# Scattering of spinning bodies with worldline quantum field theory

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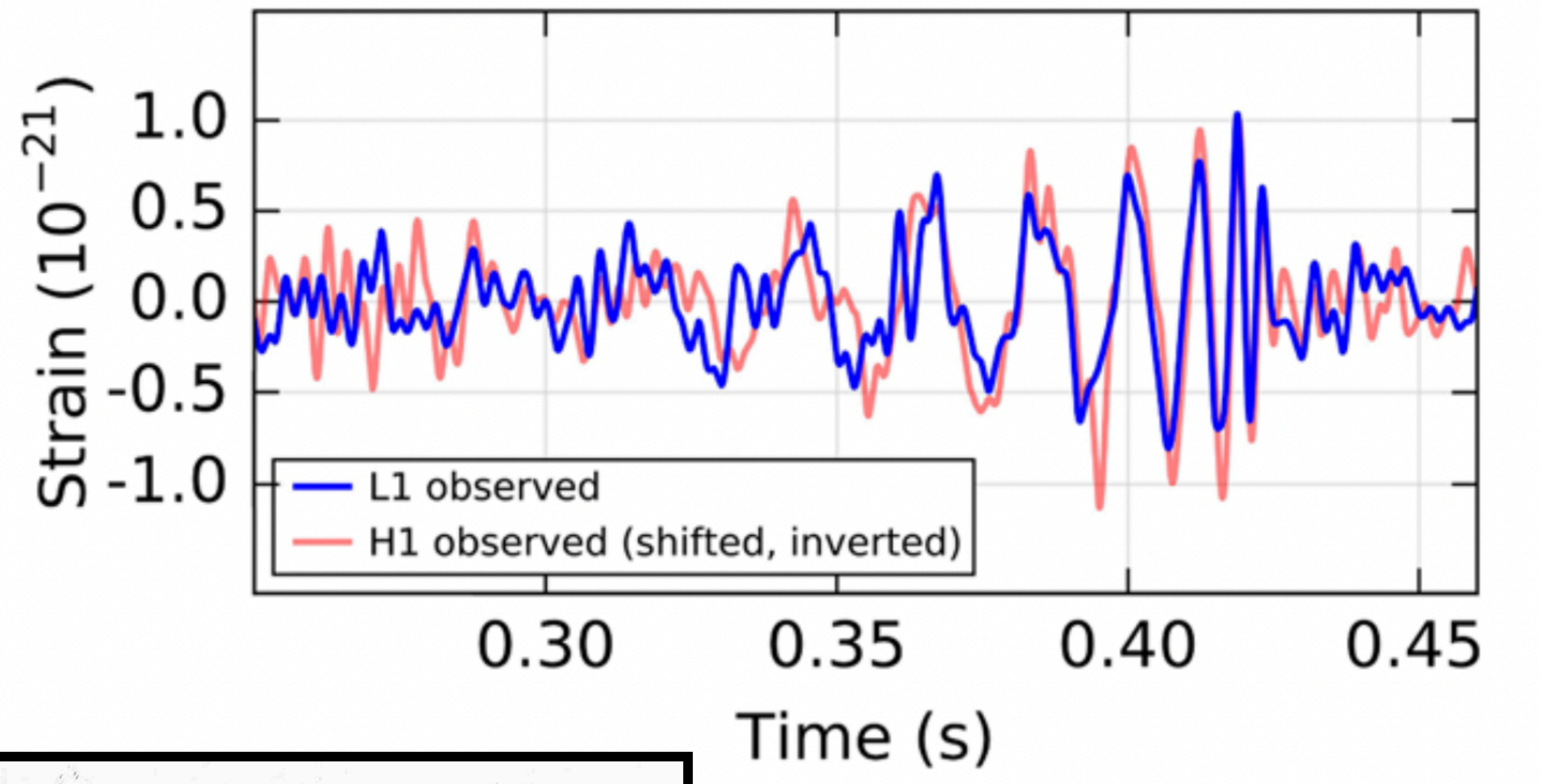
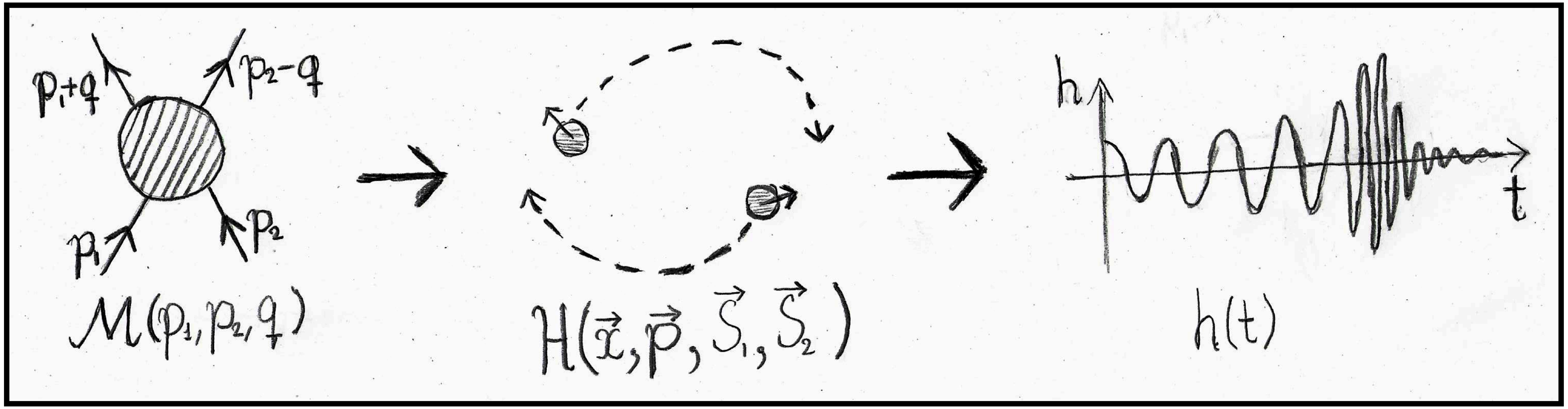
RTG 2575: Rethinking Quantum Field Theory

# Gravitational Waves from Compact Binaries

## From Scattering and QFT

GW150914, LIGO and VIRGO collaborations: 1608.01940



# Worldline Quantum Field Theory

- Classical limit is manifest and simply given by tree-level
- With in-in formalism all (radiative and conservative) contributions are included
- Spin and finite size effects are simply included
- Integrand-construction is **not** a bottle-neck with WQFT Feynman diagrams
- Optimized loop integrals for classical limit

# Worldline Quantum Field Theory

- Gravitational bremsstrahlung from scattering of spinning objects
  [with G.Mogull, J.Plefka, J.Steinhoff: 2106.10256, 2101.12688]
- SUSY WQFT formalism of spinning compact bodies
  [with G.M, J.P, J.S: 2109.04465]
- Full radiative observables: in-in formulation of WQFT
  [with G.M, J.P, B.Sauer: 2207.00569]
- Full spinning observables and conservative Hamiltonian at 3PM
  [with G.M: 2201.07778, 2210.06451]

