UPPSALA UNIVERSITET

Chern-Simons meets DBI

Maor Ben-Shahar

December 12 2022

QCD meets gravity
Based on:

[M.B.S, H. Johansson, 2112.11452]

[M.B.S, M. Guillen, 2108.11708]

[M.B.S, L. Garozzo, H. Johansson, 2212.XXXX]

Outline

- BCJ relations and double copy
- Chern-Simons matter and on-shell double copy
- Off-shell color-kinematics duality for pure CS
- Kinematic algebra and off-shell double copy
- Summary and upcoming work

Background and Motivation

- Ongoing search for theories obeying the CK duality [review: 1909.01358]
- Typically conformal [Cheung, Mangan, Shen], supersymmetric [Chiodaroli, Jin, Roiban]
- Double-copy table [Chi, Elvang, Herderscheea, Jonesa, Paranjape]

	$NLSM_2$	YM_4	BAS_6
$NLSM_2$	$sGAL_{-2}$	BI_0	$NLSM_2$
YM_4	BI_0	GR_2	YM_4
BAS_6	$NLSM_2$	YM_4	BAS_6

$$D_L + D_R - 6 = D_{d.c.}$$

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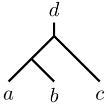
- Can we double-copy 3d CS theory?
- 3-Lie algebra in $\mathcal{N} = 6$ ABJM and $\mathcal{N} = 8$ BLG studied before [Bargheer, He, McLoughlin; Johansson, Huang].
- Off-shell color-kinematics is still an open problem.

Color-Kinematics duality

Relationship between scattering amplitude building blocks: [Bern, Carassco, Johansson]

$$\mathcal{A}_n = \sum_{i \in \Gamma_n} \frac{c_i n_i}{D_i}$$



$$= \frac{f^{abx} f^{xcd} \times n\left(\quad\right)}{(p_a + p_b)^2}$$

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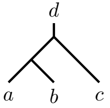

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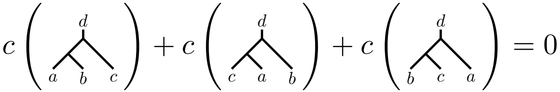
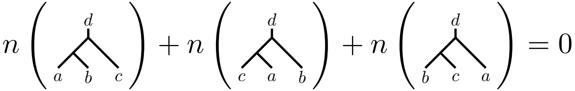
$$c_i + c_j + c_k = 0 \quad \Leftrightarrow \quad n_i + n_j + n_k = 0 \ ,$$





Double copy:

$$\mathcal{M}_n = \sum_{i \in \Gamma_n} \frac{\tilde{n}_i n_i}{D_i} \ .$$

BCJ relations and double-copy

On-shell picture, for adjoint particles:

$$\mathcal{A}_n = \sum_{\sigma} A(1, \sigma_2, \dots, \sigma_n) \mathrm{Tr}(T^{a_1} T^{a_{\sigma_2}} \dots T^{a_{\sigma_n}})$$

From DDM basis of $(n-2)!$ to $(n-3)!$ using BCJ relations:

Example ($s_{ij} \equiv (p_i + p_j)^2$):

$$s_{23} A(1234) = s_{13} A(1243) \; .$$

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KLT double copy [Kawai, Lewellen, Tye]

$$\mathcal{M}_n = \sum_{\sigma,\rho} A(1, \sigma, n-1, n) S[\sigma|\rho] \tilde{A}(1, \rho, n, n-1) \ .$$

four-point example:

$$\mathcal{M}_4 = s_{12} A(1234) A(1243)$$

Chern-Simons matter

Adjoint matter with all marginal couplings:

$$\mathcal{L} = \frac{\epsilon_{\mu\nu\rho}}{2}\left(A^{a\mu}\partial^{\nu}A^{a\rho} - \frac{gi}{3}f^{abc}A^{a\mu}A^{b\nu}A^{c\rho}\right) + (D_\mu\bar{\phi})^a(D^\mu\phi)^a$$
$$+ \alpha^{(1)}g^4\phi^a\bar{\phi}^b\bar{\phi}^d\phi^c\phi^e\bar{\phi}^h f^{abx}f^{xdy}f^{ycz}f^{zeh}$$

