Higher Spin Amplitudes from the Teukolsky Equation

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IPhT-Saclay

2212.07965 and 2107.10179

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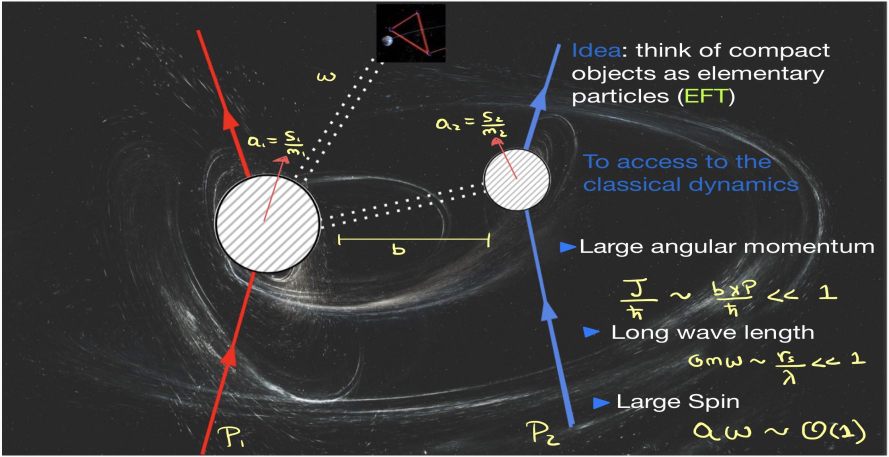
Background image source: SXS (Simulating eXtreme Spacetimes) project. Simulation of GW150914

Motivation and overview

Two-body problem: Amplitudes approach



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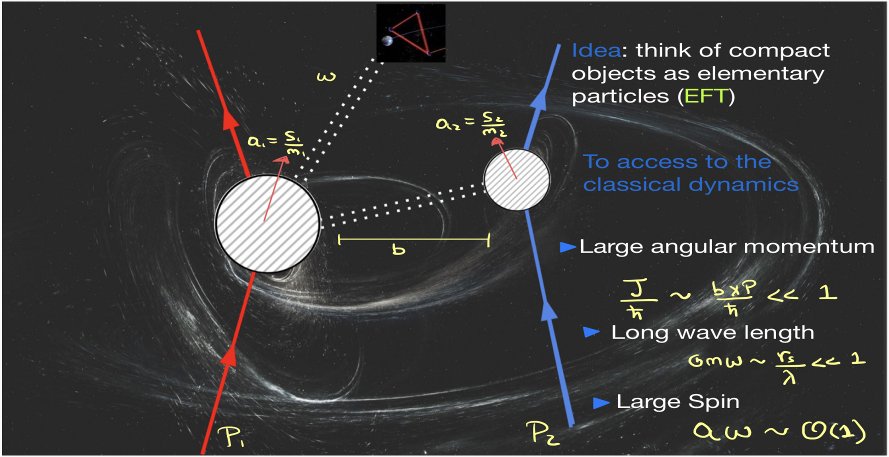

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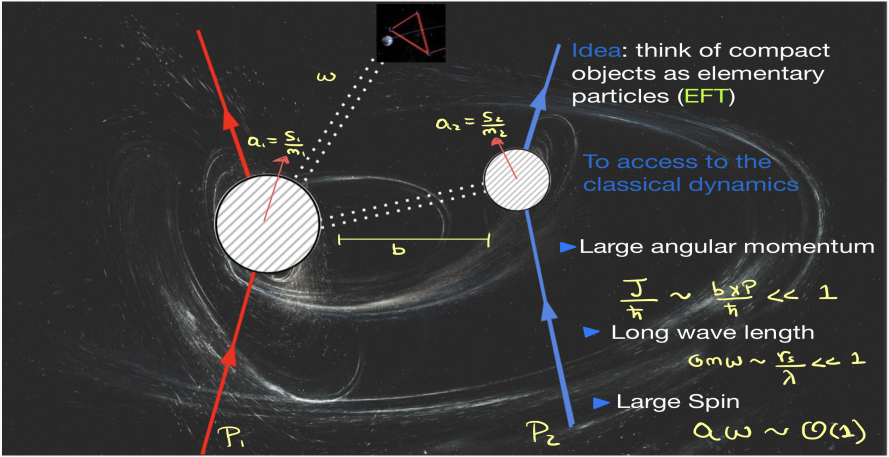

- Exploit the power of unitarity: Start from building blocks A_n
- Better make sure we have the correct building blocks !!

Do minimal coupling spinning amplitudes have actually anything to do with Kerr ?

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One can only be sure by matching amplitudes to actual GR computations!

