Non-perturbative Double Copy in Flatland

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Motivation

Double-copy (DC) – one of our best tools.

Want to do it off-shell.

Selling points

1. Easy: Would make DC trivial
2. Explain: Would help explain why DC works
3. New: Could DC new, off-shell quantities

Theories with some amount of manifest color-kinematics duality:

- Self-dual Yang-Mills (SDYM) [Monteiro & O'Connell '11] – tree level, vanishes on-shell
- 3D Chern-Simons theory [Ben-Shahar & Johansson '21] – loop level, topological
- Non-abelian fluid [Cheung & JM '20] – tree level
- Non-linear sigma model (NLSM) [Cheung & Shen '16; Cheung & JM '21] – tree level

Still more to learn

This talk

Pion-like theory in 2D that is

- interacting
- manifest color-kinematics duality
- all loop orders/off-shell/Lagrangian-level

NLSM review

Chiral current j_μ^a

(1) $F^c_{\mu\nu}(j) = \partial_{[\mu} j^c_{\nu]} + f_{ab}{}^c j^a_\mu j^b_\nu = 0$

Pure gauge, implicitly defines pion, field basis indep $j_\mu = i g^{-1} \partial_\mu g$

(2) $\partial^\mu j^a_\mu = 0$

Chiral current is conserved, Lorenz gauge, pion EOM $\partial(g^{-1}\partial g) = 0$

Cubic EOM with *manifest* BCJ

Combine (1) $F = 0$ and (2) $\partial j = 0$

$$\Box j^{c}_{\mu} + f_{ab}{}^{c} j^{a\nu} \partial_{\nu} j^{b}_{\mu} = 0$$

Manifestly satisfies kinematic Jacobi *off-shell* at tree level. Related construction for YM. Yields formula for numerators for NLSM & YM [Cheung & JM '21].

Two dimensions

Easy to satisfy (2) $\partial_\mu j^\mu = 0$ in 2D by dualizing chiral current $j^\mu = \epsilon^{\mu\nu}\partial_\nu\phi = \tilde{\partial}^\mu\phi$ where $\tilde{\partial}^\mu = \epsilon^{\mu\nu}\partial_\nu$.

$$\Box\phi^c - \frac{1}{2}f_{ab}{}^c\partial_\mu\phi^a\tilde{\partial}^\mu\phi^b = 0$$

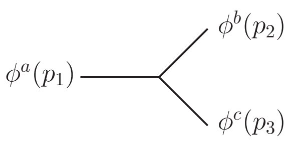
Unlike NLSM, this EOM can be *integrated* to get the Lagrangian for Zakharov-Mikhailov (ZM) theory

$$\mathcal{L}_{\text{ZM}} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi^a + \frac{1}{3!}f_{abc}\phi^a\partial_\mu\phi^b\tilde{\partial}^\mu\phi^c$$

Feynman rule

$$\mathcal{L}_{\mathrm{ZM}} = \tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi^a + \tfrac{1}{3!}f_{abc}\phi^a\partial_\mu\phi^b\tilde{\partial}^\mu\phi^c$$

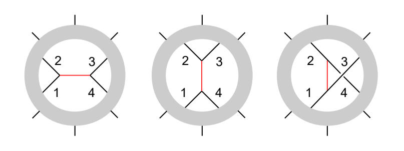
Define $\langle ij\rangle = \epsilon_{\mu\nu}p_i^\mu p_j^\nu$



$$= \quad -if_{abc}\langle 12\rangle$$

Secretly Bose sym off-shell $\langle 12\rangle = \langle 23\rangle = \langle 31\rangle$

Key step: check off-shell Jacobi



$$\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

Color-stripped vertex $\sim \langle ij \rangle = \epsilon_{\mu\nu} p_i^\mu p_j^\nu$
Off-shell Jacobi

$$n_s + n_t + n_u = \langle 12 \rangle \langle 34 \rangle + \text{cyclic}(1,2,3) = 0$$

Just Schouten identity.
ZM manifest BCJ to all loop orders.

Selling point (1): Easy

Read off replacement rules from $\mathcal{L}_{\text{ZM}}$

$$
\begin{aligned}
V^a &\quad\rightarrow\quad V \\
f_{ab}{}^c V^a W^b &\quad\rightarrow\quad \partial_\mu V \tilde{\partial}^\mu W \\
g_{ab} V^a W^b &\quad\rightarrow\quad \int VW \,.
\end{aligned}
$$

Double copy literally reduced to a replacement.

$$\mathcal{L}_{\text{BAS}} \rightarrow \mathcal{L}_{\text{ZM}} \rightarrow \mathcal{L}_{\text{SG}}$$