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$$A_4(\psi_1\bar{\psi}_2\psi_3\bar{\psi}_4) = \frac{\langle 13\rangle^3}{\langle 21\rangle\langle 14\rangle} \qquad A_4(\psi_1\bar{\psi}_2\bar{\psi}_4\psi_3) = \frac{\langle 42\rangle\langle 23\rangle}{\langle 21\rangle} \ \leftarrow \text{soft pole!}$$

Check BCJ relations, eg: $s_{24}A_4(\psi_1\bar{\psi}_2\bar{\psi}_4\psi_3) = s_{14}A_4(\psi_1\bar{\psi}_2\psi_3\bar{\psi}_4)$

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Constrain $\alpha^{(2,3)}$ from $A_4(\psi_1\bar{\psi}_2\bar{\phi}_3\phi)$ and $\alpha^{(1)}$ from A_6.

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BCJ relations imply:

- $\alpha^{(i)} = 1$
- Opposite statistics (!)
- $\mathcal{N} = 4$ SUSY ($SO(4)$ R-symmetry, no additional flavour)
- Same partial amplitudes as $\mathcal{N} = 4$ CS with bi-fundamental matter [Gaiotto, Witten], ($\mathcal{N} = 4$ truncations of ABJM)

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How to realize $SO(4)$ R-symmetry? Define $\phi_\alpha = (\bar{\phi}, \phi)$, $\psi_{\dot{\alpha}} = (\bar{\psi}, \psi)$

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$$\begin{aligned}\delta\phi_\alpha &= \bar{\xi}_\alpha^{\ \dot{\alpha}} \psi_{\dot{\alpha}} \ , \\ \delta A^a_\mu &= g\bar{\xi}^{\alpha\dot{\alpha}} \gamma_\mu \psi^b_{\dot{\alpha}} \phi^c_\alpha f^{abc} \ , \\ \delta\psi^a_{\dot{\alpha}} &= i(\not{D}\phi^\alpha)^a \xi_{\alpha\dot{\alpha}} + \dots \ ,\end{aligned}$$

Supersymmetric amplitudes with on-shell supercharges Q^α_A:

$$A(\Psi_1\Psi_2\Psi_3\Psi_4) = \delta^4(Q)\frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

Double Copy

What is the double copy of this $\mathcal{N}=4$ CS theory?

$$\mathcal{M}_4 \;=\; s_{12}A(1234)A(1243) = \delta^8(Q) \qquad \text{(no pole!)}$$

Match DBI with maximal SUSY in 3D.

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Correct statistics after double copy!

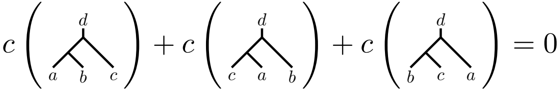
$$(\phi + \psi + c.c.) \otimes (\phi + \psi + c.c.) \to \varphi_i + \psi_i$$

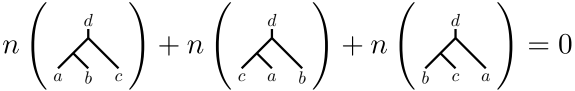
R-symmetry: $SO(4) \times SO(4) \to SO(8)$.

Off-shell Color-Kinematics duality

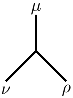
Can CK duality hold off-shell?
Is there a kinematic algebra responsible for the kinematic identities?

$$c_i + c_j + c_k = 0 \quad \Leftrightarrow \quad n_i + n_j + n_k = 0 \ ,$$





Feynman rules of pure CS in Lorenz $\partial_\mu A^\mu = 0$ gauge:



$= \epsilon^{\mu\nu\rho}$

$= \dfrac{\epsilon^{\mu\, p\, \nu}}{p^2}$

Kinematic algebra in Chern-Simons theory

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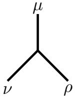

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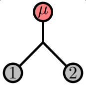
n ()

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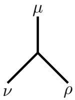

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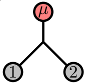
$n\Big($  $\Big) = \epsilon_{A_1 A_2 \nu}\epsilon^{\nu\, p_{12}\, \mu}$ (transverse to $p_{12} = p_1 + p_2$)

