Kinematic Hopf algebra(KiHA) in amplitude and form factor

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Brandhuber, GC, Travaglini, Wen(2104.11206, 2108.04216)
Brandhuber, GC, Johansson, Travaglini, Wen(2111.15649)
Brandhuber, Brown,GC,Gowdy, Travaglini, Wen(2208.05886)
GC, Lin, Wen(2208.05519)
Bjerrum-Bohr, GC, Skowronek (2301.?????)

QCD Meet Gravity @Zürich

Contents

- Background
- Framework: Kinematic Hopf algebra(KiHA)
- Double and applications to (spinning) heavy-mass effective theory

Motivation

- Develop a systematic framework to construct the amplitude and form factor
- Exploring the underlying algebra structures in colour-kinematic and double copy duality
- Constructing new theory(effective theory & classical/high spin theory)

Gauged Bi-adjoint scalar theory

Heavy matter coupling to gluon (Yang-Mills)

$$\mathcal{L} = \mathcal{L}^{\text{YMS}} + m^2 \phi^{I,a} \phi^{I,a} + \frac{g'}{3!} f^{IJK} f^{abc} \phi^{I,a} \phi^{J,b} \phi^{K,c}$$

Motivation

Chiodaroli,Jin,Radu 2013,
Chiodaroli,Gunaydin, Hohansson, Roiban2014,
Cachazo, He and Yuan2014

- Have c-k duality
- Double copy to graviton-scalar, EYM, EM amplitude
- Closely related to gravitational scattering(including, binary black hole, Compton scattering, light bending et al)
- It’s fun(Novel relations among the BCJ numerator)

Background on color-kinematic(CK) and double copy (DC)

Bern, Carrasco, Johansson 2008,2010

CK duality

Sum over all the cubic graphs

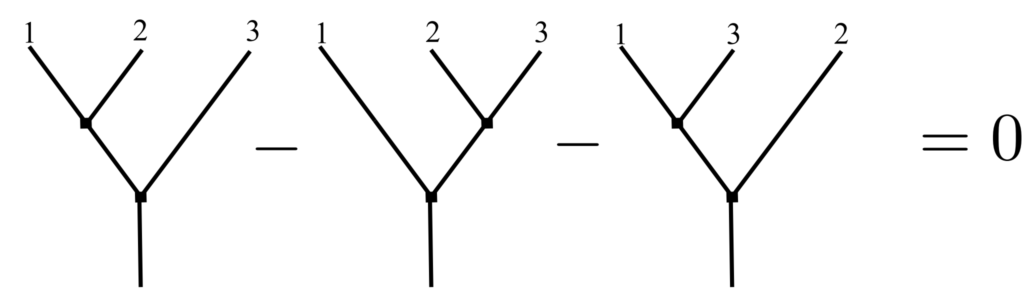
$$\mathcal{A}_n^{\text{tree}} = g^{n-2} \sum_{I=1}^{(2n-5)!!} \frac{c_I N_I}{D_I}$$

Scalar propagator

c_I Color factor

N_I kinematic numerator

$$f^{abe} f^{ecd} + f^{ace} f^{edb} + f^{ade} f^{ebc} = 0$$



DC duality

$$\mathcal{A}_n^{\text{tree}} = g^{n-2} \sum_{I=1}^{(2n-5)!!} \frac{\bar{N}_I N_I}{D_I}$$