Double-copy replacement

$f_{ab}{}^{c} V^{a} W^{b} \quad \rightarrow \quad \partial_\mu V \tilde{\partial}^\mu W$

$$\mathcal{L}_{\mathrm{BAS}} = \tfrac{1}{2}\partial_\mu \phi_{a\bar{a}} \partial^\mu \phi^{a\bar{a}} + \tfrac{1}{3!} f_{abc} f_{\bar{a}\bar{b}\bar{c}} \phi^{a\bar{a}} \phi^{b\bar{b}} \phi^{c\bar{c}}$$

$$\downarrow$$

$$\mathcal{L}_{\mathrm{ZM}} = \tfrac{1}{2}\partial_\mu \phi_{a} \partial^\mu \phi^{a} + \tfrac{1}{3!} f_{abc} \phi^{a} \partial_\mu \phi^{b} \tilde{\partial}^\mu \phi^{c}$$

$$\downarrow$$

$$\mathcal{L}_{\mathrm{SG}} = \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi + \tfrac{1}{3!} \phi \partial_\mu \partial_\nu \phi \tilde{\partial}^\mu \tilde{\partial}^\nu \phi$$

Selling point (2): Explains

Kinematica algebra is area-preserving diffs on torus (Poisson). Isomorphic to large N limit of color algebra $U(N)$ [Hoppe '88].

$$\lim_{N\to\infty} U(N) \sim \text{Diff}_{S^1\times S^1}$$

Selling point (3): new interesting things

Masses & Higher Dim Ops

Masses

Masses are fine, only affect kinetic term

Higher dim ops

ZM^2 = special Galileons are secretly free in 2D [de Rham, Fasiello & Tolley '14].

No worries, infinitely many higher dim ops double-copy

Higher Dim Ops Continued

Can double-copy any operator except

1. No closed color loops
2. No multiple color traces

because $g_{ab} V^a W^b \to \int VW$.

Example ϕ^4:

$$\mathcal{O}_{\rm BAS} = f_{abe} f_{cde} f_{\bar{a}\bar{b}\bar{e}} f_{\bar{c}\bar{d}\bar{e}} \phi^{a\bar{a}} \phi^{b\bar{b}} \phi^{c\bar{c}} \phi^{d\bar{d}}$$
$$\mathcal{O}_{\rm ZM} = f_{abe} f_{cde} \partial_\mu \phi^a \tilde{\partial}^\mu \phi^b \partial_\nu \phi^c \tilde{\partial}^\nu \phi^d$$
$$\mathcal{O}_{\rm SG} = \partial_\mu \partial_\nu \phi \tilde{\partial}^\mu \tilde{\partial}^\nu \phi \partial_\rho \partial_\sigma \phi \tilde{\partial}^\rho \tilde{\partial}^\sigma \phi.$$

Moyal deformation

Moyal deformation, also in SDYM [Chacón, García-Compeán, Luna, Monteiro & White '14]

$$
\begin{aligned}
f_{ab}{}^{c} V^{a} W^{b} &\rightarrow \partial_{\mu} V \tilde{\partial}^{\mu} W \quad \text{(above)} \\
f_{ab}{}^{c} V^{a} W^{b} &\rightarrow \frac{N}{2\pi} \sin\left(\frac{2\pi}{N} \partial_{V} \tilde{\partial}_{W}\right) VW \quad \text{(Moyal)} \\
&\sim \sin\left(\langle 12 \rangle\right)
\end{aligned}
$$

Still satisfies Jacobi at full off-shell Lagrangian level

Integrability & Wilson line

- ZM known to be integrable
- Lax pair formed from Wilson line
- Infinite tower of off-shell currents
- Everything double-copies to SG via replacement rules

Similar story for SDYM [Chacón, García-Compeán, Luna, Monteiro & White '14]

Details of currents 1/3

Color $\to$ diff

$$\underline{\mathrm{V}} = V^a \underline{\mathrm{T}}_a \quad \to \quad \underline{\mathrm{V}} = \partial_\mu V \tilde{\partial}^\mu$$

$$[V^a \underline{\mathrm{T}}_a, W^b \underline{\mathrm{T}}_b] = f_{ab}{}^c V^a W^b \underline{\mathrm{T}}_c$$
$$\to \quad [\partial_\mu V \tilde{\partial}^\mu, \partial_\nu W \tilde{\partial}^\nu] = \partial_\mu (\partial_\nu V \tilde{\partial}^\nu W) \tilde{\partial}^\mu$$

$$c.f. \quad f_{ab}{}^c V^a W^b \quad \to \quad \partial_\mu V \tilde{\partial}^\mu W$$