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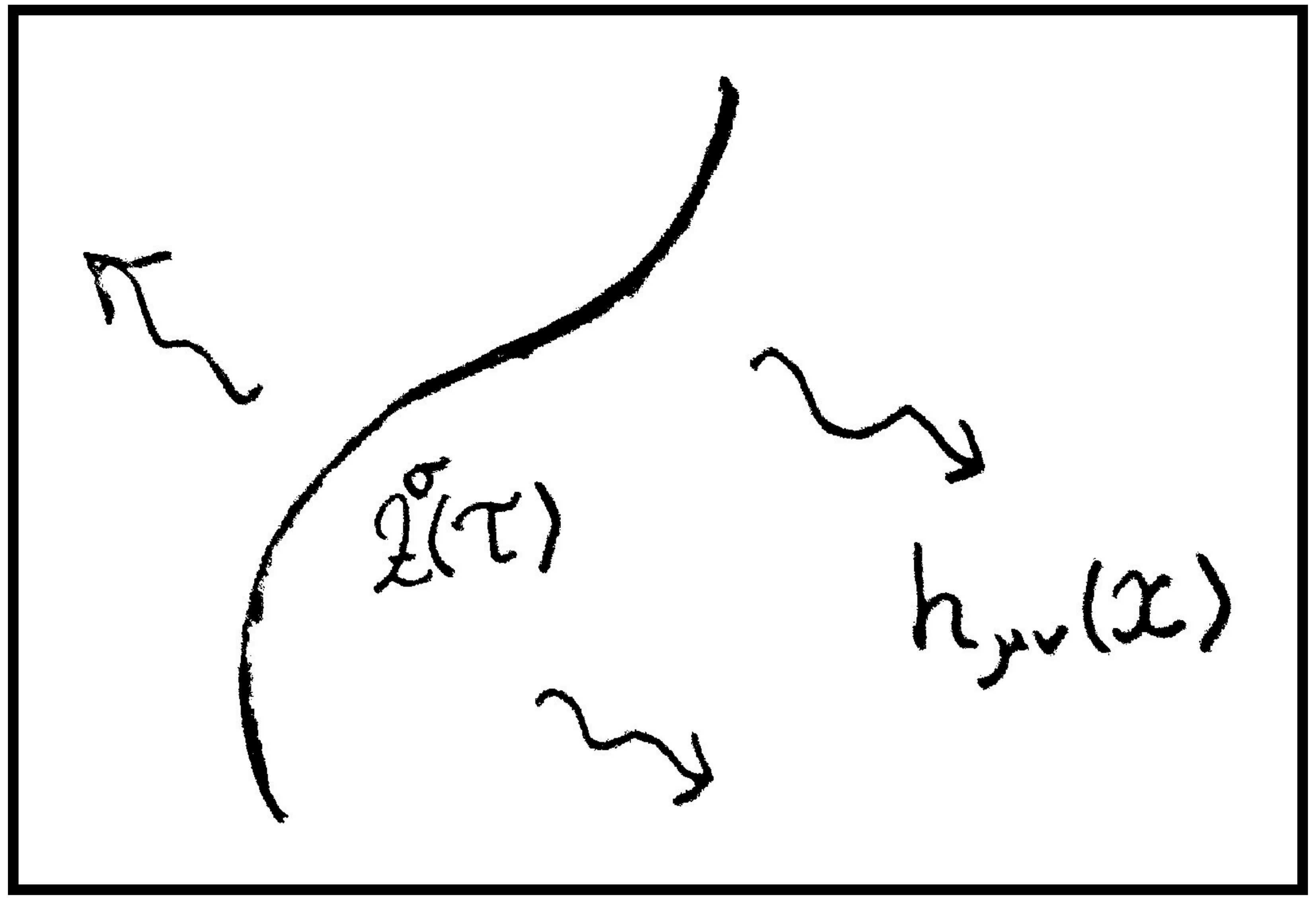
# WQFT, brief review

- Worldline effective field theory formalism with quantized worldline fields.
- One-dimensional worldline fields $z_a^\sigma(\tau)$ and D-dimensional bulk field $h_{\mu\nu}(x)$.
- Classical limit is tree-level but lack of (spacial) translational symmetry introduces loop-like integrals.



# WQFT in-in formalism

- In-in expectation values instead of in-out “transition” amplitudes.
  [Schwinger ; Keldysh ; Galley, Tiglio ; …]

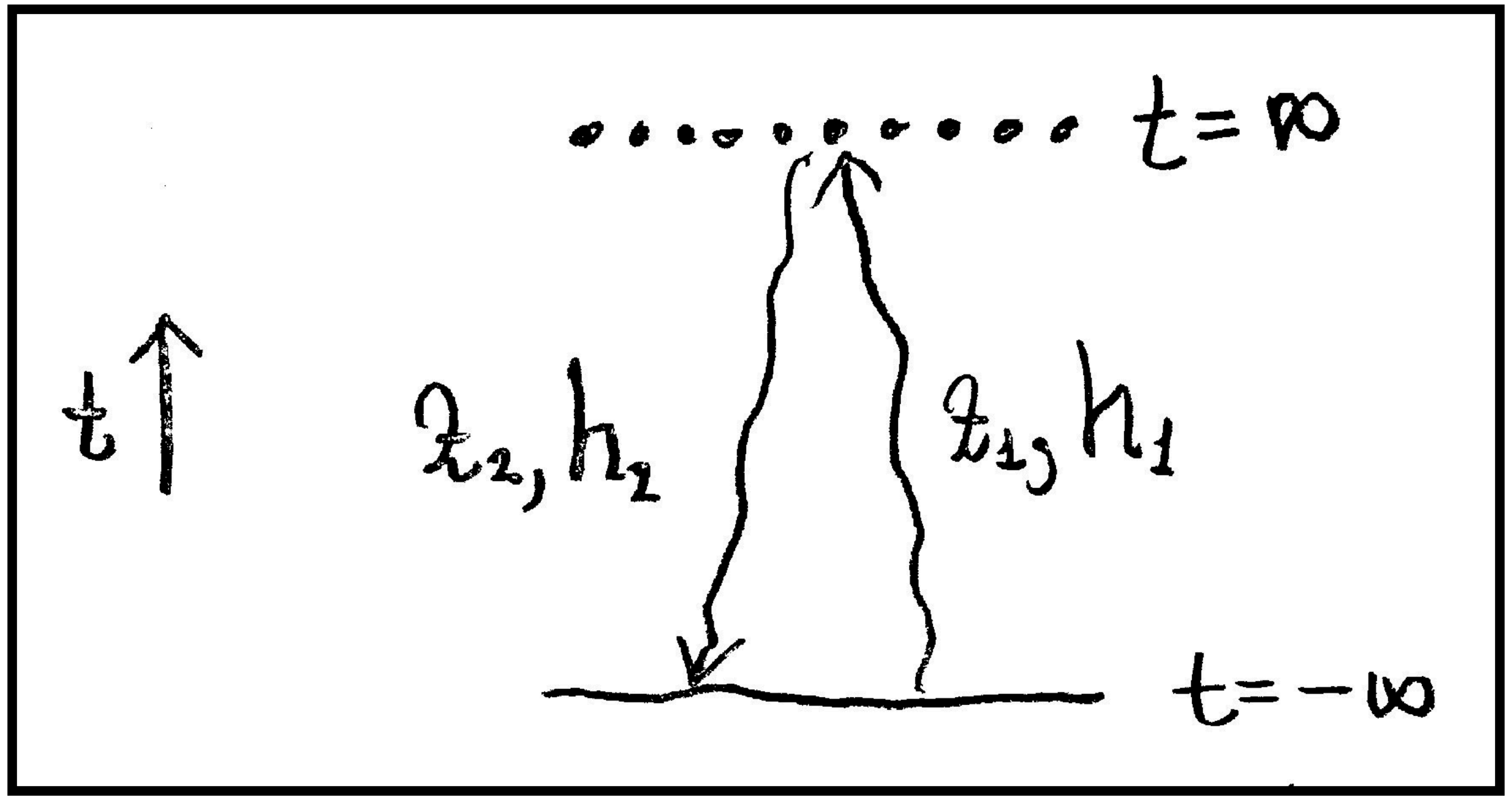
- Time integral on $t$ follows closed contour from $-\infty$ to $+\infty$.

