Feynman Integrals

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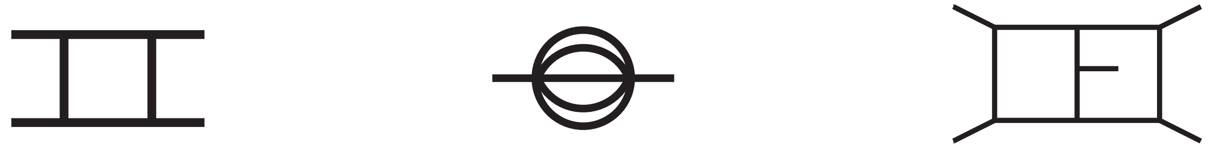
Introduction

$$I(p_1,\ldots,p_E;m_1^2,\ldots,m_p^2;\nu;D) = \int \left(\prod_{j=1}^{L} e^{\gamma_E \epsilon} \frac{d^D k_j}{i\pi^{D/2}} \right) \frac{\mathcal{N}(\{k_j \cdot k_l, k_j \cdot p_l\};D)}{\prod_{j=1}^{p} (m_j^2 - q_j^2 - i\varepsilon)^{\nu_j}}$$

- Ubiquitous in any perturbative QFT calculation
  - ✓ Truly where QCD meets gravity
- Major bottleneck when number of scales/loops increases
- Diagrammatic representation with associated Feynman rules



- In this talk:
  - ✓ The ν_j are integers (see tomorrow for other cases)
  - ✓ Use dimensional regularisation $D = 4 - 2\epsilon$ to regulate all divergences
- Lorentz invariant quantities with well defined mass dimension
  - ✓ Scaleless integrals vanish in dimensional regularisation

Outline

- Parametric representations
- Linear relations between Feynman integrals
- Differential equations
- Numerical evaluation of Feynman integrals

Reviews

- *Analytic Tools For Feynman Integrals*, V.A. Smirnov (Springer, 2012)

- *Feynman Integrals*, S. Weinzierl, 2201.03593

- *Sagex Review on Scattering Amplitudes*, 2203.13011
  - Chapter 3: *Mathematical Structures in Feynman integrals*, S. Abreu, R. Britto, C. Duhr
  - Chapter 4: *Muti-loop Feynman integrals,* J. Blümlein, C. Schneider

- … many other lecture notes (references found in above reviews)

PARAMETRIC REPRESENTATIONS

Feynman parameter integrals

Cutkosky-Baikov representation

Direct integration and types of functions

Parametric Representations — Feynman parameters

$$I(x;\nu;D) = \int \left(\prod_{j=1}^{L} e^{\gamma_E \epsilon} \frac{d^D k_j}{i\pi^{D/2}} \right) \frac{\mathcal{N}(\{k_j \cdot k_l, k_j \cdot p_l\}; D)}{\prod_{j=1}^{p} (m_j^2 - q_j^2 - i\varepsilon)^{\nu_j}}$$

$$I(x;\nu;D) = e^{\gamma_E L \epsilon}\,\Gamma\left(|\nu| - \frac{LD}{2}\right) \prod_{j=1}^{p} \int_0^\infty d\alpha_j \frac{\alpha_j^{\nu_j - 1}}{\Gamma(\nu_j)}\, \delta\left(1 - \sum_{j=1}^{p} \alpha_j\right) \frac{\mathcal{U}(\alpha)^{|\nu| - \frac{(L+1)D}{2}}}{\mathcal{F}(\alpha;x)^{|\nu| - \frac{LD}{2}}}$$

- Feynman-parameter representation (similar to *Schwinger*, *Lee-Pomeranski*, …)
- $\mathcal{U}$ and $\mathcal{F}$ are (graph) polynomials in kinematics and the α_j
- Potential alternative definition of Feynman integrals in dim reg
- Important observation: very similar dependence on ν and $D/2$
- Defines a projective integral over (positive) real projective space