Factor out the propagators

Kinematic Hopf algebra(KiHA) in Bi-adjoint scalar

Wen 2022, GC, Lin, Wen 2022

Building Blocks in KiHA

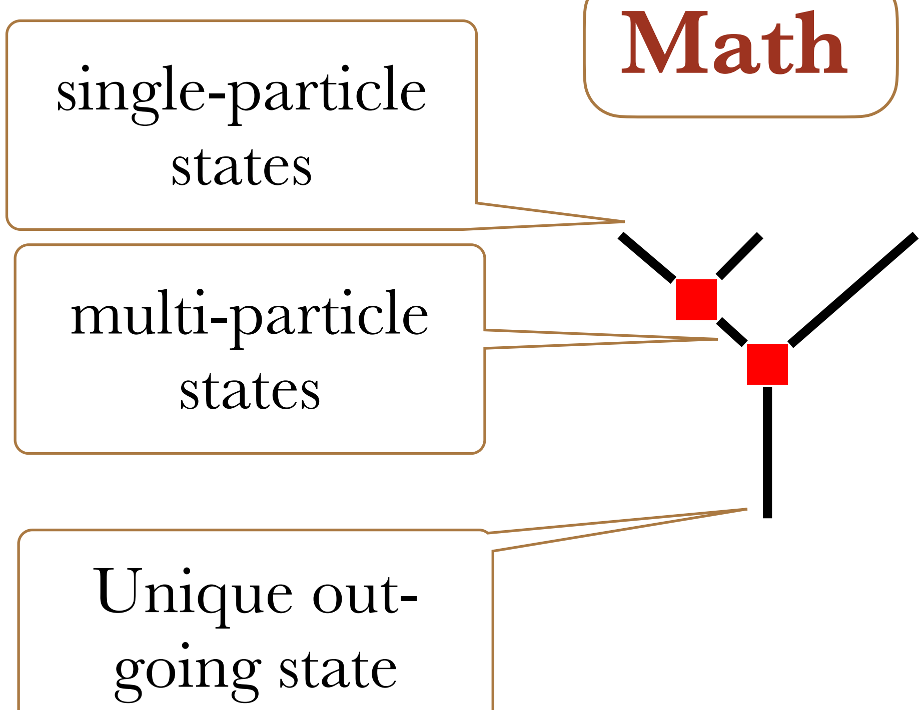
Brandhuber, GC, Johansson, Travaglini, Wen 2021-11
Brandhuber, Brown, GC, Gowdy, Travaglini2022;
GC,Johansson, Teng, Wang 2019; GC,Johansson,Teng,Wang 2021

1. Generators T
2. Fusion product $\star$
3. Evaluating Map $\langle \bullet \rangle$

Feynman Rule

- Quantum Field
- Vertex: Interaction
- Contract with external states

Math



Evaluating Map

$\longrightarrow$

$\langle \bullet \rangle$

Physics

Prenumerator:
Gauge invariant
Lorentz invariant
rational functions

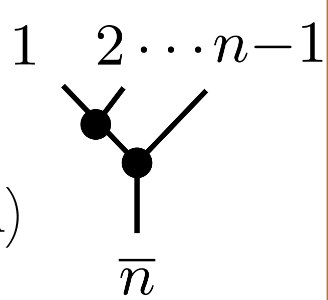
Construct amplitude

KiHA

- Defined the commutator

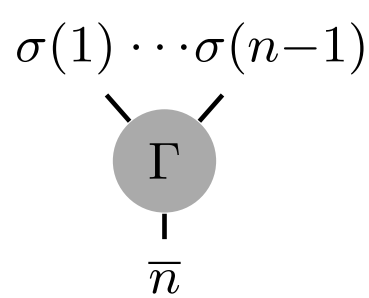
$$[X, Y] \equiv X \star Y - Y \star X$$

- Perform commutators
in a given order(a cubic graph)



- Sum over all the cubic graph, e.g. colour ordered amplitude

$$\mathcal{A}(\sigma, \overline{n}) = \sum_{\Gamma \in \mathrm{R}_\sigma} \frac{\langle \widehat{\mathcal{N}}(\Gamma) \rangle}{d_\Gamma}$$



