposes on you to average over the internal graviton legs: the $1/n!$ factor comes from expanding the coherent state, as well as all the eikonal diagrams.
- New physical principle: the space of classical conservative 4-pt amplitudes is

$$\boxed{\mathcal{H}_{4,\mathrm{cl}} := \mathcal{H}_4/ \sim_{\mathrm{cl}}}$$

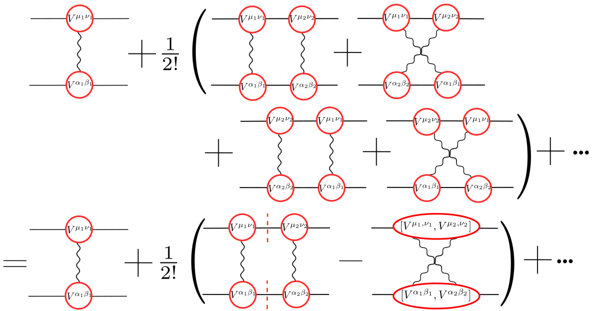
i.e. the quotient space of 4-pt amplitudes $\mathcal{H}_4$ under the equivalence relation

$$\mathcal{M}_4 \sim_{\mathrm{cl}} \mathcal{M}_4' \quad \text{iff they differ by a permutation of an internal graviton exchange}\,.$$

Note: also applies to real emission amplitudes! [Cristofoli et al.,Britto et al.]

Spinning eikonal resummation at leading order

- We can also consider the eikonal resummation for spinning particles using the HEFT [Aoude,Haddad,Helset]



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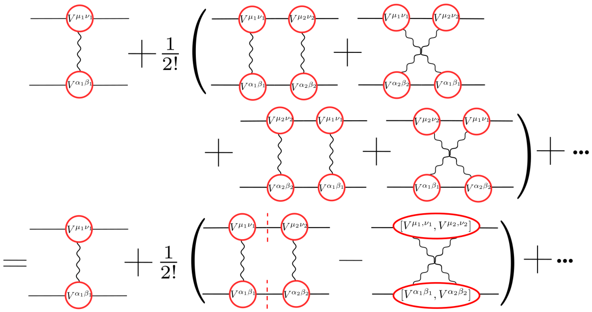
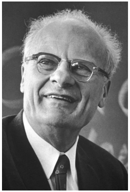
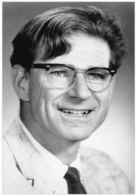
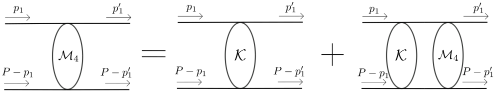


- There is an extra double commutator term, which is suppressed for Kerr black hole! [Haddad] If Kerr can be thought as a point particle at large distances, we expect these terms to be always suppressed.

- The Bethe-Salpeter equation is a non-perturbative recursion relation for 4-pt amplitudes, which generate the bound state energy poles via the iteration of a two-massive particle irreducible kernel $\mathcal{K}$







Bethe- Salpeter equation

$$\mathcal{M}_4(p_1, p_1'; P) = \mathcal{K}(p_1, p_1'; P) + \int \hat{\mathrm{d}}^4 l\, \mathcal{K}(p_1, l; P) G(l, P) \mathcal{M}_4(l, p_1'; P)\,,$$

where $G(l, P)$ is the two-body propagator.

- How can we take the classical limit?

The classical Bethe-Salpeter equation

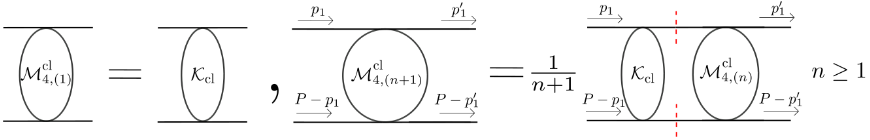
- We need to quotient by symmetrization over the internal graviton exchanges: the result is the classical Bethe-Salpeter equation in the space $\mathcal{H}_{4,\text{cl}}$