- In-in action: $S_{\text{in}-\text{in}} = S[z_1, h_1] - S[z_2, h_2]$

- Schwinger-Keldysh basis:

$$h_- = h_1 - h_2$$

$$h_+ = \frac{1}{2}(h_1 + h_2)$$



# WQFT in-in formalism

- Spinless WQFT with fields $z_a^\sigma(\tau)$ and $g_{\mu\nu}(x)$:

$$S[z_a, g] = S_{\rm EH} + S_{\rm gf} - \sum_a \frac{m}{2} \int_\tau \dot{z}_a^\mu(\tau)\dot{z}_a^\nu(\tau) g_{\mu\nu}(z_a(\tau))$$

after doubling of fields, Schwinger-Keldysh basis and classical limit:

$$S_{\rm in-in} = \sum_a \int_\tau z_{a,-}^\mu(\tau) \frac{\delta S[z_{a,+}, g_+]}{\delta z_a^\mu(\tau)} + \int_x g_{-\mu\nu}(x) \frac{\delta S[z_+, g_+]}{\delta g_{\mu\nu}(x)}$$

# WQFT in-in Path-integral

- Post-Minkowskian expansion:

$$z_a^\sigma(\tau) = b_a^\sigma + \tau v_a^\sigma + \Delta z_a^\sigma(\tau)$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sqrt{G}\ h_{\mu\nu}(x)$$

- In-in path integral:

$$\langle \mathcal{O} \rangle_{\mathrm{WQFT}} = \int D[h_+, h_-, z_{a,+}, z_{a,-}] e^{iS_{\text{in}-\text{in}}} \mathcal{O}$$

with:

$$S_{\text{in}-\text{in}} = \int_\tau \Delta z_-^\sigma(\tau) \frac{\delta S_{\text{in}-\text{out}}(z_+, g_+)}{\delta \Delta z^\sigma(\tau)} + \int_x h_-^{\mu\nu}(x) \frac{\delta S_{\text{in}-\text{out}}(z_+, g_+)}{\delta h^{\mu\nu}(x)}$$

# Observables and correlation functions

- One-point functions $h_c^{\mu\nu} = \langle h_+^{\mu\nu}(k) \rangle_{\text{WQFT}}$ and $z_c^\mu = \langle z_+^\mu(\omega) \rangle_{\text{WQFT}}$ correspond to gravitational field and worldline trajectory and solve equations of motion:

$$\frac{\delta S[z_c, h_c]}{\delta z^\sigma} = 0$$

$$\frac{\delta S[z_c, h_c]}{\delta g_{\mu\nu}} = 0$$

# Observables and correlation functions

- Impulse:

$$\Delta p^{\mu} = \omega^2 \langle \Delta z_+^{\mu}(\omega) \rangle_{\mathrm{WQFT}} \Big|_{\omega \to 0}$$

with $\Delta p^{\mu} = p^{\mu}_{\mathrm{final}} - p^{\mu}_{\mathrm{initial}}$

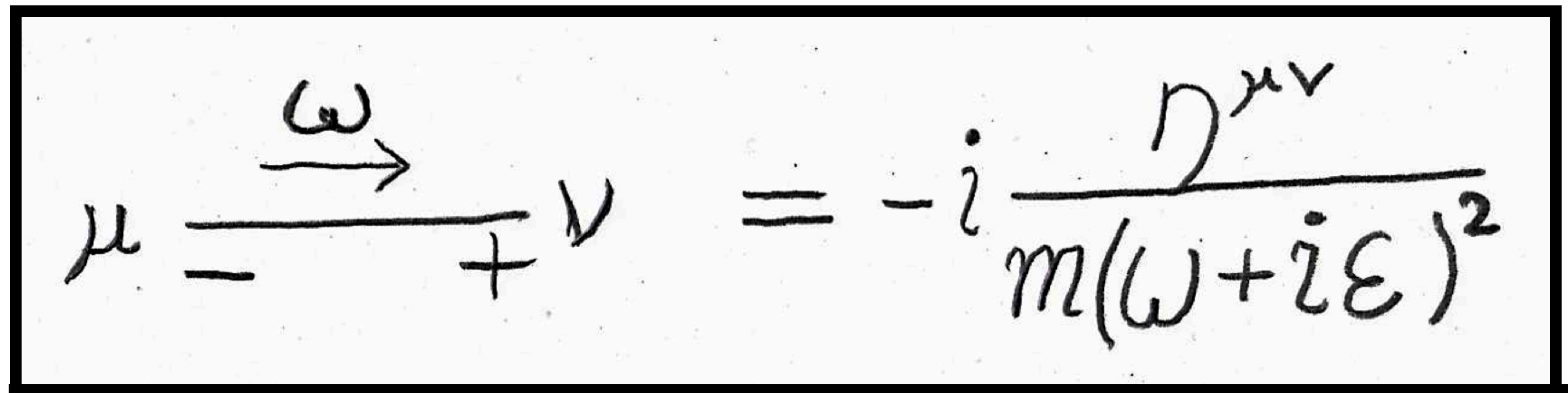
- Waveform in frequency domain:

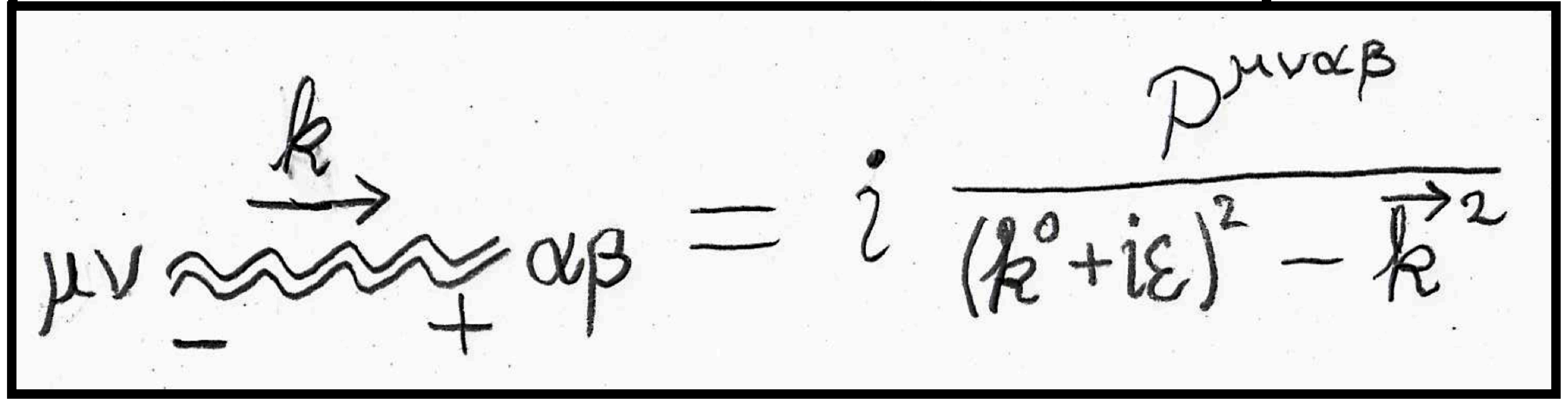
$$f(\omega, \vec{n}) = \epsilon_{\mu\nu} k^2 \langle h_+^{\mu\nu}(k) \rangle_{\mathrm{WQFT}} \Big|_{k^{\mu} \to \omega(1, \vec{n})}$$