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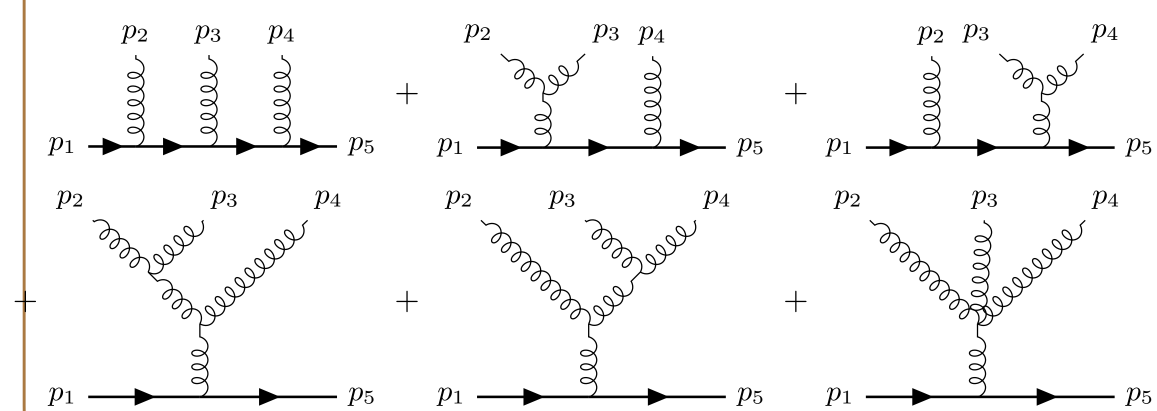
Feynman Rule

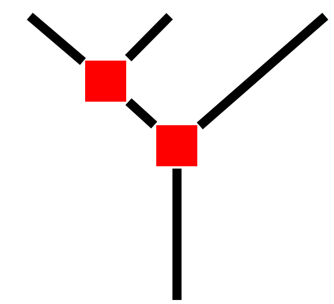
Sum over all the Feynman diagrams



Gauge invariant function from each cubic graph

Algebraic generators

$$\mathsf{K}_i = \begin{cases} T^{(i)}_{(i)} & \text{for gluons} \\ T^{(i)} t^{a_i} & \text{for scalars} \\ T^{(\alpha)}_{(\tau_1),\cdots,(\tau_r)} t^{a_i}\cdots t^{a_j} & \text{intermediate states} \end{cases},$$



α	fusion product order
t^{a_i}	flavour group generator
$(\tau_1),\cdots,(\tau_r)$	partition of gluon labels

$$T^{(6124357)}_{(12),(43),(5)} t^{a_6} t^{a_7}$$

Fusion product

Brandhuber, GC, Johansson, Travaglini, Wen 2021-11

Non-abelian quasi-shuffle Hopf algebra

$$T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \star T^{(\beta)}_{(\omega_1),\ldots,(\omega_s)} = \sum_{\substack{\pi|_\tau=\{(\tau_1),\ldots,(\tau_r)\} \\ \pi|_\omega=\{(\omega_1),\ldots,(\omega_s)\}}} (-1)^{t-r-s} T^{(\alpha\beta)}_{(\pi_1),\ldots,(\pi_t)} \,,$$

where $\pi|_\tau$ (or $\pi|_\omega$) means a restriction to the elements of π in τ (or ω), e.g. $\{(235),(4),(678)\}|_{\{2,3,4,8\}} = \{(23),(4),(8)\}$.

Example

$$T^{(12)}_{(1),(2)} \star T^{(345)}_{(34)} = T^{(12345)}_{(1),(2),(34)} + T^{(12345)}_{(1),(34),(2)} + T^{(12345)}_{(34),(1),(2)} - T^{(12345)}_{(1),(234)} - T^{(12345)}_{(134),(2)}$$