Matching to static solutions

Kerr Black hole as elementary particles

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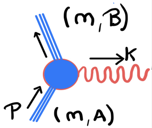

$$h^{\mu\nu}(k\pm) = p^{\mu}p^{\nu}\delta(p\cdot k)\underbrace{\overbrace{\langle\varepsilon_B|}^{\langle\varepsilon'_A|e^{\mp k\cdot a}}\; e^{\overbrace{\pm 2k\cdot a}^{\text{Pauli-Lubansky}}}\;|\varepsilon_A\rangle}_{\text{spin states}} \rightarrow p^{\mu}p^{\nu}\delta(p\cdot k)\langle\varepsilon'_A|\, e^{\pm k\cdot a}|\varepsilon_A\rangle$$

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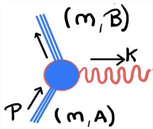

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On the gravitational side: $\mathcal{O}(G)$-Kerr metric [Vines 2017]

$$h^{\text{Kerr}}_{\mu\nu} = P_{\mu\nu\alpha\beta}p^{\alpha}\left[p^{\beta}\cosh(a\cdot\partial) + \epsilon^{\beta\gamma\rho\sigma}p_{\gamma}a_{\rho}\partial_{\sigma}\frac{\sinh(a\cdot\partial)}{a\cdot\partial}\right]\frac{Gm}{r}\xrightarrow{(2,2)}P_{\mu\nu\rho\sigma}p^{\rho}p^{\sigma}e^{\pm a\cdot\partial}\frac{Gm}{r}$$

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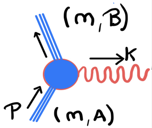

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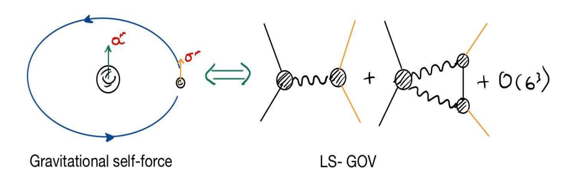
Same exponential structure as for the infinite spin amplitude!

- Linearization erases the BH horizon, $\frac{a}{GM} < 1 \Rightarrow \frac{a}{GM} \gg 1$ Not Kerr but rather a naked ring singularity

Dynamics matching

2. Linear perturbations of Kerr sourced by a small orbiting body (aligned-spins, equatorial scattering) [Siemonsen-Vines 2019]. Checks through a^3 at G^2. red-shift and procession frequency $\Rightarrow$geodesic motion deviation due to Gravitational self-force of the perturbation



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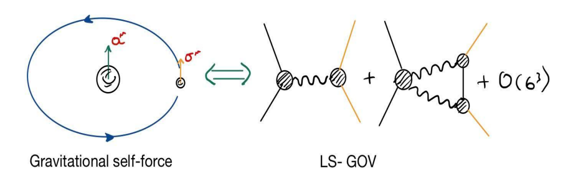
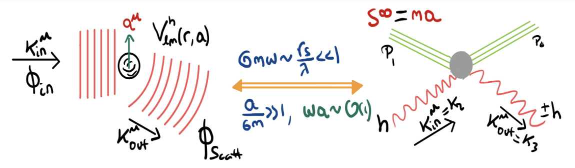


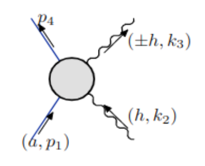
3. In this talk: BH stability under small wave perturbation (generic spin-orientation).Checks through a^6 for GWs



The higher spin Gravitational Compton amplitude

The same helicity configuration

$$A_{h=2}^{++} \sim \frac{M^4[23]^4}{(s-M^2)^2 t} e^{-(k_2-k_3)\cdot a}$$



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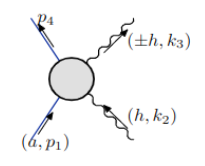

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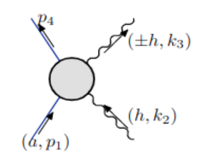

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The opposite helicity configuration $\epsilon_2 = \frac{\sqrt{2}|3]\langle 2|}{[32]} \propto \tilde{\epsilon}_3$

$$A^{+-}_{h=2} \sim \frac{(\langle 2|p_1|3])^4}{(s-M^2)^2 t} e^{(2w-k_3-k_2)\cdot a}, \quad \text{with } w^\mu = \frac{s-M^2}{2p_1\cdot\epsilon_2}\epsilon_2^\mu$$

- Unphysical pole starting at a^5.

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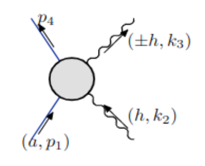

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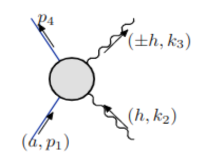

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Still useful exponential, Just need to cure uphysical behavior.