$$\mathcal{M}^{\text{cl}}_{4,(n+1)}(p_A, p_B, q) = \begin{cases} \mathcal{K}_{\text{cl}}(p_A, p_B, q) & \text{if } n = 0 \\ \frac{1}{n+1} \int \hat{\mathrm{d}}^4 l \, \mathcal{K}_{\text{cl}}(p_A, p_B, l) G_{\text{cl}}(p_A, p_B, l) \mathcal{M}^{\text{cl}}_{4,(n)}(p_A, p_B, q - l) & \text{if } n \geq 1 \end{cases} .$$

where the two-body propagator is replaced by its on-shell version

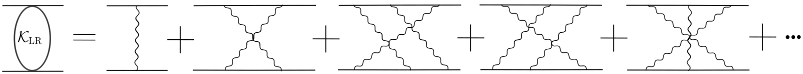
$$G_{\text{cl}}(p_A, p_B, l) = \hat{\delta}(2l \cdot p_A)\hat{\delta}(2l \cdot p_B) ,$$

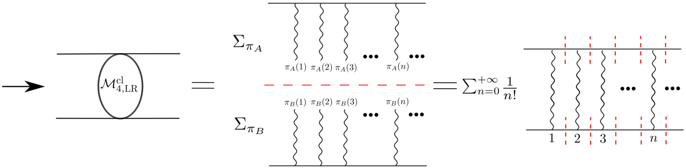
and (n) is the number of classical two-massive particle irreducible diagrams.



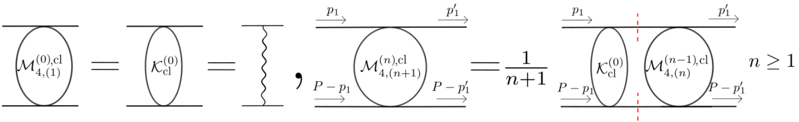
Recovering the leading eikonal resummation from BSE

- While the original iteration in the BS equation was not crossing symmetric and required an infinite number of diagrams in the kernel





- we can now recover the leading eikonal resummation with a single tree-level kernel for the new classical BSE



Exponentiation of the classical kernel: an exact solution

- The classical BSE in impact parameter space becomes

$$\widetilde{\mathcal{M}}^{\text{cl}}_{4,(n+1)}(p_A,p_B,x_\perp)=\begin{cases}\widetilde{\mathcal{K}}_{\text{cl}}(p_A,p_B,x_\perp) & \text{if } n=0\\ \frac{1}{n+1}\widetilde{\mathcal{K}}_{\text{cl}}(p_A,p_B,x_\perp)\widetilde{\mathcal{M}}^{\text{cl}}_{4,(n)}(p_A,p_B,x_\perp) & \text{if } n\geq 1\end{cases},$$

and therefore by iteration we get, schematically,

$$\widetilde{\mathcal{M}}^{\text{cl}}_{4,(n+1)}=\widetilde{\mathcal{K}}_{\text{cl}}+\frac{1}{2!}\widetilde{\mathcal{K}}^2_{\text{cl}}+\cdots+\frac{1}{(n+1)!}\widetilde{\mathcal{K}}^{n+1}_{\text{cl}}$$

which means that the final solution exponentiates exactly

$$\boxed{\widetilde{\mathcal{M}}^{\text{cl}}_{4}(p_A,p_B,x_\perp)=e^{\widetilde{\mathcal{K}}_{\text{cl}}(p_A,p_B,x_\perp)}\,.}$$

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$$\boxed{\widetilde{\mathcal{M}}^{\text{cl}}_4(p_A, p_B, x_\perp) = e^{\widetilde{\mathcal{K}}_{\text{cl}}(p_A, p_B, x_\perp)}\,.}$$

- The analytic structure (poles, etc.) in momentum space arise completely from

$$\mathrm{i}\mathcal{M}^{\text{cl}}_4(p_A, p_B; q_\perp) = \frac{4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}}{\hbar^2}\int \mathrm{d}^2 x_\perp \mathrm{e}^{-\mathrm{i}\bar{q}_\perp\cdot x_\perp}\left(\mathrm{e}^{\widetilde{\mathcal{K}}_{\text{cl}}(p_A, p_B, x_\perp)} - 1\right).$$