Parametric Representations — Cutkosky-Baikov

$$I(x;\nu;D) = \int \left(\prod_{j=1}^{L} e^{\gamma_E \epsilon} \frac{d^D k_j}{i\pi^{D/2}} \right) \frac{\mathcal{N}(\{k_j \cdot k_l, k_j \cdot p_l\};D)}{\prod_{j=1}^{p} (m_j^2 - q_j^2 - i\varepsilon)^{\nu_j}}$$

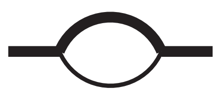
⇕

$$I(x;\nu;D) = N(D)\; G(p_1,\ldots,p_{E-1})^{\frac{E-D}{2}} \int_\Delta \prod_{a=1}^{p} dz_a\, \mathcal{B}(z)^{\frac{D-K-1}{2}} \frac{\mathcal{N}(\{z_k;x\};D)}{\prod_{c=1}^{p} z_c^{\nu_c}}$$

- ✓ Obtained by making the propagators the integration variables
- ✓ G is the Gram determinant of the external legs
- ✓ $\mathcal{B}$ is the Baikov polynomial (computable as a Gram determinant)
- ✓ Natural representation to study cuts of Feynman integrals ($\sim$ set $z_c = 0$)

Parametric Representations — Direct Integration

- ✓ Parametric representations used for direct interaction (analytic or numerical)
- ✓ One-loop bubble with one massive propagator

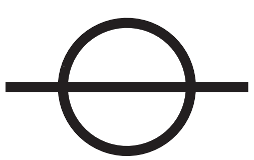


$$= I(p^2; m_1^2, 0; 1, 1; D) = e^{\gamma_E \epsilon}(m_1^2)^{-2+D/2} \frac{\Gamma(2-D/2)}{D/2-1} \, {}_2F_1\left(1, 2-\frac{D}{2}; \frac{D}{2}; \frac{p^2}{m_1^2}\right)$$

- ✓ Expansion around integer dimensions

$$I(p^2; m_1^2, 0; 1, 1; 2-2\epsilon) = \frac{1}{\epsilon(p^2-m_1^2)}\left[1 - 2\epsilon \log(1-p^2/m_1^2) + \epsilon^2\left(\frac{\pi^2}{12} + 2\log^2\left(1-p^2/m_1^2\right) + 2\,\mathrm{Li}_2\left(p^2/m_1^2\right)\right) + \mathcal{O}(\epsilon^3)\right]$$

- ✓ Types of functions that appear in evaluation of Feynman integrals
  - Hypergeometric functions (in dim reg)
  - Logarithms and Multiple Polylogarithms MPLs (expansions around integer dim)
  - Elliptic integrals and beyond (expansions around integer dim)



Functions we need to understand to compute Feynman integrals

LINEAR RELATIONS

FIXED KINEMATICS

- Integration-by-parts (IBP) relations
- Master integrals
- Dimension-shifting relations
- Laporta algorithm, intersection theory, …

Linear Relations — IBPs

- ✓ Feynman integrals with fixed kinematics and dimensions, as function of the ν_j

- ✓ Integration by parts have no boundary terms in dim. reg. For any v^μ

$$\int d^D k_i \frac{\partial}{\partial k_i^\mu}\left[v^\mu \frac{\mathcal{N}(\{k_j\cdot k_l, k_j\cdot p_l\}; D)}{\prod_{j=1}^{p}(m_j^2 - q_j^2 - i\varepsilon)^{\nu_j}}\right] = 0$$