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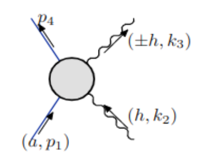

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Still useful exponential, Just need to cure uphysical behavior. At a given order in spin, we seek an ansatz of the form

$$\langle A_{h=2}^S\rangle = \underbrace{\langle A_{h=2}^0\rangle}_{\text{Helicity-weights}} \times \underbrace{\left(e^{(2w-k_3-k_2)\cdot a} + f_\xi(k_2\cdot a, k_3\cdot a, w\cdot a)\right)_{2S}}_{\text{spin-Information}}.$$

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- Crossing symmetry: Same classical amplitude in chiral and anti-chiral basis $A_n^{h=2,S} = \langle \varepsilon_n | A_n^{\text{chir.}} | \varepsilon_1 \rangle = [\varepsilon_n | A_n^{\text{antichir.}} | \varepsilon_1]$, $\Rightarrow f_\xi(k2, k3, w) = f_\xi(k3, k2, w)$

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- Laurent expansion introduces poles in ξ $(s - M^2)$, each cancelled by operator $(w{\cdot}a)^2$. Furthermore, $\langle A_4^0 \rangle$ contains a simple pole in ξ. This gives

$$f^{(m)} \propto (w \cdot a)^{2-2m} \quad \text{for} \quad m \leq 1\,.$$

- Each power of ξ introduces poles in $t = [23]\langle 32\rangle$. To cancel such poles we invoke:

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\begin{aligned}
w^{\mu} &\to k_2^{\mu} \quad \text{as} \quad \langle 23\rangle \to 0 \\
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Then, a pole in t is cancelled by $(w\cdot a - k_2\cdot a)(w\cdot a - k_3\cdot a)$: This implies

$$f^{(m)} \propto (w\cdot a - k_2\cdot a)^{m+1}(w\cdot a - k_3\cdot a)^{m+1} \quad \text{for} \quad m \geq -1 \,,$$

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- Caveat: The identity The classical identity [Gram determinant relation in Aoude's talk]

$$-\frac{(s-M^2)(u-M^2)}{4M^2}a^2 \approx \omega^2 a^2 \approx \xi(w\cdot a - k_2\cdot a)(w\cdot a - k_3\cdot a) + (w\cdot a)^2$$

$\Rightarrow$ the quadratic Casimir is not independent of our $\{k_2{\cdot}a, k_3{\cdot}a, w{\cdot}a\}$ basis. But $|a|$ is! So we can implement operators proportional to $|a|\omega$, with $\omega = \frac{s-M^2}{2M}$

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- Special role: Track BH horizon dynamics!!

After putting all the ingredients together we can parametrize the Compton ansatz via

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\begin{aligned}
f_\xi = &\sum_{m=0}^{2} \xi^{m-1}(w\cdot a)^{4-2m}(w\cdot a - k_2\cdot a)^m (w\cdot a - k_3\cdot a)^m r_{|a|}^{(m)}(k_2\cdot a, k_3\cdot a, w\cdot a)\\
&+\sum_{m=0}^{\infty}\Bigg[\frac{(w\cdot a)^{2m+6}}{\xi^{m+2}} p_{|a|}^{(m)}(k_2\cdot a, k_3\cdot a, w\cdot a)\\
&+\xi^{m+2}(w\cdot a - k_2\cdot a)^{m+3}(w\cdot a - k_3\cdot a)^{m+3}\, q_{|a|}^{(m)}(k_2\cdot a, k_3\cdot a, w\cdot a)\Bigg] \qquad (1)
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- Polynomial include linear correction in $\omega|a|$
- Contact terms starting at a^4 [Huang + 2018; Bern+ 2022; Aoude + 2022]

Compton up to sixth order in spin

The r-polynomial

$$\begin{aligned} r_{|a|}^{(m)} = {} & c_1^{(m)} + c_2^{(m)}(k_2\cdot a + k_3\cdot a) + c_3^{(m)} w\cdot a + c_4^{(m)}|a|\omega \\ & + c_5^{(m)}(w\cdot a - k_2\cdot a)(w\cdot a - k_3\cdot a) \\ & + c_6^{(m)}(2w\cdot a - k_2\cdot a - k_3\cdot a) w\cdot a \\ & + c_7^{(m)}(2w\cdot a - k_2\cdot a - k_3\cdot a)^2 + c_8^{(m)}(w\cdot a)^2 \\ & + c_9^{(m)}(k_2\cdot a + k_3\cdot a)|a|\omega + c_{10}^{(m)} w\cdot a|a|\omega + \mathcal{O}(a^3) \end{aligned} \tag{2}$$

The $p-$ and $q-$polynomilas

$$p_{|a|}^{(m)} = d_1^{(m)} + \mathcal{O}(a)\,, \qquad q_{|a|}^{(m)} = f_1^{(m)} + \mathcal{O}(a)\,, \tag{3}$$