The Hamilton-Jacobi action and observables (I)

- The classical two-body kernel contains both scattering and bound dynamics. For spinless particles, the motion is restricted to a plane and we can define the conserved quantities $(\mathcal{E}, J)$

$$\mathcal{E} := \frac{E - m_A - m_B}{\nu(m_A + m_B)}\,, \qquad J = p_\infty |x_\perp|\,, \qquad \nu = \frac{m_A m_B}{(m_A + m_B)^2}\,,$$

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- Natural connection of the kernel with Hamilton-Jacobi action [Kälin,Porto]

$$\widetilde{\mathcal{K}}_{\text{cl}}^{>/<}(p_A, p_B; x_\perp) = \frac{i}{\hbar} I^{>/<}(\mathcal{E}, J) , \quad I^{>/<}(\mathcal{E}, J) = \oint_{\mathcal{C}_{>/<}} dr\, p_r(r, \mathcal{E}, J) + J\pi ,$$

where p_r is the radial momentum and $\mathcal{C}_{>/<}$ is the contour of integration for scattering/bound orbits. Natural analytic continuation

$$I^{<}(\mathcal{E} < 0, J) = I^{>}(\mathcal{E} < 0, J) - I^{>}(\mathcal{E} < 0, -J) .$$

The Hamilton-Jacobi action and observables (II)

- The classical kernel up to 2PM is

$$\widetilde{\mathcal{K}}_{\text{cl}}^{>}(p_A, p_B, x_\perp) = \frac{i}{\hbar}\Bigg[-2G_N \log(\mu_{\text{IR}}|x_\perp|) m_A m_B \frac{2y^2-1}{\sqrt{y^2-1}} + \frac{3\pi}{4} G_N^2 m_A m_B (m_A + m_B) \frac{5y^2-1}{\sqrt{y^2-1}} \frac{1}{|x_\perp|} \Bigg],$$

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- Scattering and bound observables are derived by differentiation

$$\chi(\mathcal{E}, J) = -\frac{\partial I^{>}(\mathcal{E}, J)}{\partial J}, \qquad \Delta\Phi(\mathcal{E}, J) = -\frac{\partial I^{<}(\mathcal{E}, J)}{\partial J}.$$

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- Interested in binding energy: in the PN expansion $(\mathcal{E} \ll m_A c^2, m_B c^2)$ we get

$$I^{<}(\mathcal{E}, J) = 0 \to \epsilon = x - \frac{1}{12} x^2 (9 + \nu) + \mathcal{O}(x^3)$$

$$\epsilon = -2\mathcal{E}, \qquad x = \left(\frac{1}{G_N m_A m_B} \frac{dJ(\mathcal{E})}{d\mathcal{E}}\right)^{-\frac{2}{3}}.$$

Analytic structure of resummed classical amplitudes (I)

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- How is the classical dynamics encoded in the amplitude solution of the BSE in momentum space?
- The leading classical resummation of the tree-level kernel gives [Kabat, Ortiz]

$$i\mathcal{M}_{4,\mathrm{LR}}^{\mathrm{cl},>/<}(p_A,p_B;q_\perp)=\frac{4\pi m_A m_B\sqrt{y^2-1}}{\hbar^2\mu_{\mathrm{IR}}^2}\frac{\Gamma(1-A_0^{>/<})}{\Gamma(A_0^{>/<})}\left(\frac{4\hbar^2\mu_{\mathrm{IR}}^2}{q^2}\right)^{1-A_0^{>/<}},$$

where for $\mathcal{E}>0$ we have ("phase")

$$A_0^{>}:=i\frac{G_N}{\hbar}m_A m_B\frac{2y^2-1}{\sqrt{y^2-1}},$$

but for $\mathcal{E}<0$ (real function!!!)

$$A_0^{<}:=\frac{G_N}{\hbar}m_A m_B\frac{2y^2-1}{\sqrt{1-y^2}}.$$

Analytic structure of resummed classical amplitudes (II)