Shuffle

Stuffing

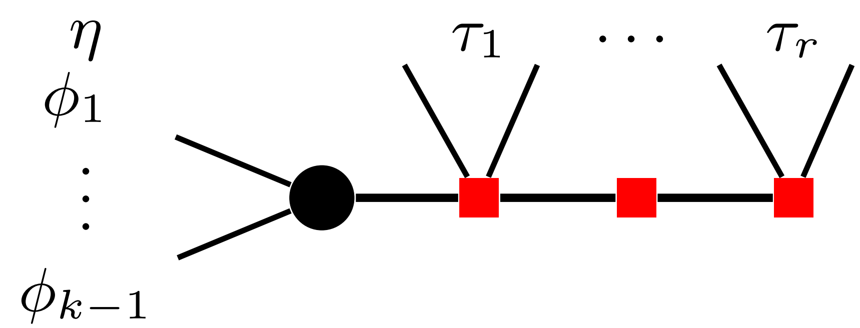
Super index is critical for non-trivial commutator

Evaluation map

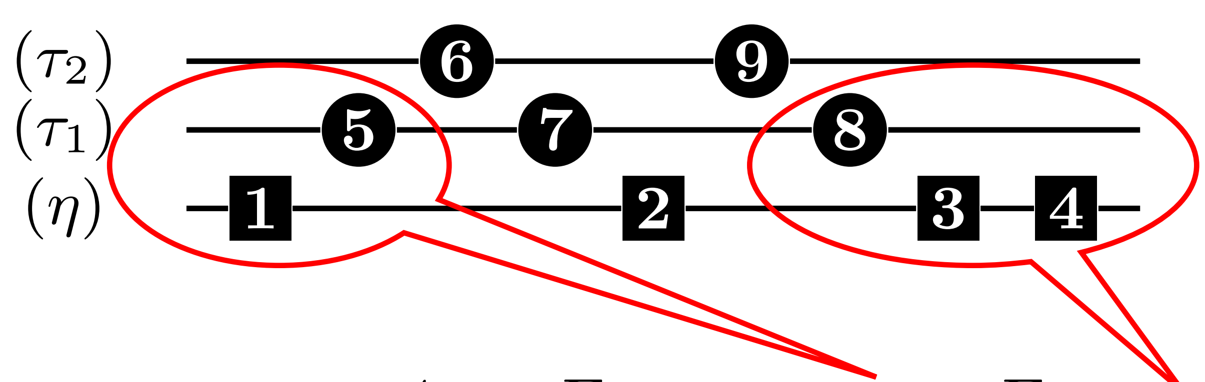
From algebraic generators to gauge invariants

$$t^{\eta} T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \xrightarrow[\text{map}]{\langle \bullet \rangle} \mathrm{tr}(t^{\eta} t^{a_n}) \langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \rangle_m$$

$$\langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \rangle_m = \left(\vphantom{\int} \right)^{(\alpha)}$$



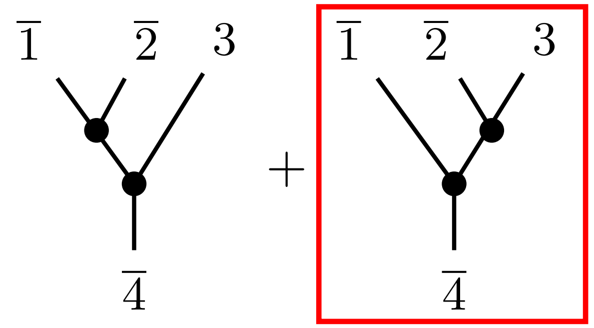
$$= \frac{2^r \prod_{i=1}^{r} \left(p_{\Theta_L^{(\alpha)}(\tau_i)} \cdot F_{\tau_i} \cdot p_{\Theta_R^{(\alpha)}(\tau_i)} \right)}{(p_{\eta}^2 - m^2)(p_{\eta\tau_1}^2 - m^2) \cdots (p_{\eta\tau_1\cdots\tau_{r-1}}^2 - m^2)}$$



$$\langle T^{(156729834)}_{(578),(69)} \rangle_m = \frac{4 p_1 \cdot F_{578} \cdot p_{34} \; p_{15} \cdot F_{69} \cdot p_{348}}{(p_{1234}^2 - m^2)(p_{1234578}^2 - m^2)} .$$