Compton up to sixth order in spin

The r-polynomial

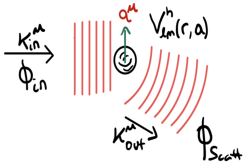
$$\begin{aligned} r_{|a|}^{(m)} = c_1^{(m)} &+ c_2^{(m)}(k_2{\cdot}a + k_3{\cdot}a) + c_3^{(m)} w{\cdot}a + c_4^{(m)}|a|\omega \\ &+ c_5^{(m)}(w{\cdot}a - k_2{\cdot}a)(w{\cdot}a - k_3{\cdot}a) \\ &+ c_6^{(m)}(2w{\cdot}a - k_2{\cdot}a - k_3{\cdot}a)w{\cdot}a \\ &+ c_7^{(m)}(2w{\cdot}a - k_2{\cdot}a - k_3{\cdot}a)^2 + c_8^{(m)}(w{\cdot}a)^2 \\ &+ c_9^{(m)}(k_2{\cdot}a + k_3{\cdot}a)|a|\omega + c_{10}^{(m)} w{\cdot}a|a|\omega + \mathcal{O}(a^3) \end{aligned} \quad (2)$$

The $p-$ and $q-$polynomilas

$$p_{|a|}^{(m)} = d_1^{(m)} + \mathcal{O}(a)\,, \qquad q_{|a|}^{(m)} = f_1^{(m)} + \mathcal{O}(a)\,, \quad (3)$$

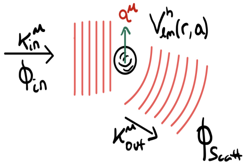
- 3-free conservative operators at a^4
- 5 conservative + 3 dissipative free operators at a^5
- 12 conservative (19 LD operators in [Aoude+2022]) +5 dissipative free operators at a^6

The gravitational setup



Parameters: $\begin{cases} \epsilon = 2GM\omega & \text{PM} \\ a^\star = \frac{a}{GM} & \text{spin} \end{cases}$ $\quad a^\star \epsilon = 2a\omega$

BHPT



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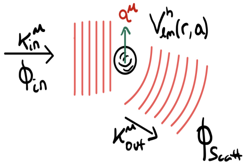
- [Teukolsky 1972]. NP formalism: Linear perturbations $\Psi_4 = \Psi_4^B + \delta\Psi_4$. $\Rightarrow$ Separation of variables in Kerr [See C. Kavanagh self-force review talk]. Radiative content in the scalar

$$ {}_h\psi \sim \sum_{\ell m} e^{-i\omega t} {}_hZ_{\ell m\omega} {}_hR_{\ell m\omega}(r) {}_hS_{\ell m}(\vartheta, \varphi, a\omega) \sim A_h $$



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- Asymptotic behavior:

$$_hR_{lm\omega} \sim \begin{cases} \frac{B^{(\text{inc})}_{lm\omega}(a^\star,\epsilon)}{r} e^{-i\omega r^*} + \frac{B^{(\text{ref})}_{lm\omega}(a^\star,\epsilon)}{r^{2h+1}} e^{i\omega r^*} & r^* \to \infty,\ (r \to \infty) \\ \underbrace{B^{\text{trans}}_{lm\omega}(a^\star, \epsilon) \frac{e^{-i\Omega_+ r^*}}{\Delta^h}}_{\text{purelly ingoing @ } r_+} & r^* \to -\infty,\ (r \to r_+) \end{cases}$$

Tree-level Scattering amplitude. Spin 4

- Remove the PW from the asymptotic Teukolsky solution ${}_{-2}\psi = \psi^{\text{PW}} + \psi^{\text{SW}}$

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$$f'^{\text{BHPT}}_{lm}(\gamma) = e^{i\Phi} \frac{\Gamma(l+1-i\epsilon)}{\Gamma(l+1+i\epsilon)} \left(1 + \beta^{(1)}_{lm}(\gamma, a^\star)\epsilon + \beta^{(2)}_{lm}(\gamma, a^\star)\epsilon^2 + \beta^{(3)}_{lm}(\gamma, a^\star)\epsilon^3 + \cdots\right)$$

- Point particle description $\Rightarrow a^\star \gg 1$. Delete BH horizon, naked singularity. keep $a^\star\epsilon = 2a\omega$ fixed.