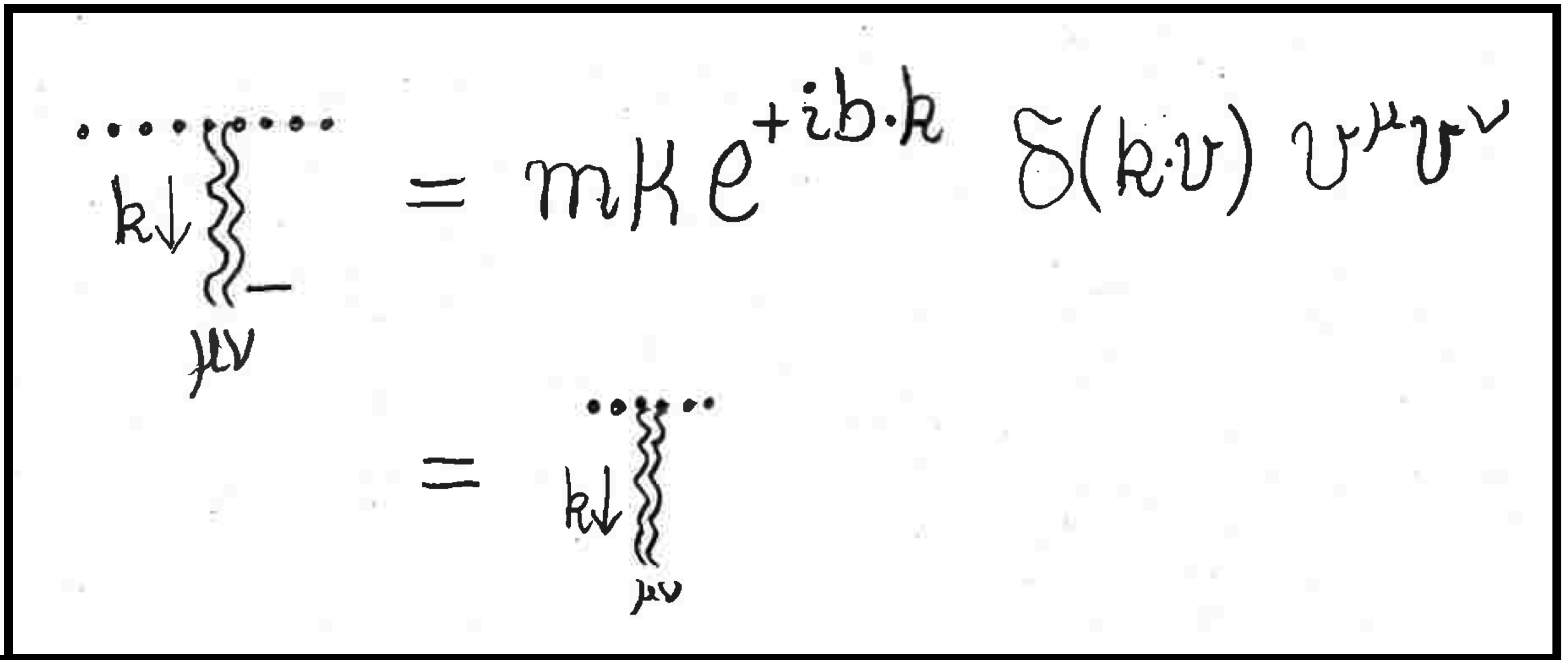
with $r h_{\mu\nu} \sim f \epsilon_{\mu\nu} + \mathcal{O}(r^{-1})$

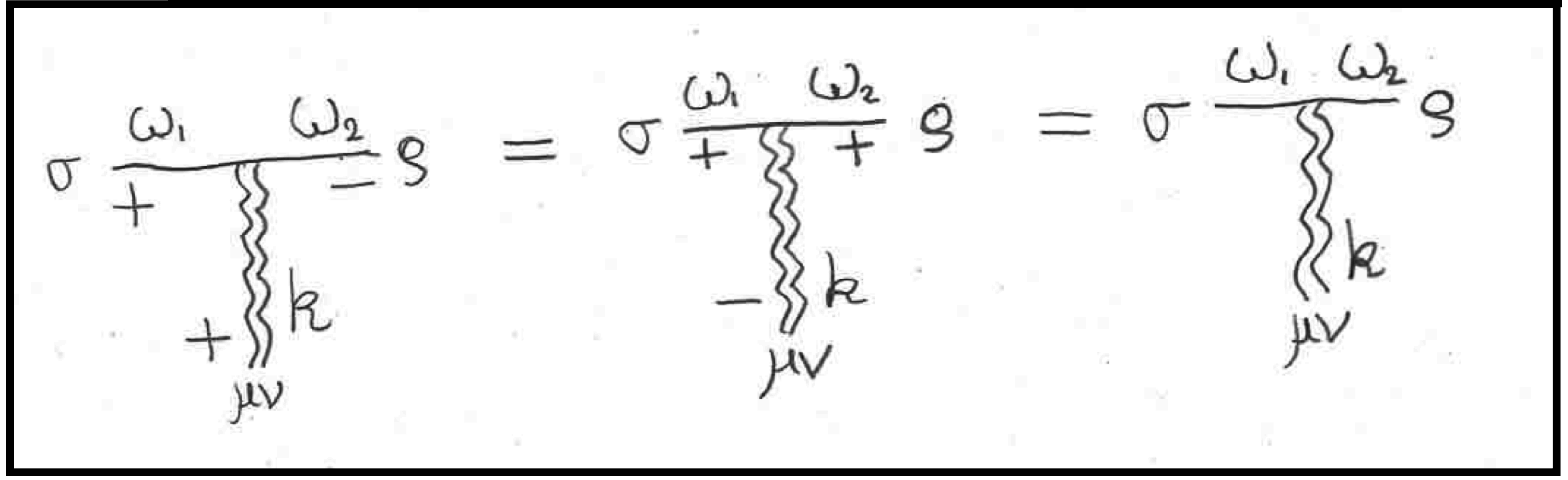
# WQFT in-in Feynman rules

- In-in Feynman rules almost identical to in-out



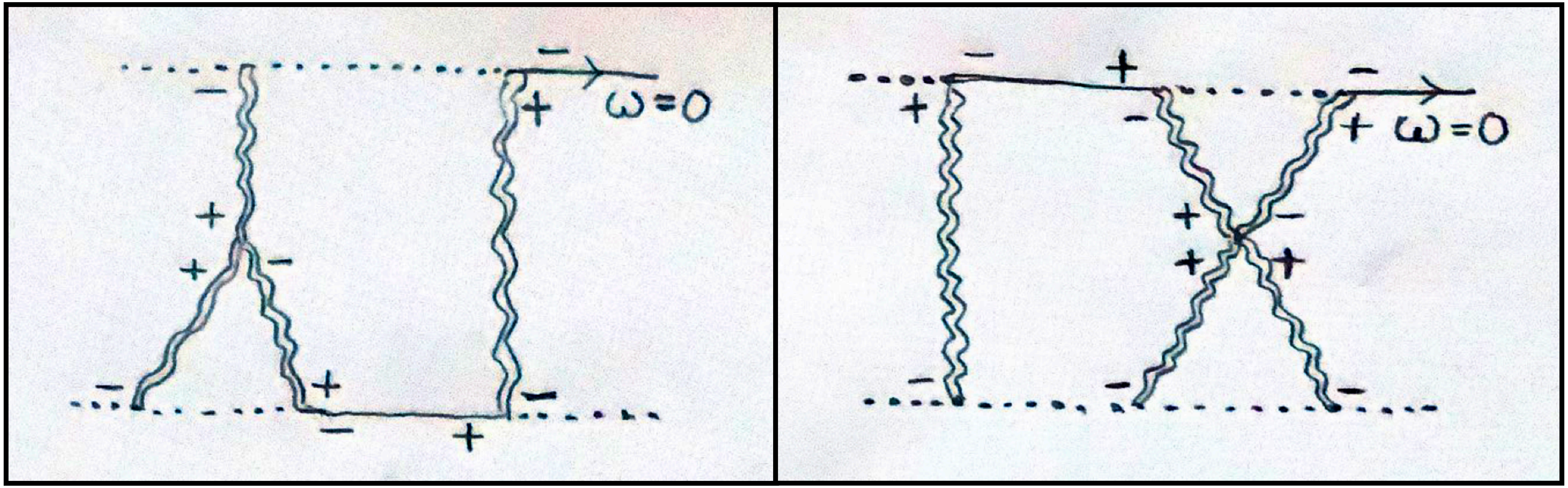
Solid lines: $\Delta z^\sigma_\pm(\omega)$

Wavy lines: $h_{\mu\nu}(k)$

Dotted lines: Background, $b^\sigma + \tau v^\sigma$

# WQFT Feynman diagrams

- WQFT diagram examples:




Solid lines: $\Delta z^{\sigma}_{\pm}(\omega)$

Wavy lines: $h_{\mu\nu}(k)$

Dotted lines: Background, $b^{\sigma} + \tau v^{\sigma}$

# PM-integration

## Many approaches, same problem

- Loop integration with retarded propagators $(\vec{k}^2 - (k^0 + i\epsilon)^2)^{-1}$ instead of Feynman $(k^2 + i\epsilon)^{-1}$