$$F_i^{\mu\nu} = p_i^{\mu} \varepsilon_i^{\nu} - \varepsilon_i^{\mu} p_i^{\nu}$$

Warm up example

$\mathcal{A}(\bar{1},\bar{2},3,\bar{4}) =$ 

1,2,4 is scalar
3 is gluon

Example KiHA

1. External state:

$$K_i = \begin{cases} T^{(1)}t^{a_1}, & i=1 \text{ scalar} \\ T^{(2)}t^{a_2}, & i=2 \text{ scalar} \\ T^{(3)}_{(3)}, & i=3 \text{ gluon} \end{cases}$$

2. Perform the commutators

$$\langle[[\mathsf{K}_1,[\mathsf{K}_2,\mathsf{K}_3]]\rangle = \langle[T^{(1)}t^{a_1},(T^{(23)}_{(3)}-T^{(32)}_{(3)})t^{a_2}]\rangle$$
$$= \langle(T^{(123)}_{(3)}-T^{(132)}_{(3)})t^{a_1}t^{a_2}-(T^{(231)}_{(3)}-T^{(321)}_{(3)})t^{a_2}t^{a_1}\rangle$$

3. Sum over the cubic graphs and mapping to kinematics

$$\mathcal{A}(\bar{1},\bar{2},3,\bar{4}) = \frac{2p_2\cdot F_3\cdot p_1}{(p_{12}^2-m^2)(p_{23}^2-m^2)}f^{a_1a_2a_4}$$

Form factor

Motivations

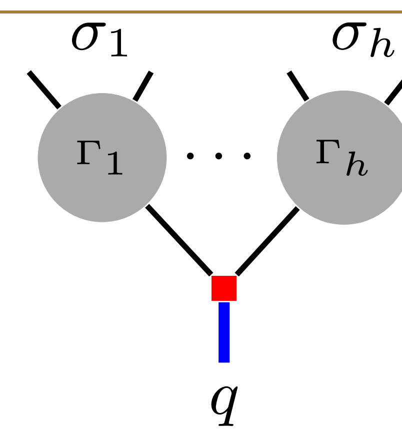
1. Does such a fusion rule have the universal behaviour as local interaction?

The operator

$$\mathrm{Tr}(\phi^h) = \mathrm{tr}(t_c^{A_1} \cdots t_c^{A_h})\mathrm{tr}(t^{a_1} \cdots t^{a_h})\phi^{a_1,A_1} \cdots \phi^{a_h,A_h}$$

Form factor

$$\mathcal{F}_{\mathrm{Tr}(\phi^h)}(\sigma) = \sum_{\Gamma \in \mathsf{R}_\sigma^{(h)}} \frac{\langle \widehat{\mathcal{N}}(\Gamma_1) \star \cdots \star \widehat{\mathcal{N}}(\Gamma_h) \rangle}{d_{\Gamma_1} \cdots d_{\Gamma_h}}$$



Fusion product (Local)

Same as amplitude

Evaluation map(global)

$$t^\eta T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \xrightarrow[\text{map}]{\langle \bullet \rangle} \mathrm{tr}(t^\eta) \quad \langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \rangle_q$$

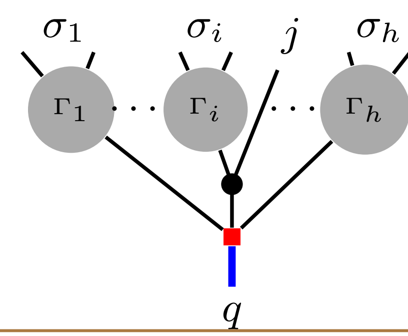
$$\langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \rangle_q = \langle T^{(\alpha)}_{(\tau_1),\ldots,(\tau_r)} \rangle_m \Big|_{m^2 \to q^2}$$

New relation(Beyond Jacobi)