- The bound state wavefunction can be computed from the conservative amplitude by matching the cross-section [Fried,Kang,McKellar]

$$\psi^{<}_{\text{bound}} := \frac{1}{8\pi E}\left(\frac{p_\infty^2}{\mu_{\text{IR}}^2}\right)^{-A_0^<} \mathcal{M}_4^{<}(p_A, p_B; q_\perp)\,.$$

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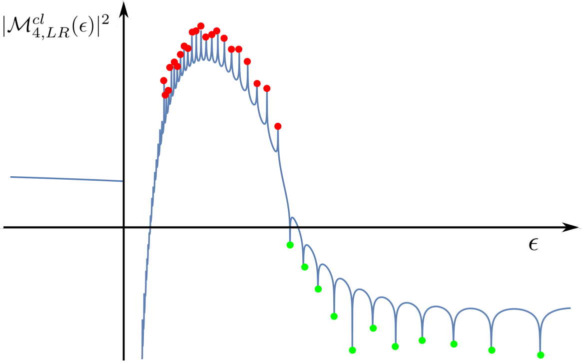
- For the leading resummation we obtain

$$\psi^{<}_{\text{bound,LR}} = \frac{2m_A m_B\sqrt{1-y^2}}{E\, q^2}\frac{\Gamma(1-A_0^<)}{\Gamma(A_0^<)}\left(\frac{4p_\infty^2}{\bar{q}^2}\right)^{-A_0^<}\,,$$

Analytic structure of resummed classical amplitudes (III)

- We can visualize the amplitude properties by looking at the cross-section



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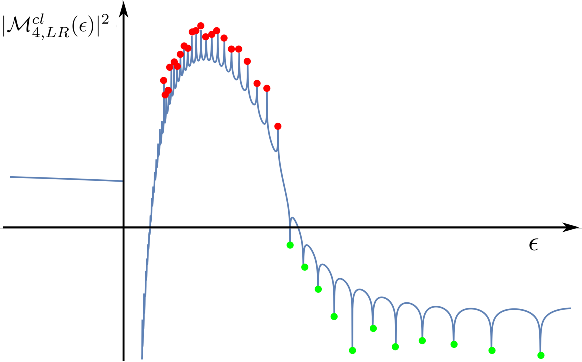

- We notice that there is a set of poles and zeros labelled by integers

poles: $1 - A_0 = 1 - n\,, n \in \mathbb{Z}_{>0}\,,$ zeros: $1 - A_0 = 1 + n\,, n \in \mathbb{Z}_{>0}\,,$

Amplitude poles and the binding energy

- The binding energy $\epsilon_n^{(1)}$ is given by

$$\epsilon_n^{(1)} = \frac{1}{2}\left[m_A + m_B - \sqrt{m_A^2 + m_B^2 + \frac{1}{\sqrt{2}} m_A m_B \sqrt{4 - \xi_n^2 + \xi_n\sqrt{8+\xi_n^2}}}\right],$$

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- We expect from the correspondence principle that

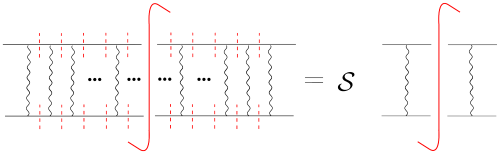
$$\hbar \to 0 \, , n \to +\infty \, , \qquad \boxed{\hbar n \overset{\hbar \to 0}{\to} J} ,$$

which means that we can recast the pole structure as $\xi_n \to J/(G_N m_A m_B)$. In particular we recover the leading PN binding binding energy

$$\epsilon^{(1)} = \frac{G_N^2 m_A^2 m_B^2}{\hbar^2 n^2} = \frac{G_N^2 m_A^2 m_B^2}{J^2} \, .$$

The (relativistic) Sommerfeld effect

- Should there be any effect of the classical resummation on the cross-section, considering in general compact objects in a bound gravitational system?