  - ▸ Linear relations with integrals with different ν_j

- ✓ Example: one-loop bubble, massless propagators $\int d^D k \frac{\partial}{\partial k^\mu}\left[v^\mu \frac{1}{(k^2)^{\nu_1}((k+p)^2)^{\nu_2}}\right] = 0$

$$\begin{cases} (D-2\nu_1-\nu_2)\, I(\nu_1,\nu_2) - \nu_2\, I(\nu_1-1,\nu_2+1) - \nu_2 p^2\, I(\nu_1,\nu_2+1) = 0 \\ (\nu_1-\nu_2)\, I(\nu_1,\nu_2) - \nu_1\, I(\nu_1+1,\nu_2-1) - \nu_1 p^2\, I(\nu_1+1,\nu_2) + \nu_2\, I(\nu_1-1,\nu_2+1) + \nu_2 p^2\, I(\nu_1,\nu_2+1) = 0 \end{cases}$$

$$\begin{cases} I(\nu_1,\nu_2) = -\dfrac{\nu_1+\nu_2-1-D}{p^2(\nu_2-1)}\, I(\nu_1,\nu_2-1) - \dfrac{1}{p^2}\, I(\nu_1-1,\nu_2) & \nu_2 \neq 1 \\ I(\nu_1,\nu_2) = -\dfrac{\nu_1+\nu_2-1-D}{p^2(\nu_1-1)}\, I(\nu_1-1,\nu_2) - \dfrac{1}{p^2}\, I(\nu_1,\nu_2-1) & \nu_1 \neq 1 \end{cases}$$

$$\Rightarrow \qquad I(\nu_1,\nu_2) = 0 \qquad \text{or} \qquad I(\nu_1,\nu_2) \propto I(1,1)$$

Linear Relations — IBPs

$$I(x;\nu;D) = \int \left(\prod_{j=1}^{L} e^{\gamma_E \epsilon} \frac{d^D k_j}{i\pi^{D/2}} \right) \frac{\mathcal{N}(\{k_j \cdot k_l, k_j \cdot p_l\}; D)}{\prod_{j=1}^{p} (m_j^2 - q_j^2 - i\varepsilon)^{\nu_j}}$$

- ✓ IBP relations can generate integrals with extra propagators
  - A *topology* contains enough propagators for this not to happen
- ✓ Integrals in a topology are related by IBP relations, which are rational in scales and D
  - Integrals in a topology related to a basis of integrals, called master integrals
- ✓ The number of master integrals is always finite
  - Can be computed from critical points, Euler characteristics, …
  - Only a finite number of integrals needs to be computed to solve a topology
- ✓ Each topologies defines a (finite dimensional) vector space
  - Like for any vector space, some bases are better than others

(See Christoph's, Pavel's, Sebastian's talks)

Linear Relations — Dimension Shifting

$$I(x;\nu;D) = e^{\gamma_E L\epsilon}\,\Gamma\left(|\nu| - \frac{LD}{2}\right)\prod_{j=1}^{p}\int_0^\infty d\alpha_j\,\frac{\alpha_j^{\nu_j-1}}{\Gamma(\nu_j)}\,\delta\left(1-\sum_{j=1}^{p}\alpha_j\right)\frac{\mathcal{U}(\alpha)^{|\nu|-\frac{(L+1)D}{2}}}{\mathcal{F}(\alpha;x)^{|\nu|-\frac{LD}{2}}}$$

- ✓ Relations between different ν_j ~ relations between different $D/2$
- ✓ Go up in dimensions, $D-2 \to D$

$$I(x;\nu;D-2) = (-1)^L\,\mathcal{U}\left(\frac{\partial}{\partial m_1^2},\ldots,\frac{\partial}{\partial m_p^2}\right)I(x;\nu;D)$$

- ✓ Go down in dimensions, $D+2 \to D$ (b_i lowers ν_i by 1)

$$I(x;\nu;D+2) = \frac{2^L\,G(p_1,\ldots,p_{E-1})}{(D-K+1)_L}\mathcal{B}(b_1,\ldots,b_K)I(x;\nu;D)$$

- ✓ Integrals in different dimensions can be used when building basis of master integrals
- ✓ Combine with IBPs to simplify r.h.s. of relations