Beyond Jacobi

$$\sum_{i=1}^{h}\langle\widehat{\mathcal{N}}(\Gamma_1)\star\cdots\star\mathcal{N}([\Gamma_i,j])\star\cdots\star\widehat{\mathcal{N}}(\Gamma_h)\rangle$$

$$=\sum_{i=1}^{h}$$



$$=0\,.$$

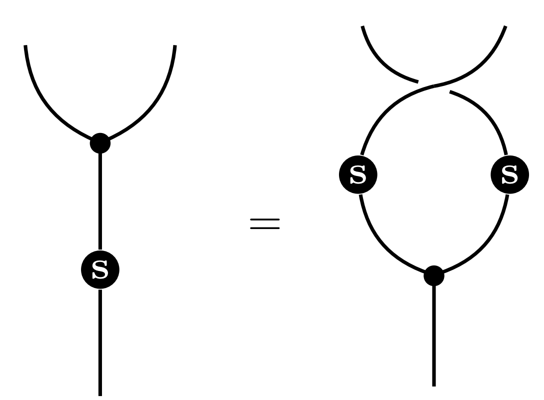
Antipode relation

$$\langle\widehat{\mathcal{N}}(12\ldots n{-}1)\rangle\Big|_{t^a\to\mathbb{I}}=\langle S(\widehat{\mathcal{N}}(12\ldots n{-}1))\rangle\Big|_{t^a\to\mathbb{I}}\,,$$

S is anti-morphism

$$S(X\star Y)=S(Y)\star S(X)$$

$$S(T^{(i)})=T^{(i)},\,S(T^{(j)}_{(j)})=-T^{(j)}_{(j)}$$



More Hopf element

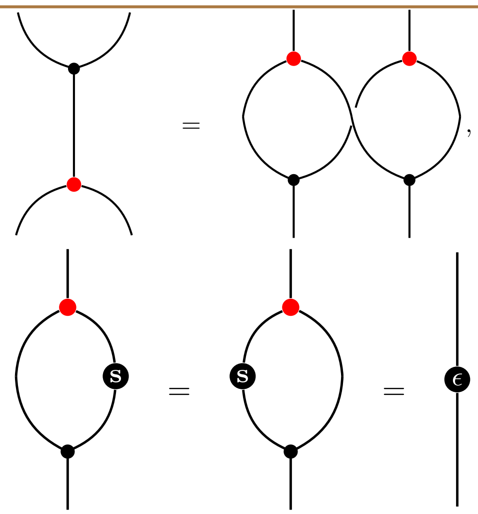
Brandhuber, Brown, GC, Gowdy, Travaglini2022;

Coproduct and counit

$$\Delta(T^{(\alpha)}) = \left(T^{(\alpha)} \otimes T^{(\alpha)}\right) \,,$$

$$\Delta(T^{(\alpha)}_{(\tau_1),\dots,(\tau_r)}) = \sum_{i=0}^{r} \left(T^{(\alpha)}_{(\tau_1),\dots,(\tau_i)} \otimes T^{(\alpha)}_{(\tau_{i+1}),\dots,(\tau_r)}\right) \,,$$

$$\epsilon(T^{(\alpha)}_{\tau}) = 0 \,, \epsilon(T^{(\alpha)}) = \mathbb{I}_{\rm qs}$$



$$\Delta(T^{(\alpha)}_{\tau}) \star \Delta(T^{(\beta)}_{\omega}) = \Delta(T^{(\alpha)}_{\tau} \star T^{(\beta)}_{\omega}) \,,$$

$$\epsilon(T^{(\alpha)}_{\tau}) \star \epsilon(T^{(\beta)}_{\omega}) = \epsilon(T^{(\alpha)}_{\tau} \star T^{(\beta)}_{\omega}) \,,$$

$$\star(\mathbb{I}_{\rm qs} \otimes S)\Delta(T^{(\alpha)}_{\tau}) = \star(S \otimes \mathbb{I}_{\rm qs})\Delta(T^{(\alpha)}_{\tau}) = \epsilon(T^{(\alpha)}_{\tau})\mathbb{I}_{\rm qs}$$