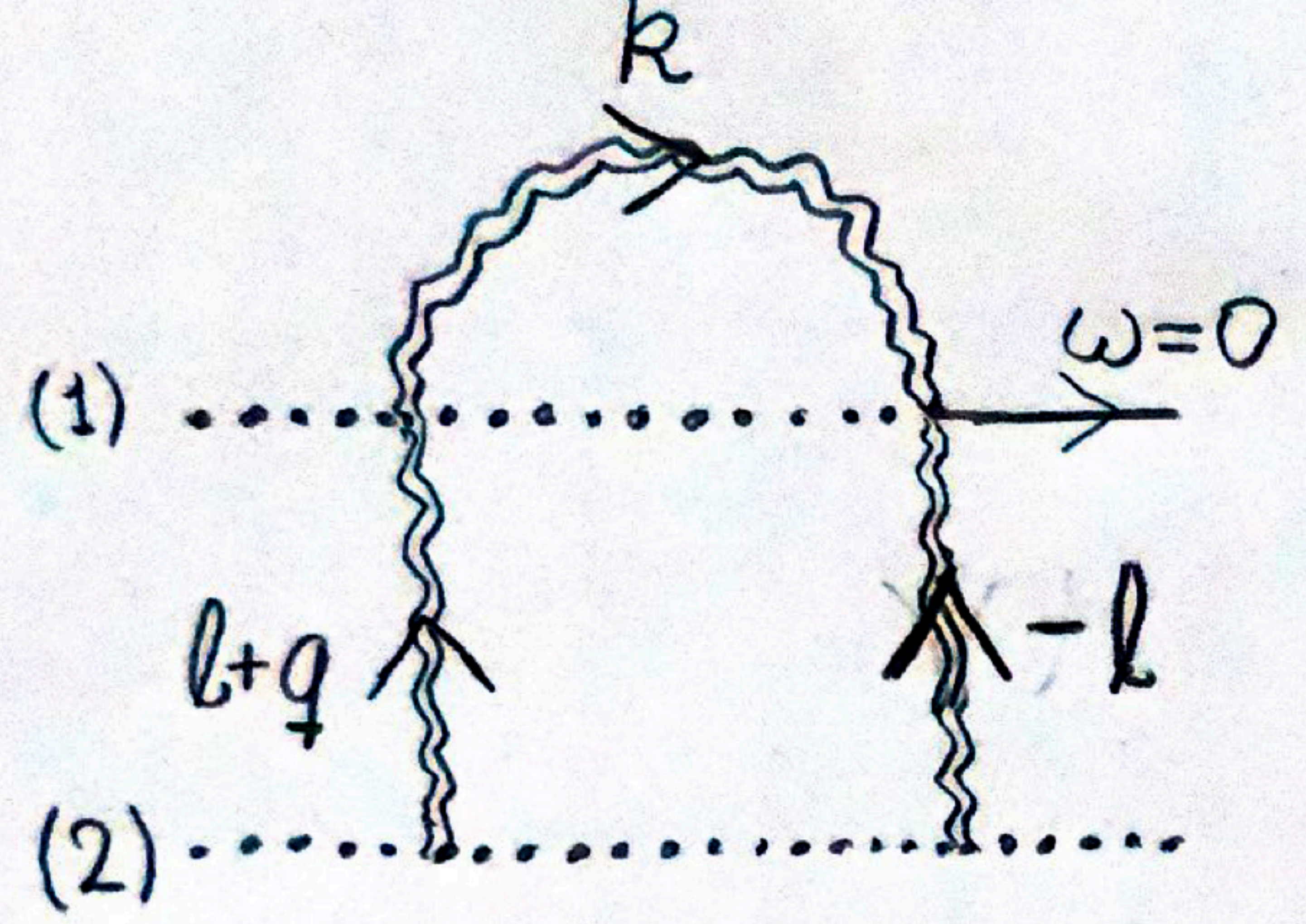
- This approach was also used to get the recent 4PM radiative results.
  [C. Dlapa, G. Kälin, Z. Liu, J. Neef, R.A. Porto: 2210.05541]

- Only difference to Feynman $i\epsilon$: symmetry relations and boundary value integrals.

# PM-integration

## Many approaches, same problem

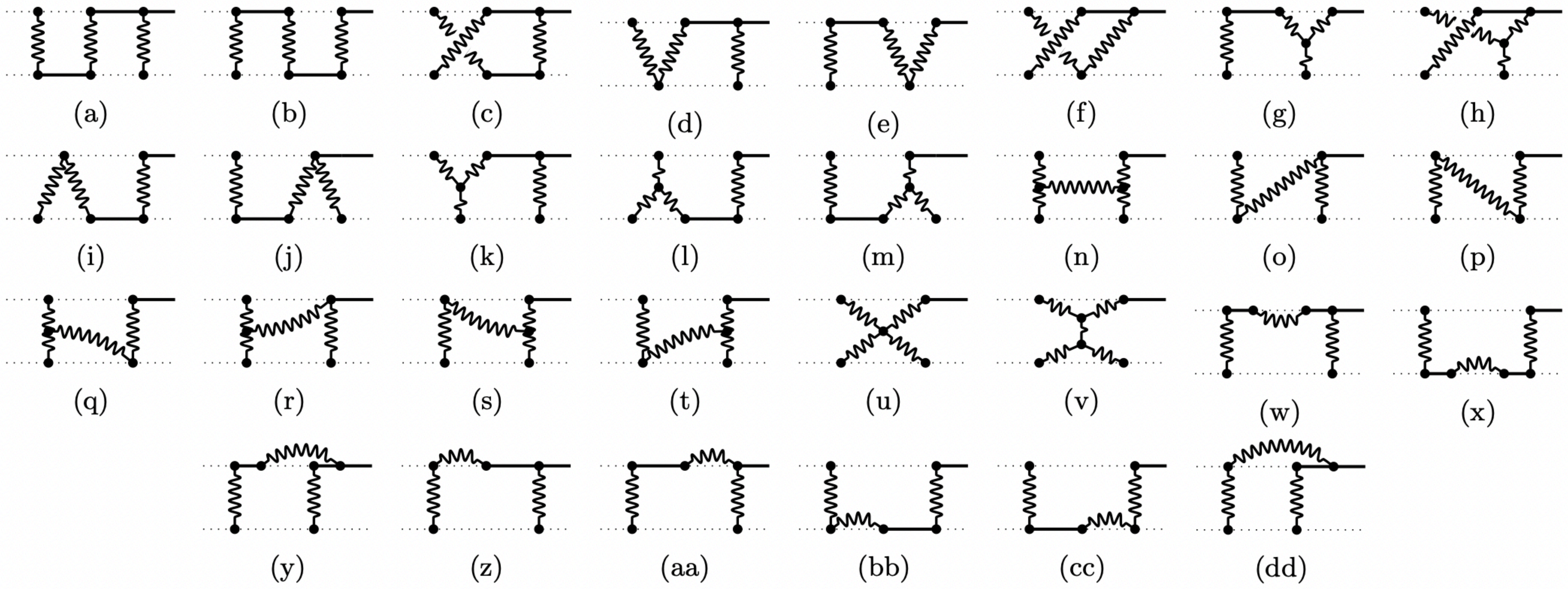
- WQFT loop integral example:



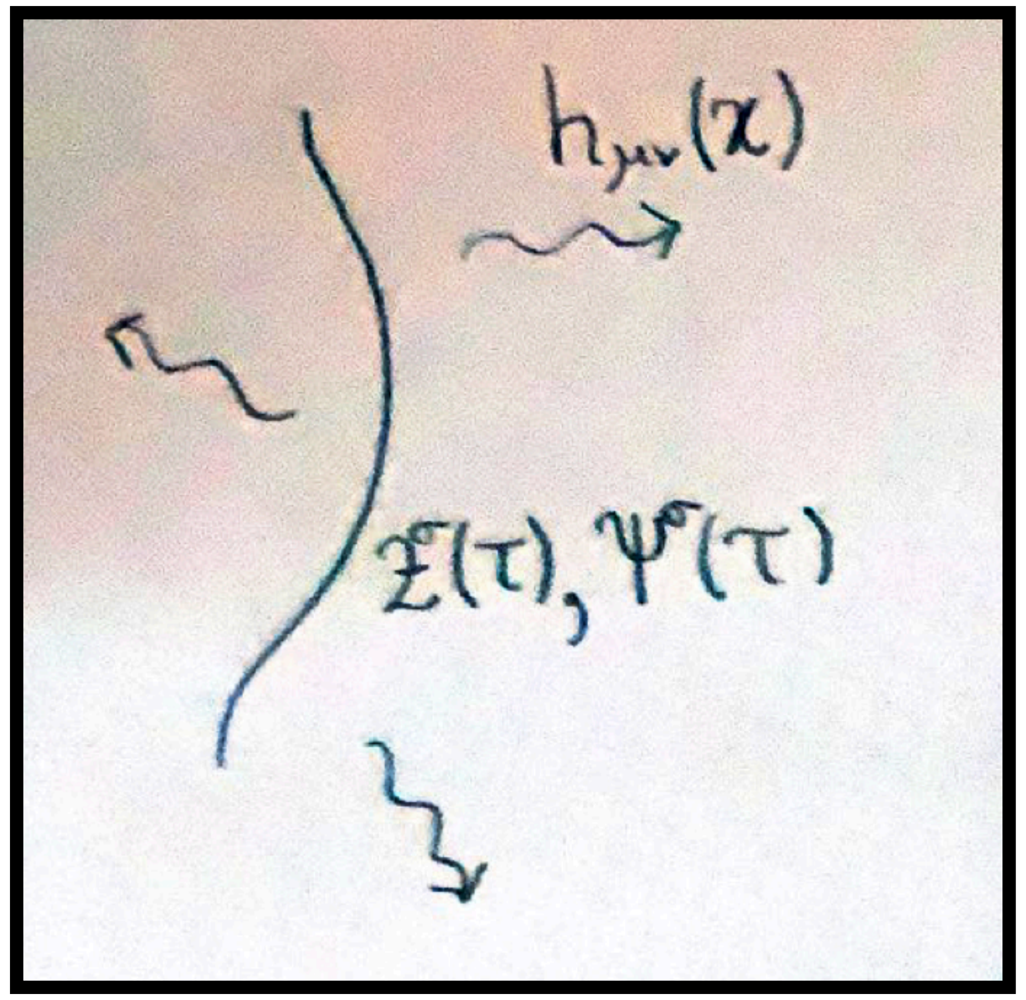

$$= \int d^D l\, d^D k \, \frac{\delta(l\cdot v_2)\,\delta(l\cdot v_1 - k\cdot v_2)}{l^2\,(l+q)^2\left((k^0+i\varepsilon)^2 - \vec{k}^{\,2}\right)}$$

# Spinning observables at 3PM order

With G. Mogull: 2201.07778 and 2210.06451



# SUSY WQFT, brief review




- Spin effects at linear order in curvature are well known.
  [M. Levi, J. Steinhoff ; R.A.Porto]

- We use Grassmann variables to describe spin degrees of freedom:
  $$S^{\mu\nu} = -\, i\psi_A^{[\mu}\psi_A^{\nu]}$$

- (Approximate) SUSY symmetry corresponds to “SSC” symmetry

- Action at $\mathcal{O}(S^2)$:
  $$S_{\mathrm{w.l.}} = -\int_\tau \left[\frac{m}{2}(\dot{z}^2 + i\bar{\psi}\dot{\psi}) + R_{\alpha\beta\gamma\delta}\bar{\psi}^\alpha\psi^\beta\bar{\psi}^\gamma\psi^\delta + C_E R_{\alpha\beta\gamma\delta}\dot{z}^\beta\dot{z}^\delta\bar{\psi}^\alpha\psi^\gamma\bar{\psi}\cdot\psi\right]$$

  [Mogull, Plefka, Steinhoff, G.U.J: 2109.04465]

# Scattering observables at 3PM

- Full results for impulse $\Delta p^{\mu}$ and spin kick $\Delta S^{\mu\nu}$ to $\mathcal{O}(S^2, G^3)$ including all conservative and radiative contributions.

- Generic spins response relations:

$$\Delta p^{\mu}_{1,\text{rad}}\Big|_{\text{even in v}^{\mu}} = \frac{\partial \Delta p^{\mu}_{1,\text{pot}}}{\partial J^{\alpha}} \Delta J^{\alpha}$$

$$\Delta S^{\mu\nu}_{1,\text{rad}}\Big|_{\text{even in v}^{\mu}} = \frac{\partial \Delta S^{\mu\nu}_{1,\text{pot}}}{\partial J^{\alpha}} \Delta J^{\alpha}$$

- Change in angular momentum with spin was computed in [G. Mogull, J. Plefka, J. Steinhoff, G.U.J: 2106.10256]