Linear Relations — Solving IBP relations

- Major bottleneck in many applications

- Laporta's algorithm, the most successful approach
  - build relations for explicit values of ν_j, within some $|\nu|$ bound
  - solve (very!) large linear system
  - new approaches based on finite fields and functional reconstruction
  - algorithmic approach, scales badly with $|\nu|$

- Solve recurrence relations (what we did for the bubble example)
  - construct all IBP relations, and solve the recurrence relations
  - full solution, not algorithmic, contains too much information (we never need to reduce integrals with very large $|\nu|$)

- Intersection theory
  - build on the vector space perspective
  - construct operators to project integrals onto a basis
  - elegant new formalism, still not competitive with Laporta's algorithm

(See Pouria's talk)

DIFFERENTIAL EQUATIONS

Compute master integrals

Pure bases (what, why, and how)

Compute integrals and organise analytic structure (symbols, special functions)

Beyond MPLs?

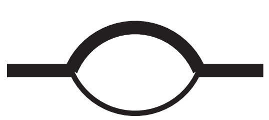
Differential Equations — Generic Basis

- Let $\vec{\mathcal{I}}$ be a vector of master integrals. It's closed under differentiation

$$\partial_{x_i} \vec{\mathcal{I}}(x,\epsilon) = A_{x_i}(x,\epsilon)\, \vec{\mathcal{I}}(x,\epsilon)$$

  - derivatives change powers of propagators ⇒ reduce to masters with IBPs
  - IBP relations are rational ⇒ $A_{x_i}(x,\epsilon)$ has rational entries

- Example: one-loop bubble with one massive propagator, $\mathcal{I} = \{I(1,1), I(1,0)\}$



$$\partial_{m_1^2} \vec{\mathcal{I}} = \begin{pmatrix} -I(2,1) \\ -I(2,0) \end{pmatrix} = \begin{pmatrix} \frac{(D-3)(m_1^2-p^2)}{(p^2-m_1^2)^2} & \frac{(D-2)(m_1^2-p^2)}{2m_1^2(p^2-m_1^2)^2} \\ 0 & \frac{D-2}{2m_1^2} \end{pmatrix} \vec{\mathcal{I}}$$

- By solving the differential equations we evaluate all master integrals

- Complicated to solve for generic basis $\mathcal{I}$

- Different orders in the ϵ expansion of the integrals mix in the differential equation

Differential Equations — Pure basis

- For large classes of integrals we can do better (e.g., those that evaluate to MPLs)!
  - find new basis $\vec{\mathcal{J}}(x,\epsilon)$ such that

$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon\, A(x)\, \vec{\mathcal{J}}(x,\epsilon)$$

$$A(x) = \sum_i A_i\, d\log W_i$$

  - A_i are matrices of rational numbers, all x dependence in W_i
  - differential equation is in canonical (dlog) form
  - only has logarithmic singularities, explicit in the differential equation
  - different orders in ϵ don't mix
  - solution trivial to write in terms of MPLs, order by order in ϵ

- Basis change between generic basis $\vec{\mathcal{I}}$ and pure basis $\vec{\mathcal{J}}$ not rational (but algebraic)

- No general algorithm to find a pure basis (but some automated codes exist)
  - leading singularities (see William's talk)
  - cuts of Feynman integrals, on-shell differential equations
  - ideas from $\mathcal{N} = 4$

Differential Equations — Pure basis example

- ✓ Pure basis: basis transformation for $\mathscr{I} = \{I(1,1), I(1,0)\}$

$$\vec{\mathscr{I}}(p^2, m_1^2; 2-2\epsilon) = \frac{1}{\epsilon}\begin{pmatrix} \frac{1}{p^2 - m_1^2} & 0 \\ 0 & 1 \end{pmatrix} \vec{\mathscr{J}}(p^2, m_1^2; 2-2\epsilon)$$

- ✓ Differential equation in canonical form ($u = p^2/m_1^2$)

$$\partial_u \vec{\mathscr{J}}(u;\epsilon) = \epsilon \left[\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \mathrm{dlog}\,(1-u) + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \mathrm{dlog}\,u \right] \vec{\mathscr{J}}(u;\epsilon)$$