Kinematic algebra is a non-abelian extended quasi-shuffle Hopf algebra

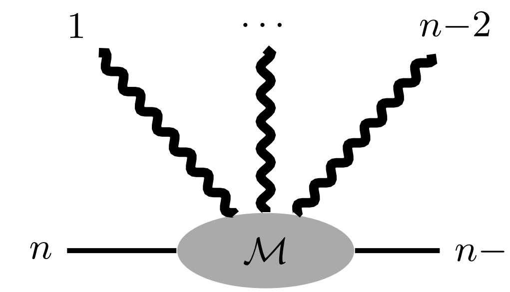
Applications: (spinning) heavy-mass effective field theory

Double copy

Brandhuber, GC, Travaglini, Wen 2021-04;
Brandhuber, GC, Johansson, Travaglini, Wen 2021-11
Brandhuber, Brown, GC, Gowdy, Travaglini2022;

Full scalar

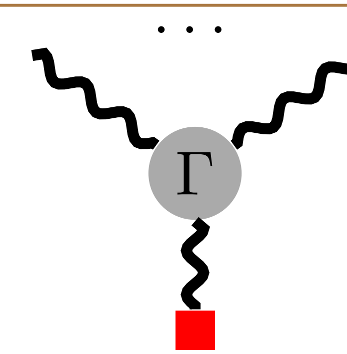
$$M(12\ldots n-2,\overline{n-1},\overline{n}) = -\sum_{\Gamma\in\tilde{\rho}} \frac{\mathcal{N}(\Gamma,\overline{n-1},\overline{n})\mathcal{N}(\Gamma,\overline{n},\overline{n-1})}{d_\Gamma},$$



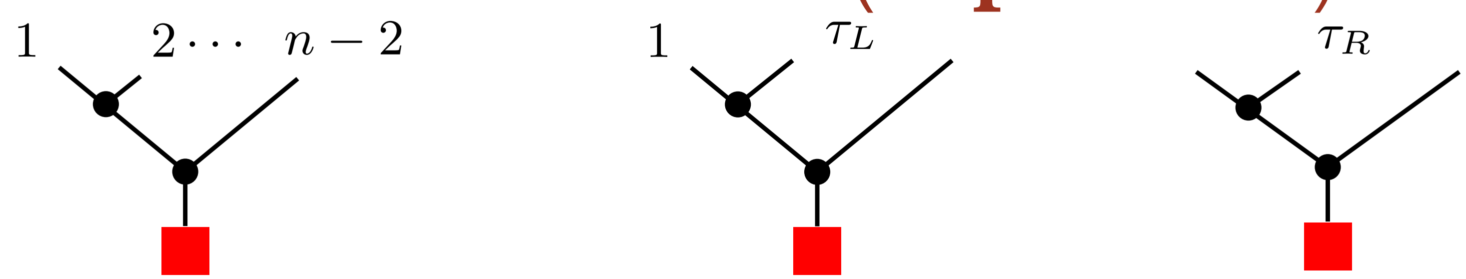
Heavy mass limit

HEFT-GR

$$M(12\ldots n{-}2,v) = \sum_{\Gamma\in\tilde{\rho}} \frac{\left[\mathcal{N}(\Gamma,v)\right]^2}{d_\Gamma}$$



Novel Factorization behaviour (coproduct)



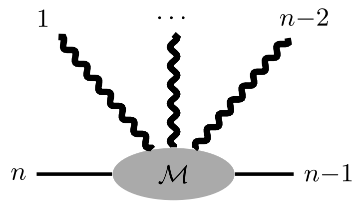
$$\mathcal{N}([12\ldots n-2],v) \xrightarrow{v\cdot p_{1\tau_L}} p_{\Theta(\tau_R)}\cdot p_{\tau_{R[1]}}\mathcal{N}([1\tau_L],v)\mathcal{N}([\tau_R],v),$$

Factorisation behaviour also holds for the Compton scattering at spin infinity order

Properties of BCJ numerator

Bjerrum-Bohr, GC, Skowronek 2023

- Manifestly Gauge invariant
- Local amplitude
- Covariant(fit the PM expansion)
- Closed form for arbitrary spin order
- Unique at four-point and finite controlled freedom at a higher point.