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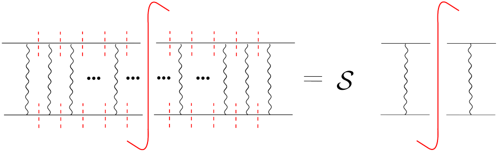
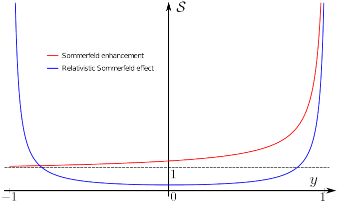


- At the Newtonian order (attractive potential) we recover the Sommerfeld enhancement, while in general relativistic physics this is not an enhancement



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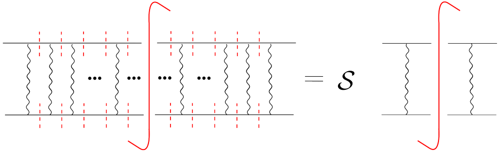

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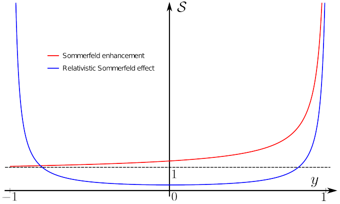


- Interesting phenomenological applications! [Slatyer;Petraki]

Next-to-leading order resummed classical amplitude (I)

- We also resum the conservative 2 PM result analytically

$$\psi^{<}_{\text{bound,NLR}} = \frac{m_A m_B \sqrt{1-y^2}}{E\hbar^2}$$

$$\times \left[2\frac{\Gamma\left(1-A_0^<\right)}{\Gamma\left(\frac{1}{2}+A_0^<\right)} \left(\frac{1}{\bar{q}}\right)^2 \left(\frac{4p_\infty^2}{\bar{q}^2}\right)^{-A_0^<} {}_0F_3\left(;\frac{1}{2},A_0^<,A_0^<;-\frac{(A_1^<)^2\bar{q}^2}{16}\right)\right.$$

$$+ \frac{A_1 \Gamma\left(\frac{1}{2}-A_0^<\right)}{\Gamma\left(\frac{1}{2}+A_0^<\right)} \frac{1}{\bar{q}} \left(\frac{4p_\infty^2}{\bar{q}^2}\right)^{-A_0^<} {}_0F_3\left(;\frac{3}{2},\frac{1}{2}+A_0^<,\frac{1}{2}+A_0^<;-\frac{(A_1^<)^2\bar{q}^2}{16}\right)$$

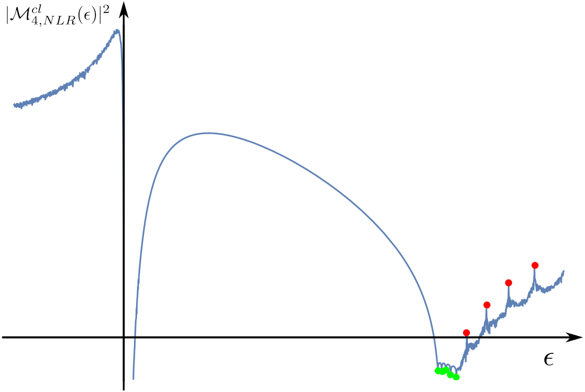
$$\left. + \left(-A_1^<\right)^{2-2A_0^<} p_\infty^{-2A_0^<} \Gamma\left(-2+2A_0^<\right) {}_0F_3\left(;1,\frac{3}{2}-A_0^<,2-A_0^<;-\frac{(A_1^<)^2\bar{q}^2}{16}\right)\right],$$

focussing on $\mathcal{E} < 0$ with $A_0^<$ defined as before and

$$A_1^< := \frac{3\pi}{4\hbar} G_N^2 m_A m_B (m_A + m_B) \frac{5y^2-1}{\sqrt{1-y^2}}.$$

Next-to-leading order resummed classical amplitude (II)

- While the function is complicated, we can find numerically poles and zeros!



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- Future directions: include radiative effects, understand better the analytic structure of spinning amplitudes, understand better the analytic continuation for generic spins and radiative effects, ...