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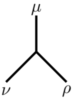

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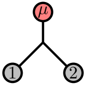
$n\Big($ 

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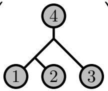
$n\Big($ 

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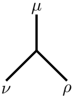

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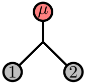
$n\Big($  $\Big) = \epsilon_{A_1 A_2 \nu} \epsilon^{\nu\, p_{12}\, \mu}$

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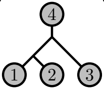
$\Big[\, n\Big($  $\Big) = \epsilon^{A_1 A_2 \sigma} \epsilon_{\sigma\, p_{12}\, \nu} \epsilon^{\nu A_3 A_4} \Big] + \text{cyclic}(1,2,3) = 0$

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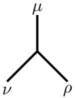

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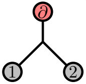
n () $= [A_1 \cdot \partial, A_2 \cdot \partial]$

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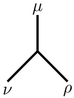

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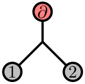
n ()

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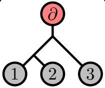
n ()

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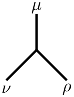
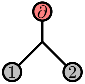
Kinematic algebra in Chern-Simons theory

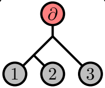
Feynman rules in Lorenz $\partial_\mu A^\mu = 0$ gauge:



$= \epsilon^{\mu\nu\rho}$

$= \dfrac{\epsilon^{\mu\, p\, \nu}}{p^2}$

n () $= [A_1 \cdot \partial, A_2 \cdot \partial]$

n () $= [[A_1 \cdot \partial, A_2 \cdot \partial], A_3 \cdot \partial]$

Algebra of volume preserving diffeos: under $x_\mu \to x_\mu + A_\mu + \ldots$, $d^3x \to d^3x$ (for $\partial \cdot A = 0$)

Kinematic algebra in Chern-Simons theory

What about ghosts?

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What about ghosts? Consider Axelrod-Singer formulation:

$$S = \frac{k}{2\pi} \int d^3x d^3\theta \, \mathrm{Tr} \left(\frac{1}{2} \Psi Q \Psi + \frac{i}{3} \Psi \Psi \Psi \right) \ ,$$

Superfield for Faddeev-Popov ghosts and vector:

$$\Psi = c + \theta_\mu A^\mu + \theta_\mu \theta_\nu \epsilon^{\mu\nu\rho} \partial_\rho \bar{c} \ ,$$

Kinetic (exterior derivative, world line BRST operator)

$$Q = \theta_\mu \partial^\mu \ .$$

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To define the propagator $b = \frac{\partial}{\partial \theta^\mu} \partial^\mu$ (co-differential, b-ghost):

- $b^2 = 0$
- $bQ + Qb = \partial_\mu \partial^\mu = \Box$

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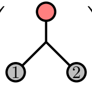
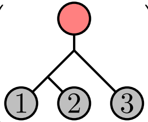
n ( $) = b(\Psi_1 \Psi_2)$

n ( $) = b(b(\Psi_1 \Psi_2)\Psi_3)$

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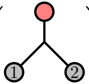
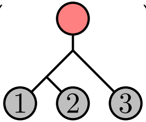
$n\Big($  $\Big) = b(\Psi_1\Psi_2)$

$n\Big($  $\Big) = b(b(\Psi_1\Psi_2)\Psi_3)$

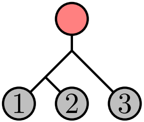
From Leibniz rule: $b = \frac{\partial}{\partial\theta^\mu}\partial^\mu$:

$$b(b(\Psi_1\Psi_2)\Psi_3) + \mathrm{cyclic}(1,2,3) = 0$$

Using $b^2 = 0$ which implies $b(\Psi_i) = 0$.