Four point example

$$\mathcal{N}_s(12,v) = -\frac{w_1\cdot F_1\cdot F_2\cdot w_2}{v\cdot p_1} + \mathcal{N}'_s(12,v).$$

$$\begin{aligned}\mathcal{N}'_s(12,v) &= (a\cdot F_2\cdot F_1\cdot v a\cdot p_1 - a\cdot F_2\cdot v a\cdot F_1\cdot p_2)G_1(x_1)G_1(x_2)\\ &+ \mathrm{i}\Big((a\cdot S\cdot F_1\cdot F_2\cdot p_1 + a\cdot S\cdot F_2\cdot F_1\cdot p_2)G_2(x_1,x_2)\\ &+ \mathrm{tr}(SF_2F_1)G_1(x_{12})\Big).\end{aligned}$$

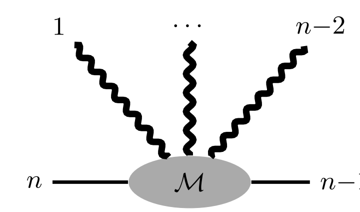
Outlook

- Geometry representation of KiHA:(ambitwistor)-string world sheet, Self-dual Yang-Mills, Chern-Simons, permutohedra

  e.g. Monteiro2022; Ben-shahar, Johansson 2021;
  Mafra, Schlotterer 2022; Cao, Dong, He, Zhang, Zeng 2022

- Classical/high spin kinematic algebra

  e.g. Chen, Chung, Huang, Kim 2021; Bern, Luna, Roiban, Shen, Zeng 2020;
  Guevara, Ochirov, Vines2018; Aoude, Haddad, Helset 2022;
  Jakobsen, Mogull 2022; Jakobsen, Mogull, Plefka, Steinhoff 2021;



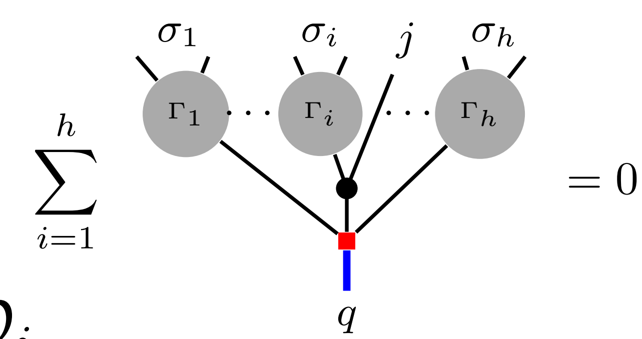
- KiHA for the interaction with fermions

  e.g. Johansson, Ochirov 2016; GC, Johansson, Teng, Wang 2019

- Generalised CK and DC in the effective theory

  e.g. Carrasco, Rodina, Yin, Zekioglu 2019,2021;
  Chi, Elvang, Herderschee, Jones, Paranjape2022

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + \sum_i c_i \mathcal{O}_i$$

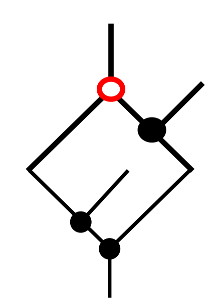


- KiHA in curved spacetime

  e.g. Chacon, Nagy, White 2021; Monteiro, O'Connell, White2014 ; Herderschee, Roiban, Teng2022

- Loop level kinematic algebra

  e.g. Borsten,Jurco,Kim,Macrelli,Saemann,Wolf 2020,2021,2022



"Thanks!"