- ✓ Boundary condition: solution should be regular at $u = 0$, fixes bubble w.r.t. tadpole

$$\mathscr{J}_2(\epsilon) = e^{\gamma_E \epsilon}\Gamma(1+\epsilon)$$

- ✓ Solution

$$\mathscr{J}_1(u;\epsilon) = 1 - 2\,\epsilon\log(1-u) + \epsilon^2\left(\frac{\pi^2}{12} + 2\log^2(1-u) + 2\,\mathrm{Li}_2(u)\right) + \mathcal{O}\left(\epsilon^3\right)$$

Differential Equations — Pure basis and analytic structure

$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon\, A(x)\, \vec{\mathcal{J}}(x,\epsilon)$$

$$A(x) = \sum_i A_i \, d\log W_i$$

- ✓ Can learn a lot without solving the equation!
- ✓ Directly read the alphabet and build the symbol of the topology $\vec{\mathcal{J}}$
  - Useful input for ansatzing coefficients of amplitudes (same singular points)
  - Study analytic properties of $\vec{\mathcal{J}}$
  - Bootstrapping approaches
  - Discontinuities, (extended) Steinmann relations, …
- ✓ Trivial to solve in terms of Chen iterated integrals, order by order in ϵ
  - Construct basis of special functions algorithmically
  - Build dedicated codes to evaluate topology $\vec{\mathcal{J}}$

Differential Equations — Beyond MPLs

- ✓ Very active area of study
- ✓ ϵ-factorisation helpful for numerical solutions
- ✓ What are pure elliptic (and beyond) functions?
- ✓ How to extract/organise analytic structure from DEs beyond MPLs? What is the symbol?

(See Christoph's, Sebastian's talks)

EVALUATING FEYNMAN INTEGRALS

From representation in terms of `known functions'

Directly from DEs

With dedicated codes

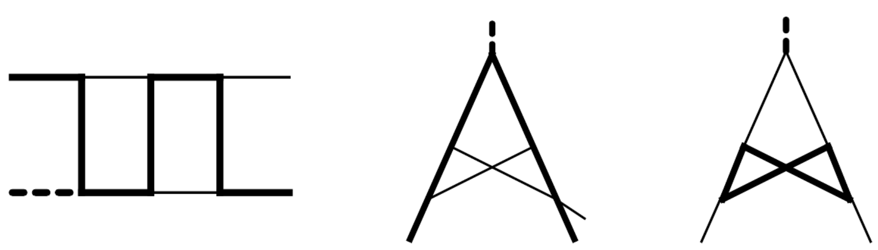
Evaluating Feynman Integrals — Known Functions

- Solve Feynman integrals in terms of known functions
  - Classical polylogarithms $\mathrm{Li}_n(x)$, MPLs $G(\vec{a}; x)$
  - eMPLs $\mathscr{E}_{3/4}/\tilde{\Gamma}$ or iterated integrals of modular forms

- Use publicly available codes (GiNaC, HandyG) when available

- Representation is region specific (branch cuts), introduces spurious poles
  - Slow convergence

- Example: Elliptic integrals in quarkonium two-loop corrections
  - Very large expressions with thousands of eMPLs
  - Several days to get ~7 digits
  - Same performance as Monte-Carlo codes like pySecDec



Evaluating Feynman Integrals — From DEs

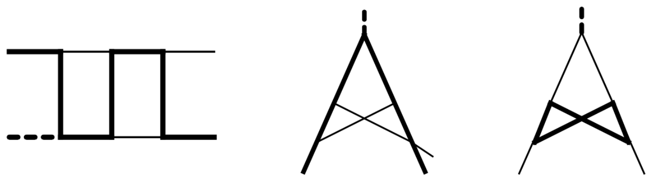
$$\partial_{x_i} \overrightarrow{\mathcal{I}}(x,\epsilon) = A_{x_i}(x,\epsilon)\,\overrightarrow{\mathcal{I}}(x,\epsilon)$$