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- Point particle description $\Rightarrow a^\star \gg 1$. Delete BH horizon, naked singularity. keep $a^\star\epsilon = 2a\omega$ fixed.
- Up to $i \leq 4$, the $\beta^{(i)}_{lm}(\gamma,a^\star)$ are REAL, and polynomial in $a^\star$. Unique analytic extension Complete agreement with the exponential

Spin 5 and 6, anomalous terms

- Non-polynomilas for $i > 4$.

$$\beta^{(5)}_{2,-1}(\gamma) = \frac{\sqrt{\pi/5}\sin^3\gamma}{42247941120}\Big[-43659(12017 + 17775\cos(2\gamma))a^{\star 5} - 1408264704\cos(\gamma)\, ia^{\star 4}\hat{\kappa}$$
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$$\beta^{(5)\eta}_{2,-1}(\gamma) = -\frac{\sqrt{\pi/5}}{967680}a^5\sin^3\gamma\Big[\Big(12017 + 17775\cos(2\gamma)\Big) + 96768\alpha - \eta 32256(1 + 4\alpha)\cos\gamma\Big].$$

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Compton Matching

Compton modes are given as integrals on the 2-sphere

$$f_{lm}^{\prime\text{QFT}}(\gamma) = \int d\Omega' \; {}_{-2}Y_{lm}^{*}(\theta, \phi') \langle A_4(\gamma, \theta, \phi') \rangle, \tag{4}$$

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For our mode example

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Agreement of the QFT and BHPT results means the equality

$$\boxed{f'^{\text{QFT}}_{lm}(\gamma) = f'^{\text{BHPT}}_{lm}(\gamma) ,} \tag{6}$$

is satisfied for all values of l, m.

- Linear system of equations for the Compton coefficients

Spin 6 results

Spin	Spurious-pole	Free Coeffs.	Teukolsky Solutions
a^4		$c_1^{(i)},\ i=0,1,2$	$c_1^{(i)}=0,\ i=0,1,2$
a^5	$c_3^{(2)}=4/15-c_3^{(0)}+c_3^{(1)}$	$c_2^{(i)},\ i=0,1,2$ $c_3^{(i)},\ i=0,1$ $c_4^{(i)},\ i=0,1,2$	$c_2^{(i)}=0,\ (i)=0,1,2$ $c_3^{(0)}=\alpha\frac{64}{15},\ c_3^{(1)}=\alpha\frac{16}{3},$ $c_3^{(2)}=\frac{4}{15}(1+4\alpha),$ $c_4^{(0)}=\eta\alpha\frac{64}{15},$ $c_4^{(1)}=\eta\alpha\frac{16}{5},\ c_4^{(2)}=\eta\frac{4}{15}$
a^6	$c_{10}^{(2)}=c_{10}^{(1)}-c_{10}^{(0)}$ $d_1^{(0)}=-\frac{8}{45}+\sum_{j=5}^{7}\sum_{i=0}^{2}(-1)^i c_j^{(i)}$ $f_1^{(0)}=\frac{4}{45}+c_6^{(0)}-c_6^{(1)}+\sum_{i=0}^{2}(-1)^i c_8^{(i)}$	$c_5^{(i)},\ i=0,1,2$ $c_6^{(i)},\ i=0,1,3$ $c_7^{(i)},\ i=0,1,2$ $c_8^{(i)},\ i=0,1,3$ $c_9^{(i)},\ i=0,1,2$ $c_{10}^{i},\ i=0,1$	$c_j^{(i)}=0,\ i=0,1,2,\ j=5,7$ $c_6^{(0)}=\alpha\frac{128}{45},\ c_6^{(1)}=\alpha\frac{32}{9},$ $c_6^{(2)}=\frac{8}{45}(1+4\alpha),\ c_8^{(0)}=-\alpha\frac{512}{45},$ $c_8^{(1)}=-\alpha\frac{160}{9},\ c_8^{(2)}=-\frac{16}{45}(1+19\alpha),$ $c_9^{(0)}=-\eta\alpha\frac{128}{45},\ c_9^{(1)}=-\eta\alpha\frac{32}{15},$ $c_9^{(2)}=-\eta\frac{8}{45},$ $c_{10}^{(0)}=-\eta\alpha\frac{256}{45},$ $c_{10}^{(1)}=-\eta\alpha\frac{352}{45},\ c_{10}^{(2)}=-\eta\alpha\frac{32}{15}$ $d_1^{(0)}=0,\ f_1^{(0)}=-\frac{4}{45}(1+4\alpha)$

- Not shift symmetry, even for $\eta=0$
- up to a^5, for $\eta=\alpha=0$, Shift-symmetric amplitude, Not true at a^6 :(
- Same helicity: $e^{(k_2-k_3)\cdot a}$ does not changes up to a^6.