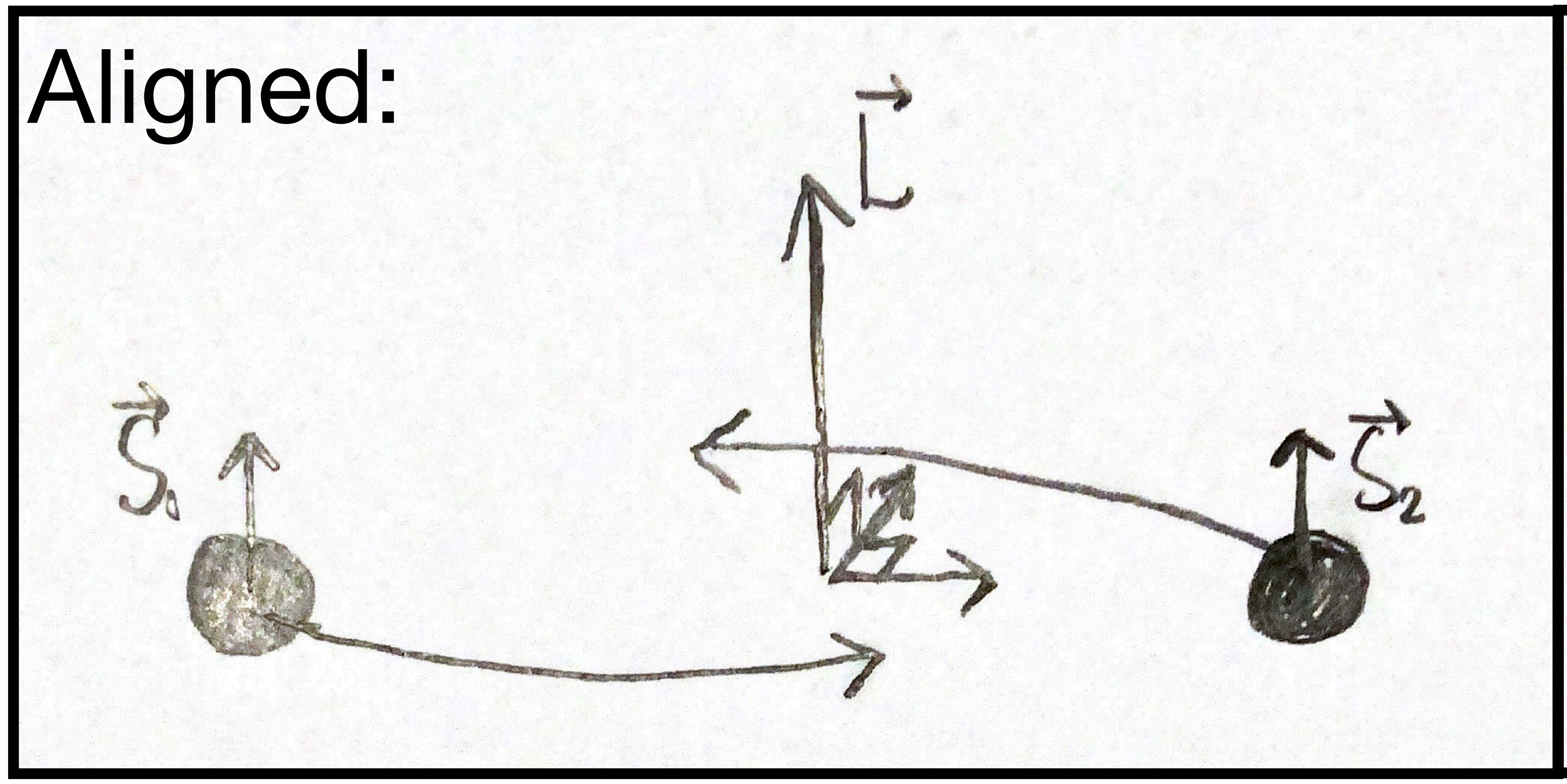
# Generic Spins Scattering angle

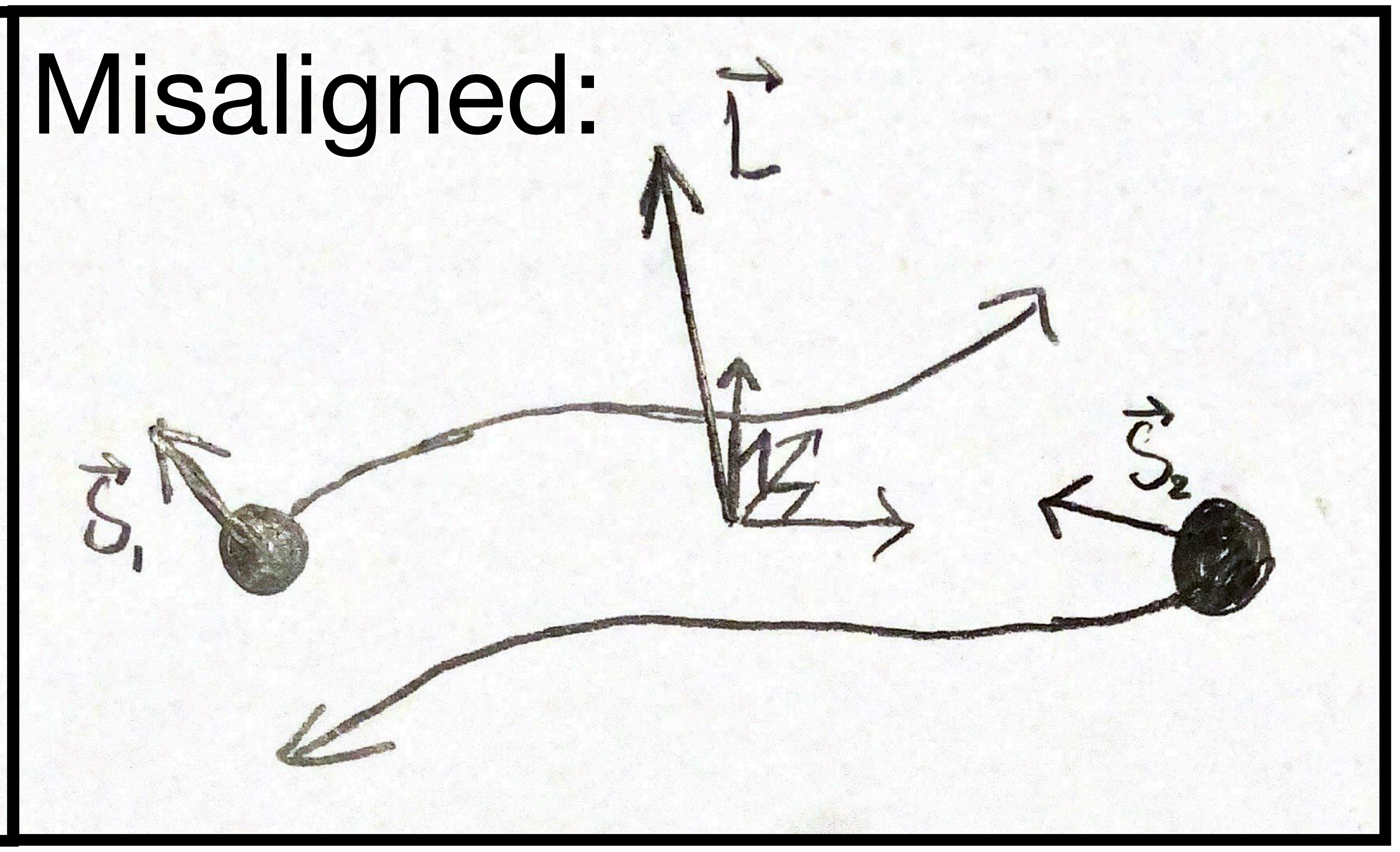
## Spinning Hamiltonian from matching

- We determine a Hamiltonian at $\mathcal{O}(G^3, S^2)$ from a matching calculation to a generic spins scattering angle $\theta$:

$$\sin\frac{\theta}{2} = \frac{|\Delta p^{\mu}_{\mathrm{pot}}|}{2p_{\infty}}$$

- The impulse and $\theta$ are SUSY invariant (SSC invariant)



# Generic Spins Scattering angle

- A simple example: $\theta_{\rm cons}$ for aligned spins with $\theta_{\rm cons}^{(n,m)}$ at order $G^nS^m$:

$$\theta_{\rm cons}^{(3,0)} = 2\frac{64\gamma^6-120\gamma^4+60\gamma^2-5}{3(\gamma^2-1)^3}\Gamma^2 - \frac{8\nu\gamma(14\gamma^2+25)}{3(\gamma^2-1)} - 8\nu\frac{4\gamma^4-12\gamma^2-3}{(\gamma^2-1)^{3/2}}\mathrm{arccosh}\gamma$$

$$\theta_{\rm cons}^{(3,1)} = 2\gamma\frac{16\gamma^4-20\gamma^2+5}{(\gamma^2-1)^{5/2}}(5\Gamma^2 s_+ - \delta s_-) - 4\nu s_+\left(\frac{44\gamma^4+100\gamma^2+41}{(\gamma^2-1)^{3/2}} + 12\gamma\frac{(\gamma^2-6)(2\gamma^2+1)}{(\gamma^2-1)^2}\mathrm{arccosh}\gamma\right)$$

$$\begin{aligned}\theta_{\rm cons}^{(3,2)} = {}& \frac{4\Gamma^2}{(\gamma^2-1)^3}\left((96\gamma^6-160\gamma^4+70\gamma^2-5)s_+^2 - \frac{1772\gamma^6-2946\gamma^4+1346\gamma^2-137}{35}s_{\rm E,+}^2\right) - 8\delta\left(\frac{16\gamma^4-12\gamma^2+1}{(\gamma^2-1)^2}s_-s_+ \right.\\ & \left. - \frac{214\gamma^4-223\gamma^2+44}{35(\gamma^2-1)^2}s_{\rm E,-}^2\right) + 8\nu\gamma\left[\frac{2\gamma^4+86\gamma^2+87}{5(\gamma^2-1)^2}s_-^2 - \frac{298\gamma^4+834\gamma^2+853}{5(\gamma^2-1)^2}s_+^2 + \frac{3244\gamma^4+7972\gamma^2+4639}{105(\gamma^2-1)^2}s_{\rm E,+}^2 \right.\\ & \left. - \left(3s_-^2(4\gamma^4+7\gamma^2+1) + 3s_+^2(8\gamma^6-68\gamma^4-63\gamma^2-9) - 2s_{\rm E,+}^2(8\gamma^6-56\gamma^4-24\gamma^2-3)\right)\frac{\mathrm{arccosh}\gamma}{\gamma(\gamma^2-1)^{5/2}}\right]\end{aligned}$$

With Lorentz factor $\gamma = v_1 \cdot v_2$, scalar spins $s_\pm = s_1 \pm s_2$, symmetric mass ratio $\nu$, and dimensionless energy $\Gamma = E/M$

# Conservative spinning Hamiltonian

$$H\left(\mathbf{x},\mathbf{p},\mathbf{S}_i\right)=\sqrt{\mathbf{p}^2+m_1^2}+\sqrt{\mathbf{p}^2+m_2^2}+V\left(\mathbf{x},\mathbf{p},\mathbf{S}_i\right)$$

$$V\left(\mathbf{x},\mathbf{p},\mathbf{S}_i\right)=\sum_A \mathcal{O}^A\,V^A(\mathbf{x},\mathbf{p})+\mathcal{O}(S^3)$$

$$=V^{(0)}+\sum_i V^{(1,i)}\mathcal{O}^{(1,i)}+\sum_{i,j,a}V^{(2,a,i,j)}\mathcal{O}^{(2,a,i,j)}+\mathcal{O}(S^3)$$

$$V^A(\mathbf{x},\mathbf{p})=\sum_n\left(\frac{GM}{|\mathbf{x}|}\right)^n c^{(n;A)}(\mathbf{p}^2)$$

$$\mathcal{O}^{(0)}=1\,,$$

$$\mathcal{O}^{(1,i)}=\frac{(\mathbf{x}\times\mathbf{p})\cdot\mathbf{a}_i}{|\mathbf{x}|^2}\,,$$

$$\mathcal{O}^{(2,1,i,j)}=\frac{\mathbf{a}_i\cdot\mathbf{a}_j}{|\mathbf{x}|^2}\,,$$

$$\mathcal{O}^{(2,2,i,j)}=\frac{\mathbf{x}\cdot\mathbf{a}_i\mathbf{x}\cdot\mathbf{a}_j}{|\mathbf{x}|^4}\,,$$

$$\mathcal{O}^{(2,3,i,j)}=\frac{\mathbf{p}\cdot\mathbf{a}_i\mathbf{p}\cdot\mathbf{a}_j}{|\mathbf{x}|^2}\,,$$