- ✓ Numerically solve differential equations (public codes: DiffExp, AMFlow)
  - Start from known initial condition, and evolve along path
  - Generalised power-series solution with finite convergence radius

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{N_{i,k}} \mathbf{c}_k^{(i,j_1,j_2)}(t-t_k)^{\frac{j_1}{2}}\log(t-t_k)^{j_2}$$

  - Match solutions along path

- ✓ Requires building differential equation (more efficient with pure basis)
- ✓ Very high-precision solution at each point
- ✓ Ideal for few dynamical scales, a bit slow for phenomenology when many scales
- ✓ Example: $\mathcal{O}(1000)$ digits for quarkonium two-loop corrections



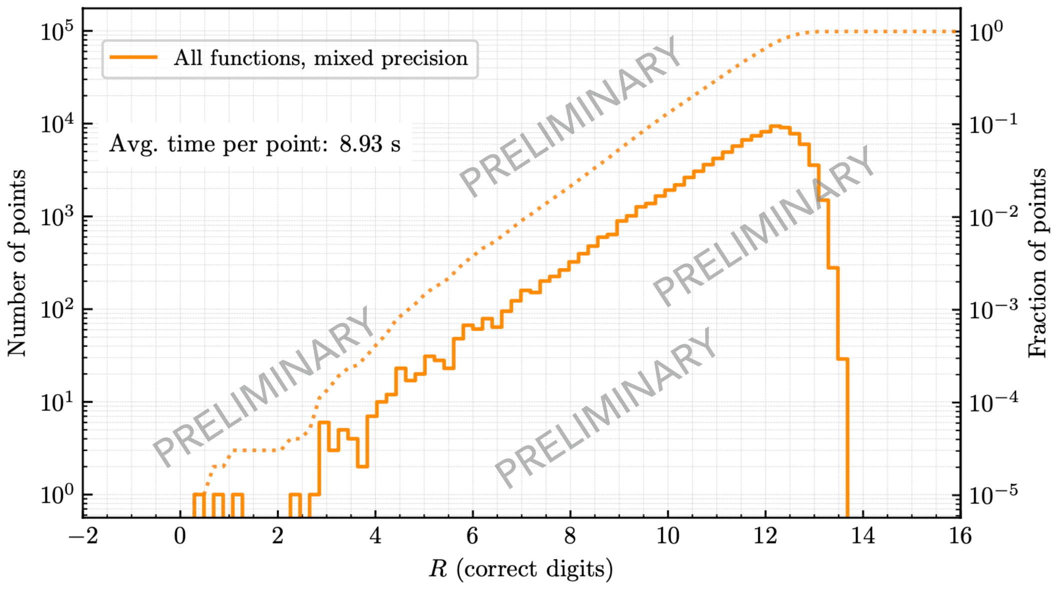
Evaluating Feynman Integrals — Dedicated Codes

- ✓ For fast evaluation in multi-dimensional phase-space
  - Complicated branch-cut structure ⇒ inefficient with known functions
  - Large phase-space ⇒ many numerical evaluations needed

- ✓ Build special basis for a given topology from differential equation
  - Pentagon functions, hexagon functions, …

- ✓ Build special basis for a given topology from differential equation

- ✓ Example: two-loop five-point one mass



SUMMARY AND OUTLOOK

Summary & Outlook

- ✓ Feynman integrals appear in all perturbative calculations
    - Several approaches to compute and study them
    - A lot of technology has been developed in the last decades

- ✓ For integrals evaluating to MPLs, we have very mature tools
    - Not yet at the edge of what can be achieved with it

- ✓ State of the art is at the evaluation of elliptic integrals and beyond
    - How to organise their analytic structure?
    - How to efficiently compute them?

- ✓ Many more interesting topics that I did not have the time to mention here…

THANK YOU!