Kinematic algebra in Chern-Simons theory

Algebraic interpretation of:



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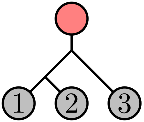

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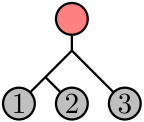

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Same Jacobi identity for diffeo generators:

$$L_\psi(f) \equiv b(\psi f) \ , \qquad [L_\psi, L_\phi] = L_{b(\psi\phi)}$$

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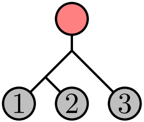

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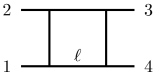
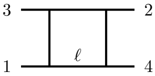
diffeos in (x,θ) space that preserve $d^3x \to d^3x$, $d^3\theta \to d^3\theta$.

One loop example:



$$= \int d^3\theta d^3\tilde{\theta}\, \Psi_4 \delta^3_{\theta,\tilde{\theta}}\, b\Big(b\big(b(\Psi_1 \tilde{b}(\delta^3_{\theta,\tilde{\theta}}))\Psi_2\big)\Psi_3\Big)\ ,$$



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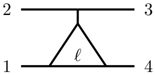

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Define $X \equiv b(\Psi_1 \tilde{b}(\delta^3_{\theta-\tilde{\theta}}))$, which satisfies $bX = 0$,

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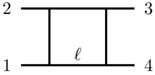

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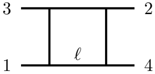

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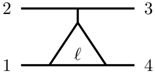

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Define $X \equiv b(\Psi_1 \tilde{b}(\delta^3_{\theta-\tilde{\theta}}))$, which satisfies $bX = 0$,

$$\begin{aligned} \rightarrow \quad & b(b(X\Psi_2)\Psi_3) - b(b(X\Psi_3)\Psi_2) - b(Xb(\Psi_2\Psi_3)) \\ = \quad & b(b(X\Psi_2)\Psi_3) + b(b(\Psi_3 X)\Psi_2) + b(b(\Psi_2\Psi_3)X) = 0 \end{aligned}$$

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Linearized cotton tensor appears in expansion of LHS

$$C^{\mu\nu} = \epsilon^{\mu\alpha\beta} D_\alpha R^\nu_\beta + (\mu \leftrightarrow \nu) \ ,$$

$C^{\mu\nu} = 0$ follows from the action

$$S_{\mathrm{CSgrav}} = \frac{1}{2\pi} \int d^3 \epsilon^{\mu\nu\rho} \left(\Gamma^\alpha_{\mu\beta} \partial_\nu \Gamma^\beta_{\rho\alpha} + \frac{2}{3} \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\gamma} \Gamma^\gamma_{\rho\alpha} \right) \ .$$

- no scattering amplitudes to test this

Conclusion

- Obtained CS-matter theories satisfying CK duality
- BCJ implied SUSY (related [Chiodaroli et. al. 1311.3600])
- Double copy to DBI, up to maximal $\mathcal{N} = 8$ SUSY

	$NLSM_2$	$CS(m)_3$	YM_4	$(DF)^2_6$
$NLSM_2$	$SGAL_{-2}$	?	DBI_0	$R + DF^2$
$CS(m)_3$	?	DBI_0	?	CSG_3
YM_4	DBI_0	?	GR_2	CG_4
$(DF)^2_6$	$R + DF^2$	CSG_3	CG_4	R^3

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- Action: $S \sim \langle \Psi Q \Psi + \Psi^3 \rangle$
- Propagator-numerator b: $b^2 = 0$, second-order wrpt vertex
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- Action: $S \sim \langle \Psi Q \Psi + \Psi^3 \rangle$
- Propagator-numerator b: $b^2 = 0$, second-order wrpt vertex
- Gauge $b\Psi = 0$.
- Algebra and CK follow from second order b operator
- See related: [Reiterer; MBS, Guillen 2108.11708]
- The algebra implies off shell CK duality, and allows to construct any loop diagram.

Future research directions

- Wilson loops?
- What are the double copy actions?
- CK duality for more $\langle \Psi Q \Psi + \Psi^3 \rangle$ actions
- Upcoming work on YM [MBS, Garozzo, Johansson]
  - Lagrangian with fields $A,\ (B, \tilde{B}),\ (Z, \tilde{Z}),\ (X, \tilde{X})$
  - tree level BCJ numerators in nMHV sector, inspired by fusion algebra [Gang Chen et. al.]

but off shell CK duality is still a mystery!