# Conservative spinning Hamiltonian

- Hamiltonian coefficients $c^{(n,A)}$ are expressed in terms of scattering angle coefficients $\theta^{(n,A)}$. Example at $G^3$ for spinless and linear in spin:

$$c^{(3;0)}(\mathbf{p}^2) = -\frac{p_\infty^2}{4E\xi}\theta_{\text{can}}^{(3;0)} + \frac{1}{2\pi p_\infty^2}\mathcal{D}\left[\frac{p_\infty^4}{E\xi}\theta_{\text{can}}^{(1;0)}\theta_{\text{can}}^{(2;0)}\right] - \frac{1}{48p_\infty^2}\mathcal{D}^2\left[\frac{p_\infty^4}{E\xi}(\theta^{(1;0)})^3\right]\Bigg|_{p_\infty\to|\mathbf{p}|}$$

$$c^{(3;1,i)}(\mathbf{p}^2) = -\frac{p_\infty}{4E\xi}\theta_{\text{can}}^{(3;1,i)} + \frac{1}{2\pi p_\infty^4}\mathcal{D}\left[\frac{p_\infty^5}{E\xi}\left(\theta_{\text{can}}^{(1;0)}\theta_{\text{can}}^{(2;1,i)} + \theta_{\text{can}}^{(2;0)}\theta_{\text{can}}^{(1;1,i)}\right)\right] - \frac{1}{16p_\infty^4}\mathcal{D}^2\left[\frac{p_\infty^5}{E\xi}(\theta_{\text{can}}^{(1;0)})^2\theta_{\text{can}}^{(1;1,i)}\right]\Bigg|_{p_\infty\to|\mathbf{p}|}$$

- Compact expressions for $\mathcal{O}(G^3, S^2)$ Hamiltonian. Special cases reproduce existing PM literature: [F.F.Cordero, M.Kraus, G.Lin, M.S.Ruf, M.Zeng: 2205.07357 ; Z.Bern, A.Luna, R.Roiban, C.-H.Shen, M.Zeng: 2005.03071 ; Z.Liu, R.A.Porto, Z.Yang: 2102.10059]
  Reproducing 4PN literature: [M.Levi, J.Steinhoff: 1607.04252]

# Spinning conservative dynamics

## From single scalar

- We have expressed the Hamiltonian, $H$, in terms of $\theta$
- We may now compute the impulse and spin kick from $H$
- Conservative impulse and spin kick are now parameterised in terms of generic spins angle $\theta$
- Similar to eikonal relations in [Z.Bern, A.Luna, R.Roiban, C.-H.Shen, M.Zeng: 2005.03071]

# Conclusions and perspectives

- Efficient in-in WQFT framework for PM-scattering
- Radiative and conservative spin observables at $\mathcal{O}(G^3, S^2)$
- Scalar quantities encapsulating conservative motion
- Gauge invariant mappings from unbound to bound with generic spins (+ tails)
- Effective one body models or other re-summation schemes

# Relation of WQFT to QFT and Amplitudes

- Dressed QFT scalar propagator becomes worldline Polyakov action
  [G.Mogull, J.Plefka and J.Steinhoff: 2010.02865]

- Velocity cuts map QFT integrals to worldline theory integrals.
  [N.E.J.Bjerrum-Bohr, P.H.Damgaard, L.Planté and P.Vanhove: 2105.05218, 2111.02976]

- Heavy-Mass Effective Theory and Heavy Black Hole Effective Theory uses linear “worldline” propagators.
  [A.Brandhuber et al: 2104.11206, K.Haddad et al: 1908.10308]

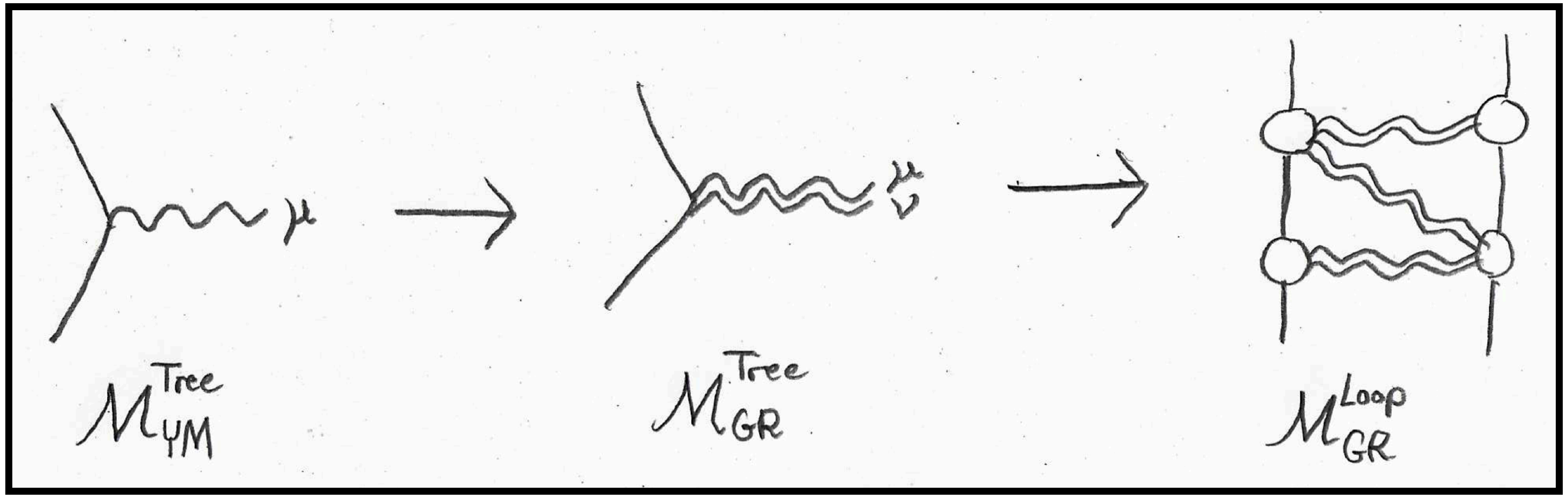
# Classical Einstein Gravity from QFT

- Effective field theory of gravity and classical limit of QFT
  [J.F.Donoghue ; N.E.J.Bjerrum-Bohr, J.F.Donoghue, B.Holstein ; C.Cheung, I.Z.Rothstein, M.P.Solon ; N.E.J.Bjerrum-Bohr, P.H.Damgaard, G.Festuccia, L.Planté, P.Vanhove: 1806.04920 ; D.A.Kosower, B.Maybee, D.O'Connell]

- Initial success with 3PM Hamiltonian [Z.Bern et al.]. Current PM state of the art is 4PM [Z.Bern et al., R.A.Porto et al.].

- Amplitudes: gauge invariance, generalized unitarity, double copy

- Integration techniques: IBPs, differential equations, reverse unitarity



# Worldline theory of compact objects

- Worldline effective field theory approach to gravitational wave sources originally designed for the PN expansion.
  [W.Goldberger, I.Rothstein: hep-th/0409156]

- Worldline Quantum Field Theory (WQFT): worldline formalism designed for PM scattering taking advantage of quantum field theoretic principles.
  [G.Mogull, J.Plefka, J.Steinhoff, B.Sauer, G.U.J: 2010.02865, 2207.00569]

- Also PM-EFT worldline formalism equally suited for PM scattering.
  [R.A.Porto, G.Kälin et al. ; M.M.Riva, F.Vernizzi et al.]