Exact Kerr matching

- No ambiguity in the interpretation of the Compton operators for exact Kerr matching

Spin	Kerr Solution
a^5	$c_2^{(i)} = 0\,,\;\; i = 0,1,2$ $c_3^{(0)} = \frac{64}{45a^{\star 4}}(1+3a^{\star 2})\Re\left(\psi_0(2i\frac{a^\star}{\hat{\kappa}})\right)$ $c_3^{(1)} = \frac{8}{45a^{\star 4}}\left((4-3a^{\star 2})\Re\left(\psi_0(i\frac{a^\star}{\hat{\kappa}})\right) + 12(1+3a^{\star 2})\Re\left(\psi_0(2i\frac{a^\star}{\hat{\kappa}})\right)\right)$ $c_4^{(0)} = \frac{32(1+3a^{\star 2})}{45a^{\star 5}}\left(i\hat{\kappa} - 2\Im\left(\psi_0(2i\frac{a^\star}{\hat{\kappa}})\right)\right)$ $c_4^{(1)} = \frac{8}{45a^{\star 5}}\left((8+9a^{\star 2})i\hat{\kappa} - a^\star(4-3a^{\star 2})\Im\left(\psi_0(i\frac{a^\star}{\hat{\kappa}})\right) - 8a^\star(1+3a^{\star 2})\Im\left(\psi_0(2i\frac{a^\star}{\hat{\kappa}})\right)\right)$ $c_4^{(2)} = \frac{4}{45a^{\star 5}}\left((2+6a^{\star 2}-3a^{\star 4})i\hat{\kappa} - a^\star(4-3a^{\star 2})\Im\left(\psi_0(i\frac{a^\star}{\hat{\kappa}})\right) - a^\star(2+3a^{\star 2})\Im\left(\psi_0(2i\frac{a^\star}{\hat{\kappa}})\right)\right)$

Table 3: Exact matching to spin operators, where coefficients are relaxed to functions of the spin norm "a". Here a^5 refers to quintic monomials in $\{k_2 \cdot a, k_3 \cdot a, w \cdot a\}$ but to all orders in the norm. In the large a limit, they reduce to the coefficients of table 2.

Conclusions

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- We have extracted a unique conservative ($\eta = 0$) amplitude up to the sixth order in spin from solutions of the Teukolsky equation. Unfortunately the solutions do not preserve spin-shift-symmetry
- BH horizon dissipative effects can be accounted for in a gravitational Compton ansatz, with operators proportional to $|a|$. Imaginary contributions: branch choice subtlety that needs further investigation.
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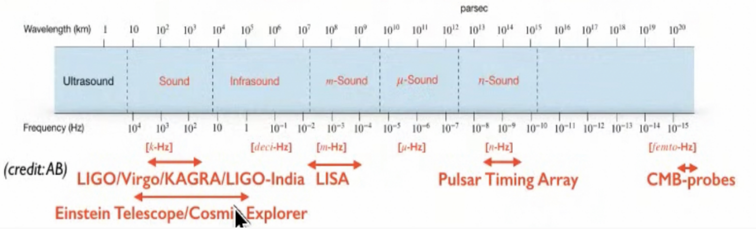
- How does the Spin supplementary condition change by allowing $|a|$ operators?
- But how about real Kerr? ($a^\star < 1$). Extract all orders in G solutions from Teukolsky.
- Future: Higher spins, higher loops.
- Double copy and the relation to $\sqrt{\text{Kerr}}$?: Linear electromagnetic perturbations of Kerr-Newman in the $GM \to 0$ limit

Thank you for your attention!

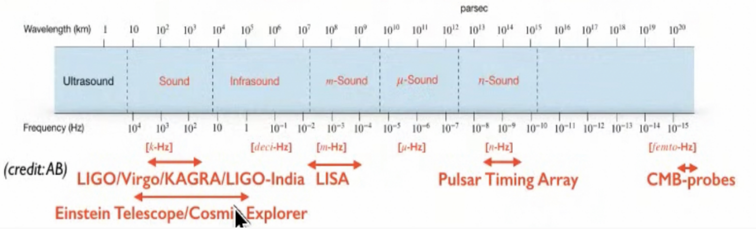
Extra slides

Gravitational waves era and templates demand



GW templates should have into account as much information about the binary as possible. In particular:

Gravitational waves era and templates demand



GW templates should have into account as much information about the binary as possible. In particular:

- spin effects: Expected to be measured with great precision in LISA for nearly extremal BH [Burke et al 2020]
- Astrophysical implications: Spin effects $\Rightarrow$ information about the Binary's formation mechanism

Spin-shift symmetry

- Low spin observations [Bern + 2022, Aoude+ 2022]: Opposite helicity amplitude $e^{(2w-k_2-k_3)\cdot a}$ invariant under (See R. Roiban review talk)

$$a^\mu \to a^\mu + \varsigma_b q^\mu / q^2 \, ,$$

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but not same helicity amplutude $e^{(k_2-k_3)\cdot a}$