Details of currents 2/3

$$\underline{A}_\mu = \frac{1}{1-\lambda^2}(\tilde{\partial}_\mu \underline{\phi} + \lambda \partial_\mu \underline{\phi})$$

$$\underline{F}_{\mu\nu} = 0$$

$$\underline{W}(x) = \mathrm{P}\exp\left[-\int^x dx'^\mu \underline{A}_\mu(x')\right]$$

$$\underline{J}_\mu = \tilde{\partial}_\mu \underline{W} = \sum_{k=0}^{\infty} \lambda^{-k} \underline{J}_\mu^{(k)}$$

Details of currents 3/3

Example ZM currents

$$\underline{\mathrm{J}}_{\mu}^{(1)} = \tilde{\partial}_{\mu}\underline{\phi}, \quad \underline{\mathrm{J}}_{\mu}^{(2)} = \partial_{\mu}\underline{\phi} + \tilde{\partial}_{\mu}\underline{\phi}\,\underline{\phi}$$

$$\underline{\mathrm{J}}_{\mu}^{(3)} = \tilde{\partial}_{\mu}\underline{\phi} + \partial_{\mu}\underline{\phi}\,\underline{\phi} + \tilde{\partial}_{\mu}\underline{\phi}\int^{x} dx'\tilde{\partial}\underline{\phi}$$

$$+ \tilde{\partial}_{\mu}\underline{\phi}\int^{x} dx'\partial\underline{\phi}\,\underline{\phi}$$

Double copy to SG $\underline{\mathrm{V}} = V^{a}\underline{\mathrm{T}}_{a} \to \underline{\mathrm{V}} = \partial_{\mu} V \tilde{\partial}^{\mu}$

$$\underline{\mathrm{J}}_{\mu}^{(1)} = \tilde{\partial}_{\mu}\partial_{\nu}\phi\tilde{\partial}^{\nu}, \quad \underline{\mathrm{J}}_{\mu}^{(2)} = \partial_{\mu}\partial_{\nu}\phi\tilde{\partial}^{\nu} + \tilde{\partial}_{\mu}\partial_{\nu}\phi\,\tilde{\partial}^{\nu}\partial_{\rho}\phi\tilde{\partial}^{\rho}$$

Can do the whole tower (closed form)

Numerical double-copy!

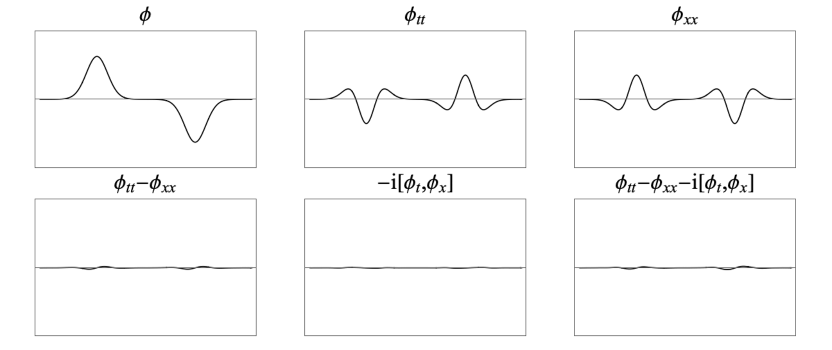
$$\phi^a_{\rm ZM}(x)\,T^a = \int d^2p\;\, e^{ipx}\,T_p\phi_{\rm SG}(p)$$

Discretize, take $N = 499$ large

Double-copy <u>from</u> SG <u>to</u> ZM (backwards)

KLT: $A_{\rm GR} \sim s^{n-3} A_{\rm YM} A_{\rm YM} + ...$

Collide two SG Gaussian wave packets.
Plot ZM sol projected onto $\sum_p T_p$.



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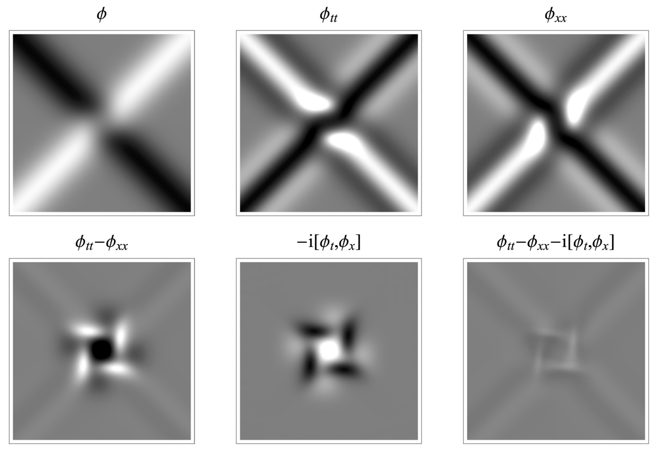

Overview

- Manifest color-kinematics duality for 2D "pion" at Lagrangian level
- Double copy is just replacement
- Kinematic algebra is diff algebra
- Color-kinematics duality – Hoppe
- Double copy masses, higher dim ops, Wilson lines, numerical solutions...

Thank you!