Spin	Shift-Sym.	Free Coeffs.	Relation to [Aoude+]
a^4	$c_1^{(i)} = 0,\ i = 1,2$	$c_1^{(0)}$	$c_1^{(0)} = -\frac{d_0^{(4)}}{4!}$
a^5	$c_j^{(i)} = 0,\ i = 1,2,\ j = 2,3$ $c_3^{(0)} = \frac{4}{15},\ c_4^{(i)} = 0\,, i = 0,1,2$	$c_2^{(0)}$	$c_2^{(0)} = \frac{32+5d_0^{(4)}-d_0^{(5)}}{5!}$
a^6	$c_j^{(i)} = 0,\ i = 0,1,2,\ j = 5,9,10$ $c_j^{(i)} = 0,\ i = 1,2,\ j = 6,7,8$ $c_8^{(0)} = -\frac{4}{45} - c_6^{(0)}$ $f_1^{(0)} = 0$ $d_1^{(0)} = -\frac{8}{45} + c_6^{(0)} + 4c_7^{(0)}$	$c_6^{(0)}, c_7^{(0)}$	$c_6^{(0)} = \frac{176 + d_0^{(4+5+6)} + d_1^{(6)}}{180}$ $c_7^{(0)} = -\frac{128 + d_0^{(4+5+6)}}{6!}$

Spin-shift symmetry

- Low spin observations [Bern + 2022, Aoude+ 2022]: Opposite helicity amplitude $e^{(2w-k_2-k_3)\cdot a}$ invariant under (See R. Roiban review talk)

$$a^\mu \to a^\mu + \varsigma_b q^\mu / q^2 \,,$$

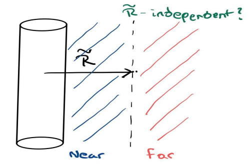
but not same helicity amplutude $e^{(k_2-k_3)\cdot a}$

Spin	Shift-Sym.	Free Coeffs.	Relation to [Aoude+]
a^4	$c_1^{(i)} = 0,\ i = 1,2$	$c_1^{(0)}$	$c_1^{(0)} = -\frac{d_0^{(4)}}{4!}$
a^5	$c_j^{(i)} = 0,\ i = 1,2,\ j = 2,3$ $c_3^{(0)} = \frac{4}{15},\ c_4^{(i)} = 0, i = 0,1,2$	$c_2^{(0)}$	$c_2^{(0)} = \frac{32+5d_0^{(4)}-d_0^{(5)}}{5!}$
a^6	$c_j^{(i)} = 0,\ i = 0,1,2,\ j = 5,9,10$ $c_j^{(i)} = 0,\ i = 1,2,\ j = 6,7,8$ $c_8^{(0)} = -\frac{4}{45} - c_6^{(0)}$ $f_1^{(0)} = 0$ $d_1^{(0)} = -\frac{8}{45} + c_6^{(0)} + 4c_7^{(0)}$	$c_6^{(0)}, c_7^{(0)}$	$c_6^{(0)} = \frac{176 + d_0^{(4+5+6)} + d_1^{(6)}}{180}$ $c_7^{(0)} = -\frac{128 + d_0^{(4+5+6)}}{6!}$

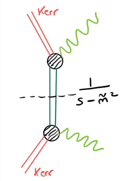
- In this talk: Does this symmetry emerge from Teukolsky solutions?

Love proposal [Ivanov +, 2022]

$$\frac{B^{(\text{refl})}_{s\ell m}}{B^{(\text{inc})}_{s\ell m}} = \frac{1}{\omega^{2s}} \underbrace{\frac{1 + ie^{i\pi\nu}\frac{K_{-\nu-1}}{K_\nu}}{1 + ie^{-i\pi\nu}\frac{\sin(\pi(\nu-s+i\epsilon))}{\sin(\pi(\nu+s-i\epsilon))}\frac{K_{-\nu-1}}{K_\nu}}}_{\text{Near zone}} \times \underbrace{\frac{A^\nu_-}{A^\nu_+} e^{i\epsilon(2\ln\epsilon-(1-\kappa))}}_{\text{Far zone}} \Rightarrow$$



Spin	"Far zone" solutions
a^4	$c_1^{(0)} = -\frac{189056}{103041},\ c_1^{(1)} = -\frac{86044}{34347},\ c_1^{(2)} = -\frac{10402}{34347},$
a^5	$c_2^{(0)} = \frac{114208}{148837},\ c_2^{(1)} = \frac{1130912}{744185},\ c_2^{(2)} = \frac{435488}{2232555}$ $c_3^{(0)} = \frac{286064608}{1157665635},\ c_3^{(1)} = -\frac{2531080196}{15049653255},\ c_3^{(2)} = -\frac{745559744}{5016551085}$ $c_4^{(i)} = 0,\ i = 0,1,2,\ e_1^{(0)} = \frac{64}{195},\ e_1^{(1)} = \frac{32}{117},\ e_1^{(2)} = \frac{8}{195}$



- Match the exponential up to a^3
- Contact deformations at a^4
- Extra, non-contact contribution to the Compton at a^5

$$\Delta f_\xi = e_1^{(0)}\frac{(w\cdot a)^5}{\xi^2} + e_1^{(1)}\frac{(w\cdot a)^5(k_2\cdot a)(-k_3\cdot a)}{\xi}$$
$$+ (w\cdot a - k_2\cdot a)(w\cdot a + k_3\cdot a)w\cdot a\left(e_1^{(2)}(k_2\cdot a)(-k_3\cdot a) - e_1^{(0)}(w\cdot a)^2\right)$$

- On the good side, only polynomials in $a^\star$

2PM aligned-spins scattering angle

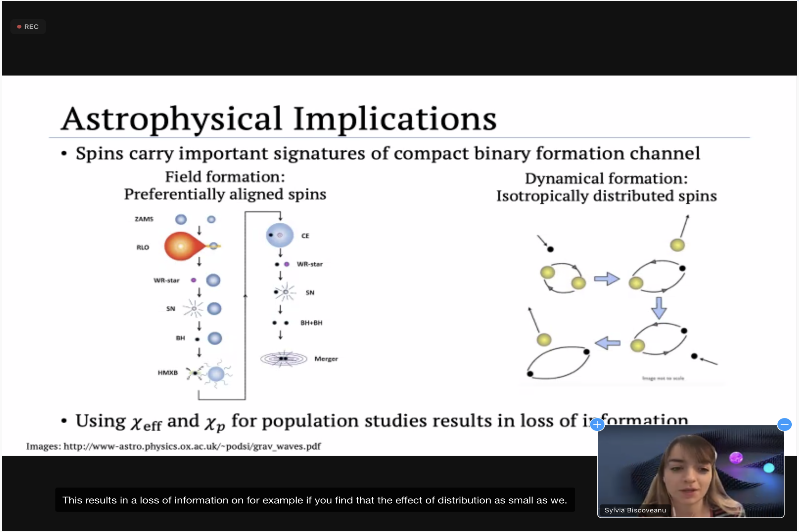
$$\theta_{\triangleleft}^{(4)} = \theta_{\triangleleft,\text{GOV}}^{(4)} - \frac{45\pi G^2 m_2 E b}{32 v^2 \gamma^2 (b^2 - a_2^2)^{7/2}} c_1^{(0+1+2)}$$

$$\theta_{\triangleleft}^{(5)} = \frac{\pi G^2 m_2 E}{v^2 (b^2 - a_2^2)^{9/2}} \Bigg[\frac{315 a_2 b}{32 \gamma^2} c_2^{(0+1+2)} - \frac{3}{16 (b^2 - a_2^2)^3 v^2} \Big(30 b^8 v (3 + v^2) + a_2^8 v (104 + 135 v^2) + 5 a_2^2 b^6 v (509 + 375 v^2) + 15 a_2^4 b^4 v (435 + 446 v^2) + 6 a_2^6 b^2 v (458 + 547 v^2) - 35 a_2 b^7 (6 + 25 v^2 + v^4) - 28 a_2^7 b (6 + 49 v^2 + 10 v^4) - 42 a_2^5 b^3 (30 + 203 v^2 + 37 v^4) - 21 a_2^3 b^5 (65 + 345 v^2 + 54 v^4) \Big) \Bigg] .$$

$$\theta_{\triangleleft}^{(6)} = - \frac{\pi G^2 m_2 E}{32 v^2 (b^2 - a_2^2)^{9/2}} \Bigg[\frac{945 b}{\gamma^2} \Big(\frac{5}{12} c_6^{(0+1+2)} + \frac{2b^2 + a_2^2}{b^2 - a_2^2} c_7^{(0+1+2)} \Big) + \frac{7}{(b^2 - a_2^2)^{11/2} v^2} \Big(80 a_2 b^8 v (11 + 5 v^2) + 4 a_2^9 v (28 + 37 v^2) + 100 a_2^7 b^2 v (38 + 47 v^2) + 10 a_2^3 b^6 v (827 + 677 v^2) + 6 a_2^5 b^4 v (2113 + 2287 v^2) + 5 b^9 (5 v^4 - 23 v^2 - 6) - 5 a_2^2 b^7 (201 + 880 v^2 + 83 v^4) - 15 a_2^4 b^5 (216 + 1219 v^2 + 202 v^4) - a_2^8 b (192 + 1690 v^2 + 353 v^4) - 2 a_2^6 b^3 (984 + 7060 v^2 + 1331 v^4) \Big) \Bigg] .$$

- Digamma contributions drop out from the scattering angle
- Kerr: $c_1^{(0+1+2)} = 0$, $c_2^{(0+1+2)} = 0$, $c_6^{(0+1+2)} = \frac{8}{45}$, and $c_7^{(0+1+2)} = 0$

Famous BBH formation mechanisms



Credits: [Sylvia Biscoveanu 2021 talk @ PI]